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On Radar Surveillance in Statistical Perspective for the Classroom

KIM, G. DANIEL
Department of Mathematics, Southern Oregon University,
Ashland, OR 97520, U. S. A.; Email: kimd@sou.edu

KIM, SUNG SOOK
Department of Mathematics Paichai University, Doma-dong, Seo-gu,
Daejeon 302-735, Korea; Email: sskim@mail.pcu.ac.kr

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Educators have found that the concept of randomness is often misunderstood by students. Chance recently pointed out that students should be introduced to the concept of randomness through the use of simulations. In this article, we studied various aspects of the probability distribution of a linear random path in a circle and introduce some related simulations to guide student exploration and discovery. Consider a random line segment that crosses a circle with a certain radius. Perhaps it can be considered to be a path that an airplane shows up and flies into a random direction in a monitor. What is the expected amount of flying distance through the monitor, and the expected variation? Are we monitoring what we see scientifically? This article studies the probability distribution and some related aspects of a linear random path within a circular monitor. Some simulative activity is also introduced which can be used in a statistics or probability classes.

Keywords: Linear random path, Adjusted frequency histogram, Probability density function, Simulation

1. INTRODUCTION

Consider a radar monitor that screens a circle area of radius r. Suppose an airplane appears on the edge of the screen, and flies to a random direction with a linear path through the zone in the radar screen. A linear path that the airplane travels through can be simulated as a line segment by connecting two randomly chosen points from the boundary of a circle with radius r. Let us call these two points on the boundary P and Q. The distance the plane travels through the zone

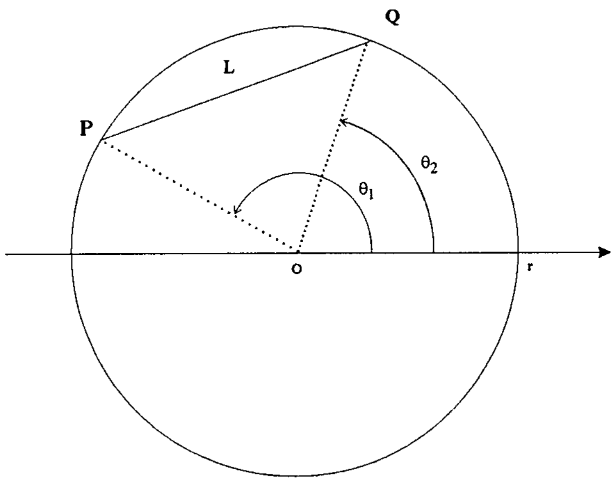


Figure 1. A linear random path

can be calculated using the distance formula. If L_r denotes the distance, L_r is a random variable that takes a value from the interval $[0, 2r]$, where r is the radius of the circle. The objective of this article is to study various aspects of the probability distribution of L_r and some related simulation activity.

For convenience, let us draw a circle that symbolizes the radar screen in a coordinate system so that the center of the circle is at the origin. As Figure 1 shows, let θ_1 and θ_2 represent the angles (in radian) of the segments OP and OQ in a standard position, respectively, where O represents the center of the circle, and equivalently origin. Then,

$$P = (r\cos\theta_1, r\sin\theta_1) \quad \text{and} \quad Q = (r\cos\theta_2, r\sin\theta_2).$$

By the distance formula the distance between P and Q is

$$L_r = \sqrt{(r\cos\theta_1 - r\cos\theta_2)^2 + (r\sin\theta_1 - r\sin\theta_2)^2}$$

To consider a simpler form of L_r above, rotate the circle clockwise by the angle QOP, which is $\theta_1 - \theta_2$, so that the side OQ lies on the horizonal axis as shown in Figure 2. Let P^* and Q^* be the corresponding points of P and Q after the rotation. Then,

$$P^* = (r\cos(\theta_1 - \theta_2), r\sin(\theta_1 - \theta_2)) \quad \text{and} \quad Q^* = (r, 0).$$

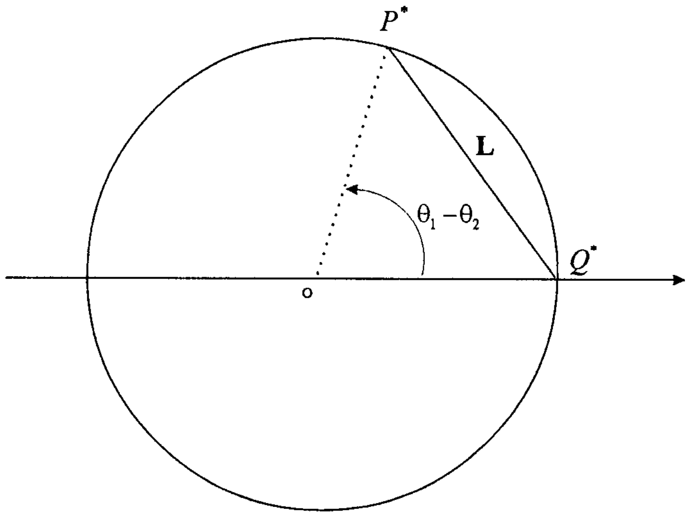


Figure 2. A rotation of the circle

Then, we have

$$\begin{aligned} L_r &= \text{the distance between P and Q} \\ &= \text{the distance between P}^* \text{ and Q}^* \\ &= \sqrt{(r\cos(\theta_1-\theta_2)-r)^2+(r\sin(\theta_1-\theta_2))^2} \\ &= r\sqrt{2-2\cos(\theta_1-\theta_2)} \qquad (1) \end{aligned}$$

2. Development

Note that since P and Q (resp.) are chosen randomly and independently from the boundary of the circle, θ_1 and θ_2 (resp.) are independent random variables and have uniform probability distribution over $[0, 2\pi]$. In order to investigate various aspects of the probability distribution of L, we first consider the probability distribution of $\theta_2 - \theta_1$ (see Lemma 1). Let $U = \theta_2 - \theta_1$ for convenience.

Lemma 1. *Let θ_1 and θ_2 be independent random variables having uniform probability distribution over $(0, 2\pi)$. Then $U = \theta_1 - \theta_2$ has the probability density function*

$$f_U(u) = \begin{cases} \frac{1}{2\pi} + \frac{1}{4\pi^2}u & : \quad -2\pi \le u < 0 \\ \frac{1}{2\pi} - \frac{1}{4\pi^2}u & : \quad 0 \le u \le 2\pi \\ 0 & : \quad \text{elsewhere} \end{cases}$$

Proof. Note that since $0 \leq \theta_1 \leq 2\pi$ and $0 \leq \theta_2 \leq 2\pi$, $-2\pi \leq U = \theta_1 - \theta_2 \leq 2\pi$.

Case 1: $(-2\pi \leq u < 0)$.

$$\begin{aligned} F_U(u) &= P(\theta_1 - \theta_2 \leq u) = \iint\limits_{\theta_1-\theta_2\leq u} f(\theta_1, \theta_2)\, d\theta_1\, d\theta_2 \\ &= \int_{-u}^{2\pi}\int_0^{\theta_2+u} \left(\frac{1}{2\pi}\right)^2 d\theta_1 d\theta_2 = \frac{1}{4\pi^2}\int_{-u}^{2\pi}(\theta_2+u)d\theta_2 = \frac{1}{2} + \frac{u}{2\pi} + \frac{u^2}{8\pi^2} \end{aligned}$$

Thus for $-2\pi \leq u < 0$ we have

$$f_U(u) = F_U'(u) = \frac{1}{2\pi} + \frac{1}{4\pi^2}u$$

Case 2: $(0 \leq u \leq 2\pi)$.

$$\begin{aligned} F_U(u) = P(\theta_1 - \theta_2 \leq u) &= 1 - P(\theta_1 - \theta_2 \geq u) \\ &= 1 - \iint\limits_{\theta_1-\theta_2\geq u} f(\theta_1, \theta_2)\, d\theta_1\, d\theta_2 \\ &= 1 - \int_u^{2\pi}\int_0^{\theta_1-u} f(\theta_1, \theta_2)\, d\theta_2\, d\theta_1 \\ &= 1 - \frac{1}{4\pi^2}\int_u^{2\pi}(\theta_1 - u)d\theta_1 \\ &= \frac{1}{2} + \frac{u}{2\pi} - \frac{u^2}{8\pi^2} \end{aligned}$$

Thus for $0 \leq u \leq 2\pi$ we have

$$f_U(u) = F_U'(u) = \frac{1}{2\pi} - \frac{1}{4\pi^2}u.$$

□

Recall from (1) that $L_r = r\sqrt{2 - 2\cos(\theta_1 - \theta_2)}$. Now we are in a position to study the probability distribution of L.

Theorem 1. *Let θ_1 and θ_2 be defined as lemma 1. Then, L_r has the probability density function*

$$f_{L_r}(l) = \begin{cases} \dfrac{2}{\pi\sqrt{4r^2 - l^2}} & : \quad 0 \leq l \leq 2r \\ 0 & : \quad \text{elsewhere} \end{cases}$$

Proof. We first consider the case where the radius r of the circle is 1. And then we extend the result to a case where r is any positive constant.

Case 1 ($r = 1$). It follows from (1) and Figure 3 that

$$\begin{aligned} F_{L_1}(l) &= P(L_1 \leq l) \\ &= P(\sqrt{2 - 2\cos(\theta_1 - \theta_2)} \leq l) \\ &= P(2 - 2\cos(\theta_1 - \theta_2) \leq l^2) \\ &= P(\cos(\theta_1 - \theta_2) \geq 1 - \frac{l^2}{2}) \\ &= \underbrace{P(-2\pi < U < -2\pi + C_1)}_{A_1} + \underbrace{P(-C_1 < U < C_1)}_{A_2} + \underbrace{P(2\pi - C_1 < U < 2\pi)}_{A_3} \end{aligned}$$

where $C_1 = \cos^{-1}(1 - \frac{l^2}{2})$. For convenience, let $f_1(u) = f_U(u)$ for $u < 0$, and $f_2(u) = f_U(u)$ for $u \geq 0$, where $f_U(u)$ is the probability density function of $U = \theta_1 - \theta_2$ defined in Lemma1. Now, from Figure 4 that the following can be noted.

$$A_1 = P(-2\pi < U < -2\pi + C_1) = \int_{-2\pi}^{-2\pi + C_1} f_U(u)du = \frac{1}{2}C_1 f_1(-2\pi + C_1)$$

$$A_1 = P(-2\pi < U < -2\pi + C_1) = \int_{-2\pi}^{-2\pi + C_1} f_U(u)du = \frac{1}{2}C_1 f_1(-2\pi + C_1)$$

$$A_3 = P(2\pi - C_1 < U < 2\pi) = \int_{2\pi - C_1}^{2\pi} f_U(u)du = \frac{1}{2}C_1 f_2(2\pi - C_1) = \frac{1}{2}C_1 f_1(-2\pi + C_1)$$

Therefore we have

$$\begin{aligned} F_{L_1}(l) = P(L_1 \leq l) &= A_1 + A_2 + A_3 \\ &= \left[\frac{1}{2}C_1 f_1(-2\pi + C_1)\right] + \left[2C_1 f_2(C_1) + C_1\left(\frac{1}{2\pi} - f_2(C_1)\right)\right] \\ &\quad + \left[\frac{1}{2}C_1 f_1(-2\pi + C_1)\right] \\ &= C_1 f_1(-2\pi + C_1) + C_1 f_2(C_1) + \frac{C_1}{2\pi} \\ &= C_1\left[\frac{1}{2\pi} + \frac{1}{4\pi^2}(-2\pi + C_1)\right] + C_1\left[\frac{1}{2\pi} - \frac{1}{4\pi^2}C_1\right] + \frac{C_1}{2\pi} \\ &= \frac{C_1}{\pi} \\ &= \frac{1}{\pi}\cos^{-1}(1 - \frac{l^2}{2}) \end{aligned}$$

Hence for $0 \leq l \leq 2$ we have

$$f_{L_1}(l) = F'_{L_1}(l) = \frac{d}{dl}\left(\frac{1}{\pi}\cos^{-1}(1 - \frac{l^2}{2})\right) = \frac{2}{\pi\sqrt{4 - l^2}}$$

Case 2: (r is any positive constant.).

By (2), we have

$$\begin{aligned} F_{L_r}(l) = P(L_r \le l) &= P(r\sqrt{2-2\cos(\theta_1-\theta_2)} \le l) \\ &= P(\sqrt{2-2\cos(\theta_1-\theta_2)} \le l/r) \\ &= P(L_1 \le l/r) \\ &= \frac{1}{\pi}\cos^{-1}(1-\frac{l^2}{2r^2}) \end{aligned}$$

Therefore, for $0 \le l \le 2r$,

$$\begin{aligned} f_{L_r}(l) = F'_{L_r}(l) &= \frac{d}{dl}\left(\frac{1}{\pi}\cos^{-1}\left(1-\frac{l^2}{2r^2}\right)\right) \\ &= \frac{2}{\pi\sqrt{4r^2-l^2}} \end{aligned}$$

□

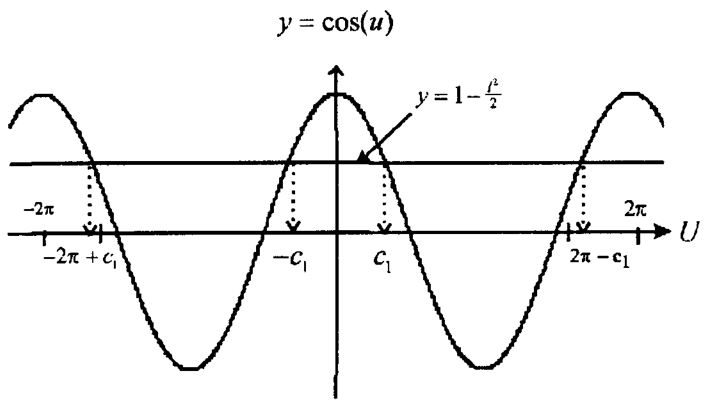


Figure 3. The graph of $y = \cos(u)$

In Theorem 2 we explore the mean and the standard deviation of L_r.

Theorem 2. $E(L_r) = \frac{4r}{\pi}$ *and* $\sigma^2_{L_r} = \frac{r^2(2\pi^2-16)}{\pi}$.

Proof. Case 1: ($E(L_r)$).

$$\begin{aligned} E(L_r) &= \int_0^{2r} x f_{L_r}(x)dx \\ &= \int_0^{2r} x\frac{2}{\pi\sqrt{4r^2-x^2}}dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\pi}\int_0^{2r} \frac{x}{\sqrt{4r^2 - x^2}}dx \\
&= \frac{4r}{\pi}
\end{aligned}
$$

Case 2: ($\sigma^2_{L_r}$).

$$
\begin{aligned}
E(L_r^2) &= \int_0^{2r} x^2 f_{L_r}(x)dx \\
&= \int_0^{2r} x^2 \frac{2}{\pi\sqrt{4r^2 - x^2}}dx \\
&= \frac{2}{\pi}\int_0^{2r} \frac{x^2}{\sqrt{4r^2 - x^2}}dx \\
&= 2r^2
\end{aligned}
$$

Therefore, we have

$$
\sigma^2_{L_r} = E(L_r^2) - E^2(L_r) = 2r^2 - \left(\frac{4r^2}{\pi}\right)^2 = \frac{\sqrt{2\pi^2 - 16}}{\pi}r \qquad \square
$$

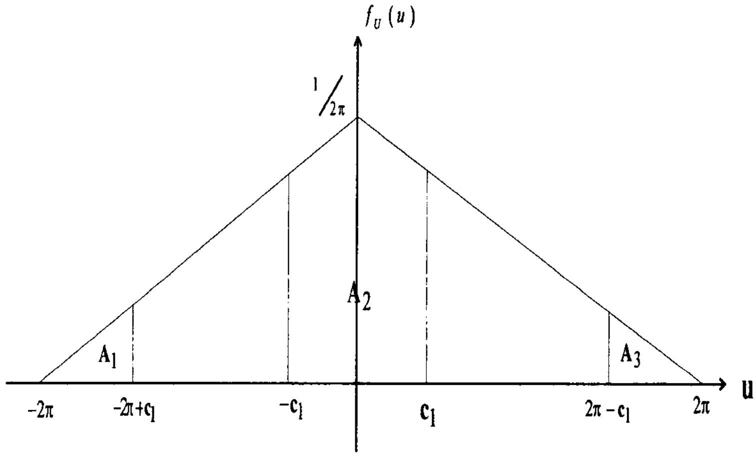


Figure 4. The graph of the density function of U

3. Simulation with a Graphing Calculator

In this section a simulation work using TI-83 is considered. The idea of the simulation is very simple. Firstly, choose two random points P and Q from the boundary of a circle with radius r. Secondly, calculate the length of the line segment L joining P and Q. Thirdly, repeat the previous two procedures many times and summarize the information of the sample obtained. The following is a TI-83 program for the simulation work. Here L_1, L_2, and L_3 are names of lists.

> `Prompt` R, N
> `ClrList` L_1, L_2, L_3
> $2\pi \times \text{rand}(N) \to L_1$
> $2\pi \times \text{rand}(N) \to L_2$
> $R \times \sqrt{((\cos(L_1) - \cos(L_2))^2 + (\sin(L_1) - \sin(L_2))^2)} \to L_3$
> (`Or equivalently,` $R \times \sqrt{2 - 2\cos(\theta_1 - \theta_2)} \to L_3$)

Using the program, data sets of L_r are simulated for various r ranging from 1 to 5. $E(L_r)$ and σ_{L_r} are calculated using the formulas in Theorem 2. Table 1 compares these parametric values with their corresponding summary statistics from the simulated data sets. The number of repeated trials N is 100.

Table 1. A Comparison of the parameter values and the observed statistics

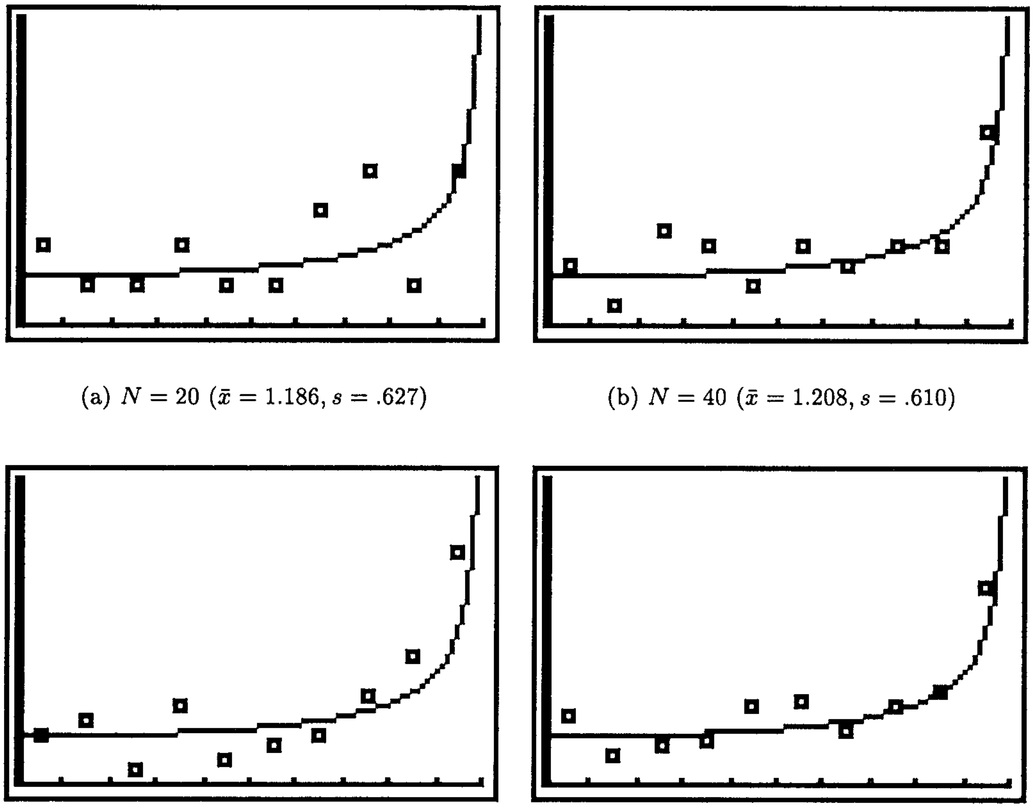
r	1	2	3	4	5
$E(L_r)$	1.273	2.546	3.820	5.093	6.366
$\bar{x}_{L_r}$	1.292	2.601	3.749	4.962	6.162
σ_{L_r}	.616	1.231	1.847	2.462	3.078
s_{L_r}	.602	1.218	1.925	2.558	3.287

The four graphs in Figure 5 drawn by TI-83 compares the probability density function f_{L_1} and the adjusted frequency histogram of a simulated data of L for various N.

To give a consistency, the range of the horizontal axis representing L_1 is set to be from 0 to 2, and the range of the vertical axis is set to be from 0 to 2. By Theorem 2, $E(L_1) = 1.273$ and $\sigma_{L_1} = .616$. The sample means and the sample standard deviations of L_1 for various N are presented in the parentheses.

4. Concluding Remarks

Chance and Rossman (2001, p. 1) recently pointed out that students should be introduced to the concept of randomness through the use of simulations. In this article we studied various aspects of the probability distribution of a linear random path in a circle and some related simulations. Although the simulation activity presented in previous section is based on a graphing calculator, it can easily be



(a) $N = 20$ ($\bar{x} = 1.186, s = .627$)

(b) $N = 40$ ($\bar{x} = 1.208, s = .610$)

(c) $N = 60$ ($\bar{x} = 1.317, s = .614$)

(d) $N = 100$($\bar{x} = 1.285, s = .612$)

Figure 5. $f_L(l)$ and the adjusted frequency histogram of the simulated L_1

extended to hands-on activities that can be conducted in a classroom environment that may not have equipped graphing calculators.

For instance, students can be provided with a circle graph and a paper clip. The circle graph with a certain radius is angle measured along the boundary. Have students spin the paper clip twice around the center of the circle graph by holing one end of the paper clip using a pencil. Through this experiment, two random points are acquired from the boundary of the circle. The distance, say L, between them can be measured using a measure stick. In repeated experiment of this procedure, various values of L can be sampled. Once a sample data of L is observed, instructors can lead students to a variety of meaningful discussions. The following list could be some examples that instructors can use for students to discuss as a group.

(1) Comparison of the density plot (of Theorem 1) and the adjusted frequency histogram: Students can discuss some issues regarding the degree of the resemblance of the two plots. Why or why not the two plots are similar can be discussed. Were there any biased activity or attitude involved during the sampling procedure?

(2) Comparison of the population mean of L (in Theorem 2) and the sample mean of L.

(3) Comparison of the population variance of L (in Theorem 2) and the sample variance of L.

(4) Comparison of the population box plots and the sample box plots: Based upon the box plots, students can predict the shape of the distribution. Discussions of how well the box plots matches to the shape of the frequency distribution would be a fun and a meaningful time to spend.

Stimulating students with some motivating contents and activities like the ones presented in this article seems to boost students interests tremendously and can change their attitudes of mathematics towards a positive direction.

The study of this article and its accompanying activity can be effectively used in upper division mathematical statistics, probability classes, or perhaps in an integral calculus classes to stimulate student's interests.

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