J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
Volume 9, Number 1 (May 2002), Pages 73–79

ISSN 1226-0657

THE EQUIVALENT CONDITIONS OF THE PETTIS INTEGRABILITY

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ABSTRACT. In this paper, we characterize the Pettis integrability for the Dunford integrable functions on a perfect finite measure space.

1. INTRODUCTION

The theory of integration of functions with values in a Banach space has long been a fruitful area of study. After Pettis [?] introduced the Pettis integral of a weakly μ-measurable function in 1938, no theorem of substance appeared until 1976. Through the twenty years since 1977, remarkable progress of the Pettis integral has been achieved by many mathematicians and the Pettis integral has begun to come into its own (*cf.* Bator [?], [?], Talagrand [?], Geitz [?], Geitz & Uhl [?], Huff [?]).

Geitz [?] gave a characterization of the Pettis integrability for the functions on a perfect finite measure space:

Geitz' Characterization. *Let (Ω, Σ, μ) be a pefect finite measure space. A bounded function $f : \Omega \to X$ is Pettis integrable if and only if there is an uniformly bounded sequence (f_n) of simple functions from Ω into X such that* $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e. *for each $x^* \in X^*$.*

In this paper, we study some necessary and sufficient conditions of the Pettis integrability for the Dunford integrable functions on a perfect finite measure space.

Let us introduce some definitions and preliminary facts. Let (Ω, Σ, μ) be a finite measure space and X denote a Banach space with dual space X^* and bidual space X^{**}.

Received by the editors April 6, 2001 and, in revised form, May 10, 2002.
2000 *Mathematics Subject Classification.* 28B05.
Key words and phrases. Pettis integrability, μ-essentially separable valued function.
This research was supported by the Catholic University of Daegu research grants 1999.

A function $f : \Omega \to X$ is called *weakly μ-measurable* if for each $x^* \in X^*$, x^*f is μ-measurable. If f is a weakly μ-measurable X-valued function on Ω such that $x^*f \in L^1(\mu)$ for each $x^* \in X^*$, then f is called *Dunford integrable.* The *Dunford integral* of f over $E \in \Sigma$ is defined by the element $x_E^{**} \in X^{**}$ such that $x_E^{**}(x^*) = \int_E x^* f d\mu$ for each $x^* \in X^*$, and we write

$$x_E^{**} = \text{(D)-}\int_E f d\mu.$$

If we define the operator $F : \Sigma \to X^{**}$ by $F(E) = \text{(D)-}\int_E f d\mu$ for each $E \in \Sigma$, the operator F is not necessarily countably additive. It can be shown that F is countably additive if and only if the operator $T : X^* \to L^1(\mu)$ defined by $T(x^*) = x^*f$ is weakly compact if and only if $\{x^*f : \|x^*\| \leq 1\}$ is uniformly integrable in $L^1(\mu)$.

In the case that $F(E) \in X$ for each $E \in \Sigma$, we write $F(E) = \text{(P)-}\int_E f d\mu$ and it is called the *Pettis integral* of f over E. It follows from Definition that a weakly μ-measurable function $f : \Omega \to X$ is Pettis integrable if and only if for every set $E \in \Sigma$ there is an element in X, denoted by $\int_E f d\mu$,which satisfies $x^* \int_E f d\mu = \int_E x^* f d\mu$ for every $x^* \in X^*$.

A finite measure space (Ω, Σ, μ) is *perfect* if for each μ-measurable function $\Psi : \Omega \to R$ and for each $E \subset R$ such that $\Psi^{-1}(E)$ in Σ, there exists a Borel set B of E such that $\mu(\Psi^{-1}(B)) = \mu(\Psi^{-1}(E))$.

Definition 1.1. A weakly μ-measurable function $f : \Omega \to X$ is separable-like if there exists a separable subspace D of X such that

$$x^* \chi_D f = x^* f \ \ \mu\text{-a.e.}$$

for every x^* in X^*.

In Huff [?] defined such a separable-like function and investigated the Pettis integrability of the function.

Fremlin's Theorem (Bator [?]). *Let (Ω, Σ, μ) be a perfect finite measure space and (f_n) be a sequence of μ-measurable extended real-valued functions on Ω. Then either (f_n) has a subsequence which converges* μ-a.e. *or (f_n) has a subsequence having no μ-measurable pointwise cluster points.*

Let (Ω, Σ, μ) be a perfect finite measure space and $f : \Omega \to X$ be Dunford integrable. If (x_n^*) is any bounded sequence in X^*, then it follows from Alaoglu's theorem that every subsequence of $(x_n^* f)$ has a pointwise cluster point. Combining this with Fremlin's theorem, we get the following Lemma.

Lemma 1.2. *If (Ω, Σ, μ) is a perfect finite measure space and $f : \Omega \to X$ is a Dunford integrable function, then the operator $T : X^* \to L^1(\mu)$ defined by $T(x^*) = x^*f$ is compact.*

Proof. Let (x_n^*) be a bounded sequence in X^*. Suppose $(x_n^* f)$ does not have a μ-a.e. convergent subsequence. By Fremlin's theorem, there is a subsequence $(x_{n_j}^* f)$ having no μ-measurable pointwise cluster points. Let (x_0^*) be a weak* cluster point of $(x_{n_j}^*)$ in X^*. Hence $x_0^* f$ is a pointwise cluster point of $(x_{n_j}^* f)$ and is therefore nonmeasurable. This contradicts the weak measurability of f. Hence some subsequence must converge μ-a.e. and by boundedness this subsequence must converge in $L^1(\mu)$, which proves the desired result. □

Recall that *if (Ω, Σ, μ) is a perfect finite measure space and $f : \Omega \to X$ is a Dunford integrable function, then f is separable-like if and only if f is Pettis integrable.*

2. Main results

Let X be a Banach space and let (Ω, Σ, μ) be a finite measure space. We define two functions f and g from Ω into X to be *weakly equivalent* if $x^*f = x^*g$ μ-a.e. for every $x^* \in X^*$.

The following Theorem offers an equivalent condition of the Pettis integrability for the Dunford integrable functions on a perfect finite measure space.

Theorem 2.1. *Let (Ω, Σ, μ) be a perfect finite measure space and $f : \Omega \to X$ be a Dunford integrable function. Then the following are equivalent:*

(i) *f is Pettis integrable.*
(ii) *There exists a Bochner integrable function $g : \Omega \to X$ which is weakly equivalent to f.*

Proof. (i) ⇒ (ii). The operator $T : X^* \to L^\infty(\mu)$ defined by $T(x^*) = x^*f$ is weakly compact and so is its adjoint T^*. Consider the restriction $T^* : L^1(\mu) \to X^{**}$. Since every weakly compact operator on $L^1(\mu)$ is representable, so is the restriction T^* on $L^1(\mu)$. Accordingly, there exists a bounded μ-measurable function $g : \Omega \to X^{**}$ such that $T^*(\tilde{f}) = \text{(B)-}\int_\Omega \tilde{f} g \, d\mu$ for all $\tilde{f}$ in $L^1(\mu)$. In particular, if $E \in \Sigma$ and $x^* \in X^*$, then $\int_E x^* f \, d\mu = \int_\Omega \chi_E \, x^* f \, d\mu = x^*(T^*(\chi_E)) = x^*(\int_E g \, d\mu) = \int_E x^* g \, d\mu$. Suppose f

is Pettis integrable. Since g is Bochner integrable, we see that

$$x^*\left((\mathrm{P})\text{-}\int_E f d\mu\right) = \int_E x^* f d\mu = \int_E x^* g d\mu = x^*\left((\mathrm{B})\text{-}\int_E g d\mu\right)$$

for all $E \in \Sigma$ and all $x^* \in X^*$. It follows that (B)-$\int_E g d\mu \in X$ for every $E \in \Sigma$. From this and the argument used to prove Diestel and Uhl [?, Theorem 3.2], we see that g has almost all of its values in X. Consequently, we have $x^*f = x^*g$ μ-a.e. for all $x^* \in X^*$.

(ii) $\Rightarrow$ (i). Suppose that there exists a Bochner integrable function $g : \Omega \to X$ which is weakly equivalent to f. Since g is Bochner integrable, x^*g is in $L^1(\mu)$ for every $x^* \in X^*$. Let E be in Σ, and let $\int_E g d\mu$ be the Bochner integral of f over E. Then for every $x^* \in X^*$, $x^*(\int_E g d\mu) = \int_E x^* g d\mu = \int_E x^* f d\mu$. This means that f is Pettis integrable. □

Recall that if (Ω, Σ, μ) is a perfect finite measure space and $f : \Omega \to X$ is a Dunford integrable function, then f is Pettis integrable if there is a sequence (f_n) of Pettis integrable functions from Ω into X such that $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e. for each $x^* \in X^*$.

Another characterization of the Pettis integrability for the Dunford integrable functions on a perfect finite measure space is shown by the next theorem.

Theorem 2.2. *Let (Ω, Σ, μ) be a perfect finite measure space and $f : \Omega \to X$ be a Dunford integrable function. Then the following are equivalent:*

(i) *f is Pettis integrable.*

(ii) *There exists a sequence (f_n) of Bochner integrable functions from Ω to X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e.*

Proof. (i) $\Rightarrow$ (ii). Suppose that f is Pettis integrable. Then there exists a sequence (f_n) of simple functions from Ω into X such that for each $x^* \in X^*$,

$$\lim_{n\to\infty} x^* f_n = x^* f \ \mu\text{-a.e.}$$

by the Geitz's Theorem [?]. Since a simple function is Bochner integrable, there exists a sequence (f_n) of Bochner integrable functions from Ω into X such that $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e. for each $x^* \in X^*$.

(ii) $\Rightarrow$ (i). Suppose that there exists a sequence (f_n) of Bochner integrable functions from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e. Since a Bochner integrable function is Pettis integrable, there exists a sequence (f_n) of

Pettis integrable functions from Ω into X such that $\lim_{n\to\infty} x^*f_n = x^*f$ μ-a.e. for each $x^* \in X^*$. Thus f is Pettis integrable. □

We introduce the notion of μ-essentially separable valued function, and investigate some properties of μ-essentially separable valued function and related topics in some more detail. Recall that a function $f : \Omega \to X$ is μ-measurable if and only if it is μ-essentially separable valued and weakly μ-measurable by the Pettis Measurability Theorem (*cf.* Bator [?]).

The following definition provides the main results of this paper.

Definition 2.3. A function $f : \Omega \to X$ is said to be *μ-essentially separable valued* if there exists $E \in \Sigma$ with $\mu(E) = 0$ and such that $f(\Omega \setminus E)$ is a separable subset of X.

In particular, simple functions are μ-essentially separable valued, and there exists a function $f : \Omega \to X$ such that $x^*f{=}0$ μ-a.e. but not μ-essentially separable valued.

First of all, we obtain the following Lemma.

Lemma 2.4. *Let (Ω, Σ, μ) be a finite measure space and $f : \Omega \to X$ be a Dunford integrable function. If we define the operator $T : X^* \to L^1(\mu)$ by $T(x^*) = x^*f$, then*

(a) *T is bounded;*
(b) *if T is weakly compact and f is μ-essentially separable valued, then f is Pettis integrable, and if f is Pettis integrable, then T is weakly compact; and*
(c) *T is compact if f is Bochner integrable.*

Proof. (a) and (c) are well-known results. Directing our attention to (b), suppose f is μ-essentially separable valued. By the Pettis Measurability Theorem, f is μ-measurable. Since every μ-measurable function is separable-like, f is separable-like. Since T is weakly compact and f is separable-like, f is Pettis integrable.

For the converse, suppose f is Pettis integrable. Then the vector measure

$$F : \Sigma \to X^{**}$$

associated by $F(E) = (\mathrm{D})\text{-}\int_E f d\mu$ is countably additive. Thus T is weakly compact. □

The following Theorem 2.5 and Theorem 2.6 offer characterizations of the Pettis integrability for the Dunford integrable functions on a perfect finite measure space.

Theorem 2.5. *Let (Ω, Σ, μ) be a perfect finite measure space and $f : \Omega \to X$ be a Dunford integrable function. Then f is Pettis integrable if f is μ-essentially separable valued.*

Proof. Suppose that f is μ-essentially separable valued. From Lemma 1.2 and Lemma 2.4 (b), f is Pettis integrable. □

Theorem 2.6. *Let (Ω, Σ, μ) be a perfect finite measure space and $f : \Omega \to X$ be a Dunford integrable function. Then the following are equivalent:*

(i) *f is Pettis integrable.*

(ii) *There exists a sequence (f_n) of Dunford integrable and μ-essentially separable valued functions from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e.*

Proof. (i) $\Rightarrow$ (ii). Suppose that f is Pettis integrable. Then there exists a sequence (f_n) of simple functions from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e. Since simple function is Dunford integrable and μ-essentially separable valued, there exists a sequence (f_n) of Dunford integrable and μ-essentially separable valued functions from Ω into X such that $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e. for each $x^* \in X^*$.

(ii) $\Rightarrow$ (i) By hypothesis and Theorem 2.5, (f_n) is a sequence of Pettis integrable functions from Ω into X such that $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e. for each $x^* \in X^*$. Thus f is Pettis integrable. □

We conclude with the following summary.

Theorem 2.7. *Let (Ω, Σ, μ) be a perfect finite measure space and $f : \Omega \to X$ be a Dunford integrable function. Then the following statements are equivalent:*

(i) *f is Pettis integrable.*

(ii) *There exists a Bochner integrable function $g : \Omega \to X$ which is weakly equivalent to f.*

(iii) *There exists a sequence (f_n) of Bochner integrable functions from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e.*

(iv) *There exists a sequence (f_n) of Dunford integrable and μ-essentially separable valued functions from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e.*

(v) *There exists a sequence (f_n) of Pettis integrable functions from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e.*

Acknowledgements

The author is very grateful to the editors and anonymous referees for the valuable comments.

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