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Maximum Flow in Self-Similar Flow Networks

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ABSTRACT

Special structure of flow networks can reduce the order of the relative existent algorithms. In this paper a special structure of flow network called self-similar flow networks is introduced. After describing such networks an efficient algorithm for finding maximum flow is presented. It is shown that this algorithm runs in $O(m)$ time.

1. INTRODUCTION

In the theory of network flows various types and structures of networks have been considered. Some types have been only theoretically assumed, but most of them were born from a variety of applications in the real world [1, 2, 3]. There have been some considered network structures that have highly affected the order of the related problem solving algorithms. Layered [6] and serial-parallel [4] networks have such structures. Here we introduce a special structure of network flow which exists in some natural systems and has also applications in computer networks. We call this structure as self-similar flow network.

Consider the flow network $G = (N, A)$ with nodes s and t as the source and sink nodes, respectively. Suppose that η is an operation that replaces each arc $(i,j) \in A$ by G taking s as i and t as j. The consequential flow network after ap-

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plying η for k times is called the k^{th} *self-similar flow network* and is denoted by $G_k = (N_k, A_k)$. The source and sink of G_k are denoted by s_k and t_k respectively. The flow network $G_0 = (N_0, A_0)$, which is the primary flow network, $G = (N, A)$ is called *basic network*. The nodes of basic network which are remained in all of the subsequent networks are called *basic nodes*.

Example 1
Consider the basic network in Figure 1.

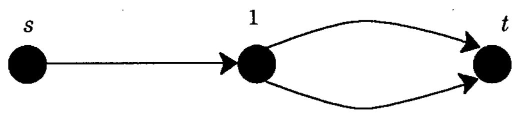

Figure 1. Basic network

Applying η on this network results in the network of Figure 2. Basic nodes in Figure 1 and Figure 2 are bold.

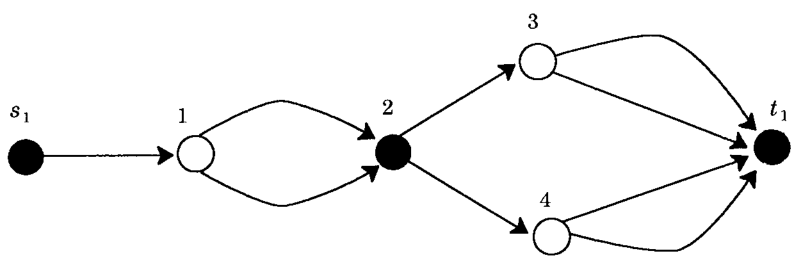

Figure 2. A self-similar flow network

Similarly, the 2^{nd} self-similar flow network will be obtained by applying η on the 1^{st} self-similar network. It can be shown that applying η on G_{k-1} is equivalent to applying η on the basic network for k times.

2. DEFINITIONS & BASIC CONCEPTS

Now we first state and then prove the relation between the number of nodes and arcs in the k^{th} self-similar and basic networks. To do this, suppose that m_j and

n_j are the number of arcs and nodes in the j^{th} self-similar flow network respectively, that is $|A_j| = m_j$ and $|N_j| = n_j$. Furthermore let $n_0 = n$, $m_0 = m$ and the arcs of the basic network are numbered 1 to m.

Theorem 1: a) $m_k = m^{k+1}$

b) $n_k = n + (n-2)\sum_{i=1}^{k} m^i \quad k \geq 1$

Proof: a) By induction on k if $k = 1$ then by definition of η we have:

$$m_1 = \sum_{(i,j)\in A} m = m\,|A| = m^2$$

Now suppose that $m_j = m^{j+1}$ $(j \geq 2)$ *then* $m_{j+1} = \sum_{(u,v)\in A_j} m = m\,|A_j| = m^{j+2}$.

b) Regarding the η operation to construct G_k, each arc of the basic network is replaced with G_{k-1}. Since nodes i and j are taken as s and t each arc (i,j) of the basic network adds $n_{k-1} - 2$ nodes to G_0. Therefore

$$n_k = m(n_{k-1} - 2) + n$$

and by repeating the evaluation of n_{k-i} for $i = 1, \cdots, k$ we get

$$n_k = n + (n-2)\sum_{i=1}^{k} m^i. \;\blacksquare$$

Lemma 1: The number of G_l in G_k is m^{k-1}, where $l = 0, 1, \cdots, k-1$.

Proof: Using Theorem 1, we have:

$$m_k = m^{k+1} = m^{k-1}.m^{l+1}$$

Considering η and this fact that the number of arcs in G_l is $m^{l+1} (l = 0, 1, \cdots, k-1)$, the above expression implies that the number of G_l in G_k is m^{k-1}. $\blacksquare$

Theorem 2: If G_0 does not include a path from t to s , then there is no path form t_k to s_k in G_k for $k = 1, 2, \cdots$.

Proof: Suppose that k is the smallest index such that G_k contains a path from t_k to s_k. Let $P_k = (t_k = i_1, i_2 \cdots, i_q, \cdots, i_l = s_k)$ be one of these paths. Suppose that i_q is a basic node (note that such node exist). Regarding η there exist two cases:

Case 1. $(i_q, t_k) \in A_0$. In this case i_q is the source node in G_{k-1}. But G_{k-1} is between two basic nodes in G_k thus the subpath $(t = i_1, i_2, \cdots, i_q = s)$ is in G_{k-1}. That is there exist a path from t_{k-1} to s_{k-1} which is contradiction.

Case 2. $(t_k, i_q) \in A_0$ Suppose that j is the smallest index greater than q such that i_j is a basic node. Then either $(i_j, i_q) \in A_0$ or $(i_q, i_j) \in A_0$. If $(i_j, i_q) \in A_0$ then i_q and i_j are the source and sink of G_{k-1}, respectively. So there exist a path from t_{k-1} to s_{k-1} in G_{k-1}, which is a contradiction. Otherwise, if $(i_q, i_j) \in A_0$ then there is a path from t_k to i_j. By repeating this reasoning at most to the number of basic nodes in P_k we find a path from t to s in G_0, a contradiction. ∎

It is easy to show that if G contains a path from t to s then, corresponding to this path, there exists a path from s_1 to t_1 in G_1.

Theorem 3: If G_0 does not include a directed cycle then there is no directed cycle in G_k for $k \geq 1$.

Proof: Suppose that k is the smallest index such that G_k contains a directed cycle. Let $C_k = (i_1, i_2, \cdots, i_q, i_1)$ be one of theses cycle. Without lost of generality, suppose that i_1 belong to the l^{th} G_{k-1}. In G_k there exist an integer j such that $2 \leq j \leq q$ and does not belong to the l^{th} G_{k-1}. Otherwise all nodes of C_k. belong to the l^{th} G_{k-1} which is a contradiction. Note that i_j is a basic node. Define $P = (i_1, i_j)$ as a path from i_1 to i_j. By repeating this process at most to the number of basic nodes in C_k, we find a $P = (i_1, i_j, \cdots, i_1)$ that all of it's nodes are basic. So P belongs to the basic network. But P becomes a directed cycle in G_0 that is a contradiction. ∎

All of the self-similar subnetworks of G_k are isomorphic [5] with G_{k-1}. Suppose that $i \in N_{k-1}$, the set of nodes in G_k which are projected on node i are called the *corresponding nodes to node i* and denoted by M_i^k. For example the corresponding nodes to node 1 in Figure 1 is $M_1^1 = \{1, 3, 4\}$. With this definition we have the following properties:

Property 1: If $i \in N_{k-1}$, and $i \neq s_{k-1}, t_{k-1}$ then all members of M_i^k are non-basic.

Property 2: If $i, j \in N_{k-1}, i \neq j$ and $i, j \neq s_{k-1}, t_{k-1}$ then $M_i^k \cap M_j^k = \Phi$.

Property 3: If $i \in N_k$ and i is a basic node then $i \in M_{s_{k-1}}^k$ or $i \in M_{t_{k-1}}^k$.

Property 4: $\left| M_{s_{k-1}}^k \cup M_{t_{k-1}}^k \right| = n.$

Theorem 4: Suppose that $i \in M_{s_{k-1}}^k \cup M_{t_{k-1}}^k$ and $j \in N_0$ is the corresponding node in G_0 then we have:

$$indegree(i) = indegree(j) \,.\, indegree(t_{k-1})$$
$$outdegree(i) = outdegree(j) \,.\, outdegree(s_{k-1}).$$

Proof: In using η for construction G_k we replace each arc $(i, j) \in A$ with G_{k-1} by taking $l = s_{k-1}$ and $j = t_{k-1}$. So for each incoming arc to node j in G_{k-1}, the number of incoming arc to the corresponding node j in G_k is equal to the number of incoming arc to node t_{k-1}. Thus:

$$indegree(i) = indegree(j) \,.\, indegree(t_{k-1})$$

Similarly, we can show that

$$outdegree(i) = outdegree(j) \,.\, outdegree(s_{k-1}). \quad ∎$$

Corollary 1: With condition of Theorem 4 we have:

$$indegree(i) = indegree(j) \,.\, (indegree(t_0))^k$$

$$outdegree\,(i) = outdegree\,(j)\,.\,(outdegree\,(s_0))^k.$$

3. MAXIMUM FLOW IN SELF-SIMILAR FLOW NETWORKS

In the following section the problem of maximum flow in self-similar flow networks is considered and an algorithm for computing the maximum flow is provided. It is shown that this algorithm runs in $O(m_k)$ time. In the following definitions the basic networks in G_k are numbered in the same order as arc's numbers in the basic network.

In G_k we define:

x_j^i : The amount of flow of the i^{th} arc in the j^{th} basic network.
$(i = 1, 2, \cdots, m \; \& \; j = 1, 2, \cdots, m^k)$

u_j^i : The capacity of the i^{th} arc in the j^{th} basic network.
$(i = 1, 2, \cdots, m \; \& \; j = 1, 2, \cdots, m^k)$

f_j^i : Maximum flow in the j^{th} G_j.
$(i = 1, 2, \cdots, k-1 \; \& \; j = 1, 2, \cdots, m^{k-i})$

Furthermore, let $f(u_1, u_2, \cdots, u_m)$ be the maximum flow in the basic network where u_i is the capacity of the i^{th} arc.

Theorem 5: Suppose that G contains no path from t to s. The maximum flow in : $G_k, k \geq 1$ is:

$$f^* = f(f_1^{k-1}, f_2^{k-1}, \cdots, f_m^{k-1})$$

where

$$f_j^i = f(f_{mj-(m-1)}^{i-1}, f_{mj-(m-2)}^{i-1}, \cdots, f_{mj}^{i-1}) \quad i = k-1, k-2, \cdots, 1 \quad j = 1, 2, \cdots, m^{k-i}$$

$$f_j^0 = f(u_j^1, u_j^2, \cdots, u_j^m) \quad j = 1, 2, \cdots, m^k$$

Proof: By mathematical induction on k. If $k = 0$ obviously $f(\cdot)$ is the maximum flow in G. Suppose that the result is valid for G_k. Let f_1^{k-1} for

$i = 1, \cdots, m$ be the maximum flow in the i^{th} network of G_{k-1}. Regarding the η operation, it is sufficient to replace the maximum flow in the i^{th} network of G_{k-1} with the capacity of the i^{th} arc in G. Since the basic network contains no path from t to s, and noting Theorem 2 we have:

$$f^* = f(f_1^{k-1}, f_2^{k-1}, \cdots, f_m^{k-1}). \blacksquare$$

Now, using Theorem 5 the following algorithm for computing the maximum flow in a self-similar flow networks is provided. In Theorem 6 we will show that this algorithm runs in $O(m_k)$ time.

Algorithm MFSSFN

Let $f(u_1, u_2, \cdots, u_m)$ be the maximum flow in basic network.

$f[0, j] = f(u[1, j], u[2, j], \cdots, u[m, j])$,

$j = 1, 2, \cdots, m^k$ // u_j^i is stored in $u[i, j]$ and f_j^i in $f[i, j]$.

$f[i, j] = f(f[i-1, mj-(m-1)], f[i-1, mj-(m-2)], \cdots, f[i-1, mj])$

$i = 1, 2, \cdots, k \quad j = 1, 2, \cdots, m^{k-i}$

MaxFlow = $f[k, 1]$

Theorem 6: The MFSSFN algorithm runs in $O(m_k)$ time.

Proof: Suppose that the constant p is the number of operations for computing the maximum flow in G_0. Pointing to the concept of basic nodes, the number of computations in G_1 is $(m+1)p$, because G_1 contains m basic network and p operations is needed for overall computations. Similarly the number of computations in G_2 is

$$(m(m+1)+1)p = (m^2 + m + 1)p$$

In the same manner, the number of computations in G_k will be

$$\left(\sum_{i=0}^{k} m^i\right) p = \frac{m^{k+1} - 1}{m-1} p$$

which is of $O(m_k)$. ■

Example 2
Consider the following self-similar flow network shown by Figure 3.

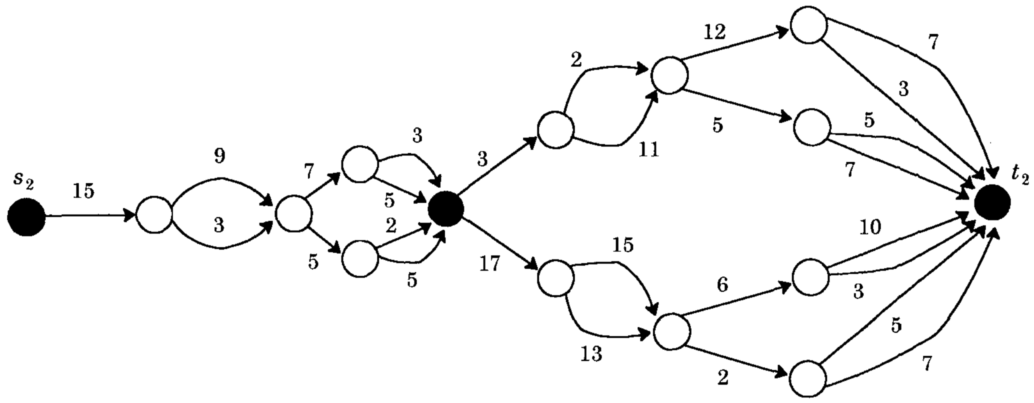

Figure 3. A self-similar flow network

The basic network of the above network is shown in Figure 4 with $f(u_1, u_2, u_3) = \min\{u_1, u_2 + u_3\}$.

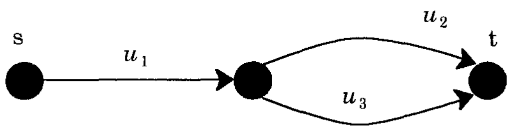

Figure 4. Basic network of figure 3

Using MFSSFN algorithm we have:

$f_1^0 = \min\{15,9+3\} = 12,\quad f_2^0 = \min\{7,3+5\} = 7,\quad f_3^0 = \min\{5,2+5\} = 5$

$f_4^0 = \min\{3,2+11\} = 3,\quad f_5^0 = \min\{12,7+3\} = 10,\quad f_6^0 = \min\{5,5+7\} = 5$

$f_7^0 = \min\{17,15+13\} = 17,\quad f_8^0 = \min\{6,3+10\} = 6,\quad f_9^0 = \min\{2,5+7\} = 2$

$f_1^1 = \min\{f_1^0, f_2^0 + f_3^0\} = \min\{12,7+5\} = 12$

$f_2^1 = \min\{3,10+5\} = 3$

$f_3^1 = \min\{17,6+2\} = 8$

$f^* = f_1^2 = \min\{f_1^1, f_2^1 + f_3^1\} = 11.$

4. CALCULATING ARC'S FLOW

Using MFSSFN algorithm, only the maximum flow value will be obtained. In order to compute the arc's flow first we find the relation between coefficient matrixes of self-similar flow networks. Suppose that $\mathbf{A}^k$ is the incidence matrix of G_k. To show the relation between $\mathbf{A}^k$ and $\mathbf{A}^{k-1}$ we consider two cases:

Case a) $i \in M_j^k$ & $j \neq s_{k-1}, t_{k-1}$ (i is a non-basic node and j is it's corresponding in G_{k-1}). Regarding to the order of enumerating the arcs and self-similar flow networks of previous stages, it is clear that:

$$\mathbf{a}_k^i = (\mathbf{0}_1, \mathbf{a}_{k-1}^j, \mathbf{0}_2)$$

where, $\mathbf{a}_q^p$ is corresponding row to node p *in* G_q, and if node j is in the l^{th} network of G_{k-1}, then $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vector in $R^{m_{k-1}(l-1)}$ and $R^{m_{k-1}(m-l)}$ respectively.

Case b) $i \in M_{s_{k-1}}^k \cup M_{t_{k-1}}^k$. Let $j \in N_0$ be the corresponding node of i. According to Theorem 4, row $\mathbf{a}_k^i$ include $\mathbf{a}_{k-1}^{t_{k-1}}$ and $\mathbf{a}_{k-1}^{s_{k-1}}$ repeated *indegree*(j) and *outdegree*(j) times, respectively.

Remembering that the number of G_{k-1} in G_k is m and considering the order of enumerating the arcs, the first m_{k-1} columns of $\mathbf{A}^k$ are correspondent to the arcs of the first G_{k-1}. In the same manner, the l^{th} group that contains m_{k-1} columns of $\mathbf{A}^k$ corresponds to the arcs of the l^{th} G_{k-1}. On the other hand, there exist n basic nodes in G_k and the $(k-1)^{th}$ self-similar flow networks can only have these nodes in common. So these m partitions of $\mathbf{A}^k$ columns have at most n rows in common. Thus we have:

$$\mathbf{A}^k = \left[\overline{\mathbf{A}}_1^{k-1}, \overline{\mathbf{A}}_2^{k-1}, \cdots, \overline{\mathbf{A}}_m^{k-1}\right]$$

where $\overline{\mathbf{A}}_j^{k-1} (j = 1, \cdots, m)$ is the same as $\mathbf{A}^{k-1}$ plus $n_k - n_{k-1}$ zero rows. In other words, if we omit all zero rows of $\overline{\mathbf{A}}_j^{k-1} (j = 1, \cdots, m)$ we will obtain $\mathbf{A}^{k-1}$.

Now we define g_j^i as the flow which must blow over the j^{ih} G_i of G_k. With this definition the goal is to find g_j^0 for $j = 1, 2, \cdots, m^k$.

For computing the arc's flow it is sufficient to solve the following system:

$$\begin{aligned} &\mathbf{A}^k \mathbf{X} = \mathbf{F} \\ &\mathbf{X} + \mathbf{S} = \mathbf{U} \\ &\mathbf{X} \geq \mathbf{0}, \mathbf{S} \geq \mathbf{0} \end{aligned} \tag{1}$$

where

$$\mathbf{F} = \begin{bmatrix} f^* \\ \mathbf{0}_{n_k - 2} \\ -f^* \end{bmatrix}, \mathbf{X} = (x_j^i), \mathbf{U} = (u_j^i), \ \mathbf{S} = (S_j^i), \mathbf{0}_{n_k-2} \text{ is a zero vetro in } \mathbf{R}^{n_k - 2}$$

and $\mathbf{S}$ is the slack vector.

We define:

$$\mathbf{A}^k = \begin{bmatrix} \mathbf{A}^k & 0_{m_k} \\ \mathbf{I}_{m_k} & \mathbf{I}_{m_k} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{X} \\ \mathbf{S} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{F} \\ \mathbf{U} \end{bmatrix}$$

So system (1) is equivalent to the following system:

$$\begin{aligned} &\mathbf{A}^k \mathbf{X} = \mathbf{F} \\ &\mathbf{X} \geq \mathbf{0} \end{aligned} \tag{2}$$

Theorem 7 (Decomposition Theorem): Consider the following system:

$$\begin{aligned} &\mathbf{A}^k \mathbf{X} = \mathbf{F} \\ &\mathbf{X} \geq \mathbf{0} \end{aligned}$$

Solving this system is equivalent to solving the following subsystems:

$$\begin{aligned}
&\mathbf{A}^0\, G_j^i = \mathbf{F}_j^i \quad i = k, k-1, \cdots, 1 \quad j = 1, 2, \cdots, m^{k-i} \\
&\mathbf{G}_j^i \geq \mathbf{0} \\
&\qquad \& \\
&\mathbf{A}^0\, \mathbf{X}_j = \mathbf{B}_j \quad j = 1, 2, \cdots, m^k \\
&\mathbf{X}_j \geq \mathbf{0}
\end{aligned}$$

where

$$\mathbf{G}_j^i = \begin{bmatrix} g_{1+m(j-1)}^{i-1} \\ g_{2+m(j-1)}^{i-1} \\ \vdots \\ g_{mj}^{i-1} \\ S_{1+m(j-1)}^{i-1} \\ S_{2+m(j-1)}^{i-1} \\ \vdots \\ S_{mj}^{i-1} \end{bmatrix}, \quad \mathbf{F}_j^i = \begin{bmatrix} g_j^i \\ \mathbf{0}_{n-2} \\ -g_j^i \\ f_{1+m(j-1)}^{i-1} \\ f_{2+m(j-1)}^{i-1} \\ \vdots \\ f_{mj}^{i-1} \end{bmatrix}, \quad \mathbf{X}_j = \begin{bmatrix} x_j^1 \\ \vdots \\ x_j^{mj} \\ d_j^1 \\ \vdots \\ d_j^{mj} \end{bmatrix}, \quad \mathbf{B}_j = \begin{bmatrix} g_j^i \\ \mathbf{0}_{n-2} \\ -g_j^i \\ u_j^1 \\ \vdots \\ u_j^m \end{bmatrix}$$

s_j^i and d_j^i are slack variables and $\mathbf{0}_{n-2}$ is a zero vector in $\mathbf{R}^{n-2}$.

Proof: We replace each G_{k-1} in G_k by an arc. Regarding to the definition of self-similar flow networks, the consequential network is a basic network. We enumerate the arcs of the consequential basic network. Therefore the i^{th} arc is corresponding to the i^{th}, G_{k-1} $(i = 1, 2, \cdots, m)$. Using the definition of f_j^i we let the capacity of the i^{th} arc equal to f_i^{k-1}. On the other hand, by definition of g_j^i we have $g_1^{k-1} = f^*$. So by solving the following system we obtain the flow of the consequential basic network:

$$\mathbf{A}^0\, \mathbf{G}_1^k = \mathbf{F}_1^k$$

Obviously, the flow of the i^{th} arc is equal to $g_i^{k-1}(i = 1, 2, \cdots, m)$ which is obtained by solving the above system. To determine the flow of arcs in G_{k-1} it is sufficient to solve the following systems:

$$\mathbf{A}^{k-1}\,\mathbf{X}_j = \mathbf{G}_j^{k-1} \quad j = 1, 2, \cdots, m$$

For solving each one of these systems we can use the above process. In other word, we replace each G_{k-2} in all G_{k-1} by an arc. After using this process, $g_j^0 (j = 1, 2, \cdots, m^k)$ will be obtained. At the end, for finding x_j^i it is sufficient to solve the following systems:

$$\mathbf{A}^0\,\mathbf{X}_j = \mathbf{B}_j \quad j = 1, 2, \cdots, m^k. \ \blacksquare$$

Now, using the above Theorem the following algorithm computes the maximum flow in a self-similar flow network.

Algorithm MXSSFN

Let subroutine **Mat_Opr** $(a_1, a_2, \cdots, a_m, b, c_1, c_2, \cdots, c_m)$ solves system $\mathbf{A}^0 [a_1 \cdots a_m \ s_1 \cdots s_m]^T = [b \cdots -b \ c_1 \cdots c_m]^T$.

Call Mat_Opr $(g_{1+m(j-1)}^{i-1}, \cdots, g_{mj}^{i-1}, g_j^i, f_{1+m(j-1)}^{i-1}, \cdots, f_{mj}^{i-1})$

$i = k, k-1, \cdots, 1 \ \ j = 1, 2, \cdots, m^{k-i}$

Call Mat_Opr $(x_j^1, \cdots, x_j^m, g_j^0, u_j^1, \cdots, u_j^m)$

$j = 1, 2, \cdots, m^k$

Theorem 8: The MXSSFN algorithm runs in $O(m_k)$ time.

Proof: Suppose that p is the necessary operation for solving the subsystem $\mathbf{A}^0\,\mathbf{X} = \mathbf{B}$. In running MXSSFN algorithm we need to solve the system $\mathbf{A}^0\,\mathbf{X} = \mathbf{B}$ for $\sum_{i=0}^{k} m^i = \frac{m^{k+1} - 1}{m - 1}$ times. The number of operations p and m are constant and $m_k = m^{k+1}$, so the algorithm runs in $O(m_k)$ time. $\blacksquare$

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