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Pitman Nearness for a Generalized Stein-Rule Estimators of Regression Coefficients

R. Karan Singh[1] and N. Rastogi[1]

ABSTRACT

A generalized Stein-rule estimator of the vector of regression coefficients in linear regression model is considered and its properties are analyzed according to the criterion of Pitman nearness. A comparative study shows that the generalized Stein-rule estimator representing a class of estimators contains particular members which are better than the usual Stein-rule estimator according to the Pitman closeness.

Keywords. Generalized Stein-rule estimator, Pitman closeness, maximum likelihood estimator.

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1. Introduction

Rao (1981) has argued that the risk criterion based on a quadratic loss function may not necessarily exhibit the intrinsic qualities of the estimators and has gone in favour of the criterion like the probability of the closeness of the estimators to the true parameter in the sense of Pitman (1937). Rao, Keating and Mason (1986), Keating and Mason (1988) and Sen, Kubokawa and Saleh (1989) used Pitman closeness criterion to analyze the performance of estimators over the maximum likelihood estimator. Srivastava and Srivastava (1993), using asymptotic approximation for Pitman closeness criterion, derived the dominance condition for Stein-rule estimator over the maximum likelihood estimator. We attempt here to analyse the properties of the generalized Stein-rule estimator similar to Singh *et al.* (1993) according to the criterion of Pitman closeness and derive the dominance condition for the generalized estimator over Stein-rule estimator and the maximum likelihood estimator.

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[1]Department of Statistics, Lucknow University, Lucknow 226 007, India

2. The Model and the Estimators

Let the linear regression model be

$$y = X\beta + u \tag{2.1}$$

where y is an $n \times 1$ vector of observations on the variable to be explained, X is an $n \times p$ full column rank matrix of observations on p explanatory variables, β is a $p \times 1$ vector of regression coefficients and u is an $n \times 1$ vector of disturbances having multivariate normal distribution with mean vector 0 and variance-covariance matrix $\sigma^2 I$.

We know that the unbiased maximum likelihood estimator of β is

$$b = (X'X)^{-1}X'y \tag{2.2}$$

and Stein-rule estimator of β is

$$\hat{\beta} = \left[1 - \frac{k(y - Xb)'(y - Xb)}{(n - p + 2)b'X'Xb}\right] b \tag{2.3}$$

where $k(> 0)$ is a characterizing scalar.

Similar to Singh *et al.* (1993), the generalized Stein-rule estimator is

$$\hat{\beta}_g = g(v)b \tag{2.4}$$

where $v = \dfrac{(y - Xb)'(y - Xb)}{(n - p + 2)b'X'Xb}$, $g(v)$ satisfying the validity conditions of Taylor's (Maclaurin's) series expansion is a bounded function of v such that $g(v = 0) = 1$ and $g(v) = O_p(n^{-1})$.

It may be mentioned here that different forms of $g(v)$ in (4.3) and (4.4) are chosen having the value equal to unity at $v = 0$ similar to Stein-rule estimator $(1 - kv)b$ with $g(v) = (1 - kv)$ having the value unity at $v = 0$.

For comparing $\hat{\beta}_g$ and b, let us consider

$$D = (b - \beta)'X'X(b - \beta) - (\hat{\beta}_g - \beta)'X'X(\hat{\beta}_g - \beta). \tag{2.5}$$

The estimator $\hat{\beta}_g$ is superior to b in the sense of Pitman closeness if, for all β and σ^2, the probability $P(D > 0)$ exceeds 0.5. In the next section, the asymptotic approximation for $P(D > 0)$ is derived.

3. The Estimator $\hat{\beta}_g$ and Pitman Closeness Criterion

Let

$$Z = \frac{1}{\sigma}(X'X)^{-1/2}X'u,\ \alpha = \frac{1}{(\beta'X'X\beta)^{1/2}}(X'X)^{1/2}\beta,$$

$$\theta = \frac{\beta'X'X\beta}{n\sigma^2},\ w = \frac{1}{\sigma^2 n}u'u - 1$$

and $(n^{-1}X'X)$ tends to a finite nonsingular matrix as $n \to \infty$.

We can easily see that

$$\begin{aligned}(y - Xb)'(y - Xb) &= u'u - u'X(X'X)^{-1}X'u \\ &= n\sigma^2(1 + w - n^{-1}Z'Z)\,, \end{aligned} \tag{3.1}$$

$$\begin{aligned}b'X'X(b - \beta) &= \beta'X'u + u'X(X'X)^{-1}X'u \\ &= \sigma^2\left[(n\theta)^{1/2}\alpha'Z + Z'Z\right]\,, \end{aligned} \tag{3.2}$$

$$\begin{aligned}\frac{1}{(n-p+2)} &= \frac{1}{n}\left(1 - \frac{p-2}{n}\right)^{-1} \\ &= \frac{1}{n} + O(n^{-2})\,, \end{aligned} \tag{3.3}$$

$$\begin{aligned}\frac{1}{b'X'Xb} &= \frac{1}{\beta'X'X\beta + 2\beta'X'u + u'X(X'X)^{-1}X'u} \\ &= \frac{1}{n\sigma^2\theta}\left(1 + \frac{2\alpha'Z}{(n\theta)^{1/2}} + \frac{Z'Z}{n\theta}\right)^{-1} \\ &= \frac{1}{n\sigma^2\theta}\left[1 - \frac{2\alpha'Z}{(n\theta)^{1/2}} + O_p(n^{-1})\right]. \end{aligned} \tag{3.4}$$

Noting that $g(v = 0) = 1$, expanding $g(v)$ in second order Taylor's series about $v = 0$, we have

$$\begin{aligned}\hat{\beta}_g &= \left[g(0) + vg'(0) + \frac{v^2}{2!}g''(v_*)\right]b \\ &= \left[1 + vg'(0) + \frac{v^2}{2}g''(v_*)\right]b\,, \end{aligned} \tag{3.5}$$

where $g'(0)$ and $g''(v_*)$ are the first and second order derivatives of $g(v)$ at the points $v = 0$ and $v = v_*$ respectively, and $v_* = hv$ with $0 < h < 1$.

Further,

$$\hat{\beta}_g - \beta = (b - \beta) + \left[\frac{(y - Xb)'(y - Xb)}{(n-p+2)b'X'Xb}g'(0) + \frac{v^2}{2!}g''(v_*)\right]b. \tag{3.6}$$

Now

$$D_g = (b-\beta)'X'X(b-\beta) - (\hat{\beta}_g - \beta)'X'X(\hat{\beta}_g - \beta)$$

$$= -\frac{(y-Xb)'(y-Xb)}{(n-p+2)b'X'Xb} g'(0)$$

$$\times \left[2b'X'X(b-\beta) + \frac{(y-Xb)'(y-Xb)g'(0)}{(n-p+2)} + O_p(n^{-1})\right]$$

$$= -\frac{(1+w-n^{-1}Z'Z)g'(0)}{n\theta}\left\{1 - \frac{2\alpha'Z}{(n\theta)^{1/2}} + O_p(n^{-1})\right\}$$

$$\times \left[2\sigma^2\{(n\theta)^{1/2}\alpha'Z + Z'Z\} + \sigma^2(1+w-n^{-1}Z'Z)g'(0) + O_p(n^{-1})\right]$$

$$= -\frac{2\sigma^2 g'(0)}{(n\theta)^{1/2}}\left[\alpha'Z + w\alpha'Z + \frac{1}{(n\theta)^{1/2}}\left\{Z'(I-2\alpha\alpha')Z + \frac{g'(0)}{2}\right\}\right.$$

$$\left. + O_p(n^{-1})\right]$$

$$= -\frac{2\sigma^2 g'(0)}{(n\theta)^{1/2}}\left[V_0^* + V_{-1/2}^* + O_p(n^{-1})\right] \quad (3.7)$$

where $-g'(0) > 0$, $V_0^* = \alpha'Z$ and

$$V_{-1/2}^* = w\alpha'Z + \frac{1}{(n\theta)^{1/2}}\left\{Z'(I-2\alpha\alpha')Z + \frac{g'(0)}{2}\right\},$$

and the suffixes of V^* indicate the order of magnitude in probability. Approximating D_g by first two terms on the right hand side of (3.7) and writing

$$V^* = V_0^* + V_{-1/2}^*, \quad (3.8)$$

we see that $P(V^* > 0)$ becomes an asymptotic approximation of the probability $P(D_g > 0)$.

We now consider the characteristic function of V^* given by

$$\Phi(t) = E(e^{itV^*})$$

$$= E\left[e^{itV_0^*}(1 + itV_{-1/2}^* + \cdots)\right]$$

$$= E\left[e^{itV_0^*} + it\left\{w\alpha'Z + \frac{1}{(n\theta)^{1/2}}\left(Z'[I-2\alpha\alpha']Z + \frac{g'(0)}{2}\right)e^{it\alpha'Z}\right\}\right]$$

$$+ O(n^{-1}). \quad (3.9)$$

Since $Z \sim N(0, I)$,

$$E(e^{itV_0^*}) = E(e^{it\alpha' Z}) = \frac{1}{(2\pi)^{p/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{it\alpha' Z - \frac{1}{2}Z'Z} dz = e^{-t^2/2} \tag{3.10}$$

and

$$E\left[Z'(I - 2\alpha\alpha')Ze^{it\alpha' Z}\right] = (2\pi)^{-p/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} Z'(I - 2\alpha\alpha')Ze^{it\alpha' Z - \frac{1}{2}Z'Z} dz = e^{-t^2/2}\left[(p-2) + t^2\right]. \tag{3.11}$$

Since w and Z are uncorrelated and $E(w) = 0$, we have $E(w\alpha' Z) = \alpha' E(wZ) = 0$ so that, using (3.10) and (3.11), the characteristic function $\Phi(t)$ up to order $O(n^{-1/2})$ is

$$\Phi(t) = \left[1 + \frac{it}{(n\theta)^{1/2}}\left(p - 2 + t^2 + \frac{g'(0)}{2}\right)\right] e^{-t^2/2}. \tag{3.12}$$

By inversion theorem, the probability density function of V^* to order $O(n^{-1/2})$ is

$$g(v^*) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itv^*}\left[1 + \frac{it}{(n\theta)^{1/2}}\left(p - 2 + t^2 + \frac{g'(0)}{2}\right)\right] e^{-t^2/2} dt = f(v^*) + \frac{1}{(n\theta)^{1/2}}\left[\left(p + 1 + \frac{g'(0)}{2}\right)v^* - v^{*3}\right] f(v^*) \tag{3.13}$$

where $f(v^*) = e^{-\frac{1}{2}v^{*2}}/\sqrt{2\pi}$.

An asymptotic approximation for Pitman closeness criterion is

$$P(V^* > 0) = \int_0^{\infty} g(v^*)dv^* = 0.5 + \frac{1}{(2\pi n\theta)^{1/2}}\left[(p-1) + \frac{g'(0)}{2}\right] \tag{3.14}$$

which is greater than 0.5 if

$$0 < -g'(0) < 2(p-1), \quad p > 1. \tag{3.15}$$

4. Concluding Remarks

The superiority condition for $\hat{\beta}_g$ over b in the sense of Pitman closeness is

$$0 < -g'(0) < 2(p-1). \tag{4.1}$$

Stein-rule estimator $\hat{\beta}$ is a particular case of $\hat{\beta}_g$ for $g(v) = 1 - kv$ with $g'(0) = -k$ which, when substituted in (4.1), gives the same condition

$$0 < k < 2(p-1) \tag{4.2}$$

for the superiority of $\hat{\beta}$ over b as obtained by Srivastava and Strivastava (1993).

Satisfying the regularity conditions, some more particular estimators of $\hat{\beta}_g$ other than Stein-rule and maximum likelihood estimators are

$$\text{(i) } \hat{\beta}_{g_1} = \left[1 - k\{(1+v)^{k_1} - 1\}\right] b, \tag{4.3}$$

$$\text{(ii) } \hat{\beta}_{g_2} = \left[1 - k\{(1+k_1 v)^{k_2} - 1\}\right] b, \tag{4.4}$$

where k, k_1 and k_2 are the characterizing scalars to be chosen suitably.

For $\hat{\beta}_{g_1}$, $g(v) = 1 - k\{(1+v)^{k_1} - 1\}$ with $g'(0) = -kk_1$ which, when substituted in (4.1), gives the condition

$$0 < kk_1 < 2(p-1) \tag{4.5}$$

for the superiority of $\hat{\beta}_{g_1}$ over b in the sense of Pitman closeness.

Comparing (4.2) with (4.5), for $0 < k_1 < 1$, we see that the range of the superiority condition (4.5) of $\hat{\beta}_{g_1}$ over b, is wider than that of the superiority condition (4.2) of $\hat{\beta}$ over b, hence in the extended range of the superiority condition (4.5) over the condition (4.2), $\hat{\beta}_{g_1}$ is superior to both estimators $\hat{\beta}$ and b in the sense of Pitman closeness. Similar results hold for $\hat{\beta}_{g_2}$ also.

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