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Construction of Second Order Slope Rotatable Designs Using Symmetrical Unequal Block Arrangements with Two Unequal Block Sizes

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ABSTRACT

A new method of construction of second order slope rotatable designs (SOSRD) using symmetrical unequal block arrangements (SUBA) with two unequal block sizes is suggested. The proposed method may sometimes lead to designs with less number of design points than those available in the literature. Further, bounds for the parameters of SOSRD are also obtained.

Keywords. Response surface design, slope rotatability, slope rotatable central composite design, second order slope rotatability.
AMS 2000 subject classifications. Primary 62K05; Secondary 05B05.

1. Introduction

The study of the rotatable designs mainly emphasized on the estimation of response and its precision. Estimation of difference in response at two different points in the factor space will be often of great importance. If difference at two points close together is involved, estimation of the local slope (rate of change) of the response surface is of interest.

The problem of estimation of slopes occurs frequently in practical situations. For instance, there are cases in which one wants to estimate rate of reaction in chemical experiment, rate of change in the yield of a crop to fertilizer doses, rate of disintegration of radioactive material in an animal, *etc.* Hader and Park (1978) introduced slope rotatable central composite designs (SRCCDs). Victorbabu and Narasimham (1991) studied in detail the conditions to be satisfied by a general second order slope rotatable designs (SOSRD) and constructed SOSRD using balanced incomplete block designs (BIBD). Park and Kim (1992) suggested a

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measure of slope rotatability for second order response surface designs. Jang and Park (1993) suggested a measure and a graphical method for evaluating slope rotatability in response surface designs. Victorbabu and Narasimham (1993) constructed SOSRD using pairwise balanced designs (PBD). Further, Victorbabu and Narasimham (1994) constructed a new type of slope rotatable central composite design.

In this paper, a new method of construction of SOSRD using symmetrical unequal block arrangements (SUBA) with two unequal block sizes is suggested. It is found that in some cases this new method leads to SOSRD with lesser number of design points than the number required in those associated with SRCCD and SOSRD constructed with the help of BIBD and PBD. Further, bounds for the parameters of SOSRD using SUBA with two unequal block sizes are also obtained.

2. Conditions for SOSRD

Suppose we want to use the second order response surface design $D = ((x_{iu}))$ to fit the surface,

$$Y_u(x_{1u}, x_{2u}, \cdots, x_{vu}) = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i<j}\sum b_{ij} x_{iu} x_{ju} + e_u$$

where x_{iu} denotes the level of the i^{th} factor $(i = 1, 2, \cdots, v)$ in the u^{th} run $(u = 1, 2, \cdots, N)$ of the experiment, e_u's are uncorrelated random errors with mean zero and variance σ^2. A design D is said to be SOSRD if D satisfies the following conditions (*cf.* Hader and Park, 1978; Victorbabu and Narasimham, 1991):

1. $\sum_{u=1}^{N} \prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0$ if any α_i is odd, for $\sum \alpha_i \leq 4$
2. (i) $\sum_{u=1}^{N} x_{iu}^2 = \text{constant} = N\lambda_2$, (ii) $\sum_{u=1}^{N} x_{iu}^4 = \text{constant} = cN\lambda_4$
3. $\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \text{constant} = N\lambda_4$, for $i \neq j$
4. $(c + v - 1)\lambda_4 > v\lambda_2^2$
5. $\{v(5 - c) - (c - 3)^2\}\lambda_4 + \{v(c - 5) + 4\}\lambda_2^2 = 0$ (2.1)

where c, λ_2 and λ_4 are constants and the summation is over the design points.

3. New Method of Construction of SOSRD

The method of construction of SOSRD using symmetrical unequal block arrangements (SUBA) with two unequal block sizes is given in Theorem 3.1. Here we use the notations of Raghavarao (1971, pp. 291-292 and 298-300), Victorbabu and Narasimham (1991, pp. 2467-2472).

SUBA with two unequal block sizes (*cf.* Raghavarao, 1962). The arrangement of v-treatments in b blocks where b_1 blocks of size k_1, and b_2 blocks of size k_2 is said to be a symmetrical unequal block arrangement with two unequal block sizes, if

(i) every treatment occurs $\frac{b_i k_i}{v}$ blocks of size k_i $(i = 1, 2)$ and

(ii) every pair of first associate treatments occurs together in u blocks of size k_1 and in $(\lambda - u)$ blocks of size k_2 while every pair of second associate treatments occurs together in λ blocks of size k_2.

From (i) each treatment occurs in $(b_1k_1/v) + (b_2k_2/v) = r$ blocks among all. Note that $(v, b, r, k_1, k_2, b_1, b_2, \lambda)$ are known as the parameters of the SUBA with two unequal block sizes.

Let $(v, b, r, k_1, k_2, b_1, b_2, \lambda)$, $k = \sup\{k_1, k_2\}$ and $b_1 + b_2 = b$ be a SUBA with two unequal block sizes. Let $2^{t(k)}$ denote a resolution V fractional factorial of 2^k in ± 1 levels such that no interaction with less than five factors is confounded. Let n_0 denote the number of central points in the design.

Let $\{1 - (v, b, r, k_1, k_2, b_1, b_2, \lambda)\}$ denote the design points generated from the transpose of incidence matrix of SUBA with two unequal block sizes, $\{1 - (v, b, r, k_1, k_2, b_1, b_2, \lambda)\}\, 2^{t(k)}$ be the $b\, 2^{t(k)}$ design points generated from SUBA with two unequal block sizes by multiplication (*cf.* Raghavarao, 1971, pp. 298-300), $(a, 0, 0, \cdots, 0)\, 2^1$ denote the design points generated from $(a, 0, 0, \cdots, 0)$ point set, and $\bigcup$ denote combination of the design points generated from different sets of points.

Theorem 3.1. *The design points,*

$$\{1 - (v, b, r, k_1, k_2, b_1, b_2, \lambda)\}\, 2^{t(k)} \bigcup (a, 0, 0, \cdots, 0)\, 2^1 \bigcup (n_0)$$

give a v-dimensional SOSRD in $N = b\, 2^{t(k)} + 2v + n_0$ *design points.*

Proof. For the design points generated from SUBA with two unequal block sizes, the condition (1) of (2.1) is true. The conditions (2) and (3) of (2.1) are true as follows :

2. (i) $\sum x_{iu}^2 = r\,2^{t(k)} + 2a^2 = N\lambda_2$ (3.1)

   (ii) $\sum x_{iu}^4 = r\,2^{t(k)} + 2a^4 = cN\lambda_4$ (3.2)

3. $\sum x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)} = N\lambda_4.$ (3.3)

The equations (3.2) and (3.3) lead to

$$a^4 = (c\lambda - r)\,2^{t(k)-1} \tag{3.4}$$

(putting $c_1 = c - 3$), and (5) of (2.1) and (3.4) lead to

$$\frac{v(c_1-2)+4}{N\{v(c_1-2)+c_1^2\}} = \frac{\lambda 2^{t(k)}}{(r\,2^{t(k)}+2a^2)^2} \tag{3.5}$$

$$a^4 = \{(c_1+3)\lambda - r\}2^{t(k)-1}. \tag{3.6}$$

Equation (3.5) in the unknowns c_1 or a^2 can be solved iteratively.
[Alternatively eliminating c_1(or a^2) from (3.5) or (3.6), we get a 4^{th} degree equation in a^2 (or c_1).] □

Corollary. *If $k_1 = k_2 = k$, then Theorem* 3.1 *reduces to the method of construction of SOSRD using BIBD.*

Example 1. We illustrate Theorem 3.1 by constructing a SOSRD for 12−factors with the help of a SUBA with two unequal block sizes ($v = 12, b = 13, r = 4, k_1 = 3, k_2 = 4, b_1 = 4, b_2 = 9, \lambda = 1$). The design points,

$$\{1 - (12,13,4,3,4,4,9,1)\}\,2^4 \cup (a,0,0,\cdots,0)\,2^1 \cup (n_0 = 1)$$

will give a SOSRD in $N = 233$ design points for 12−factors (taking one central point). Here (2) and (3) of (2.1) are

2. (i) $\sum x_{iu}^2 = 64 + 2a^2 = N\lambda_2$

   (ii) $\sum x_{iu}^4 = 64 + 2a^4 = cN\lambda_4$

3. $\sum x_{iu}^2 x_{ju}^2 = 16 = N\lambda_4.$

Here

(3.6) implies $a^4 = 8(c_1 - 1)$, (3.7)

(3.5) implies $\dfrac{12c_1 - 20}{N(c_1^2 + 12c_1 - 24)} = \dfrac{16}{(64+2a^2)^2}\,.$ (3.8)

Solving (3.7) and (3.8) using an iterative technique, we get $a^2 = 5.5863$ and $c = 7.9008$.

Here we may point out that the SOSRD for 12 factors has only 233 design points, whereas the corresponding SRCCD of Hader and Park (1978) needs 281, SOSRD constructed using BIBD and PBD of Victorbabu and Narasimham (1991) and (1993) need 377 and 537 design points respectively. Thus this new method leads to a 12−factor SOSRD in less number of design points than those available in the literature.

Example 2. Here we construct a SOSRD for $v = 6$ factors with the help of a SUBA with two unequal block sizes $(v = 6, b = 7, r = 3, k_1 = 2, k_2 = 3, b_1 = 3, b_2 = 4, \lambda = 1)$. The design points,

$$\{1 - (6,7,3,2,3,3,4,1)\}\, 2^3 \cup (a,0,0,\cdots,0)\, 2^1 \cup (n_0 = 1)$$

will give a SOSRD in $N = 69$ design points. Here (2) and (3) of (2.1) are

2. (i) $\sum x_{iu}^2 = 24 + 2a^2 = N\lambda_2$,
   (ii) $\sum x_{iu}^4 = 24 + 2a^4 = cN\lambda_4$.
3. $\sum x_{iu}^2 x_{ju}^2 = 8 = N\lambda_4$.

Here

$$\text{(3.6) implies } a^4 = 4c_1, \tag{3.9}$$

$$\text{(3.5) implies } \frac{6c_1 - 8}{N(c_1^2 + 6c_1 - 12)} = \frac{8}{(24 + 2a^2)^2}\ . \tag{3.10}$$

solving (3.9) and (3.10) using an iterative technique, we get $a^2 = 4.5314$ and $c = 8.1333$.

The appendix gives the slope rotatability values for designs with minimum number of design points for each factor $6 \leq v \leq 16$.

4. Bounds for the Parameters of SOSRD

In this section the admissible values of the parameters a, c in SOSRD constructed using SUBA with two unequal block sizes are studied. Lower and upper bounds (a_L, a_U) and (c_L, c_U) for a, c are obtained.

It is found empirically that the relative behaviour of the parameters, a, c, n_0 of SOSRD constructed using SUBA with two unequal block sizes is as follows:

(i) If $n_0 \uparrow$, then $a \downarrow$ and $c \downarrow$

(ii) If $c \uparrow$, then $a \uparrow$ and $n_0 \downarrow$

(iii) If $a \uparrow$, then $c \uparrow$ and $n_0 \downarrow$ (4.1)

Upper bounds for a and c. For any v with $n_0 = 1$, upper bound of $a\,(a_U)$ and $c\,(c_U)$ will be obtained in view of (4.1) (which is justified in Hader and Park, 1978; Victorbabu and Narasimham, 1991 and 1993). With this a_U and from (3.2) and (3.3), we can obtain the upper bound for $c\,(c_U)$ from the relation

$$r2^{t(k)} + 2a_U^4 = c_U \lambda 2^{t(k)}. \tag{4.2}$$

Thus,

$$c_U = \frac{r2^{t(k)} + 2a_U^4}{\lambda 2^{t(k)}}. \tag{4.3}$$

Lower bounds for $c\,(c_L)$ for any v. From the slope rotatability condition (5) of (2.1), we have

$$\frac{\lambda_4}{\lambda_2^2} = \frac{v(c-5)+4}{(c-3)^2 - v(5-c)}\ . \tag{4.4}$$

But from non-singularity condition (4) of (2.1), we have

$$\frac{\lambda_4}{\lambda_2^2} > \frac{v}{c+v-1}\ . \tag{4.5}$$

Therefore from (4.4) and (4.5), we have

$$\frac{v(c-5)+4}{(c-3)^2 - v(5-c)} > \frac{v}{c+v-1} \tag{4.6}$$

which leads to

$$\frac{4(c-1)}{v^2(c-5) + 2v(c^2 - 6c + 7) + (c-1)(c-3)^2} > 0. \tag{4.7}$$

For $c > 1$, numerator of (4.7) is always > zero. For given 'v' the maximum value of 'c' for which the denominator of (4.7) is zero, gives the lower bound for $c\,(c_L)$.

Lower bound for $a\,(a_L)$. With this c_L which is obtained from equation (4.7) and using (3.2) and (3.3), we get the lower bound of $a\,(a_L)$ is given by

$$a_L = \left\{(\lambda c_L - r)2^{t(k)-1}\right\}^{1/4}. \tag{4.8}$$

Obviously lower bound of $n_0(n_{0_L}) = 1$ and upper bound of $n_0(n_{0_U}) = \infty$.

Example 3. We illustrate the computation of bounds for a, c. Consider, SOSRD for 6−factors using SUBA with two unequal block sizes with the following design points,

$$\{1 - (v = 6, b = 7, r = 3, k_1 = 2, k_2 = 3, b_1 = 2, b_2 = 4, \lambda = 1)\}\, 2^3$$
$$\bigcup(a, 0, \cdots, 0)2^1 \bigcup(n_0).$$

For any v with $n_0 = 1$, upper bound of $a\,(a_U)$ will be obtained by solving equations (3.9) and (3.10). Hence here,

(3.9) and (3.10) implies $a_U = 2.1287$.

(4.3) implies $c_U = 8.1333$.

(4.7) implies $c_L = 4.5826$.

(4.8) implies $a_L = 1.5862$.

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Appendix

These are SOSRDs with design points,

$$\{1-(v,b,r,k_1,k_2,b_1,b_2,\lambda)\}\,2^{t(k)}\bigcup(a,0,0,\cdots,0)\,2^1\bigcup(n_0).$$

(6,7,3,2,3,3,4,1)			(8,12,4,2,3,4,8,1)			(9,18,5,2,3,9,9,1)		
n_0	N	a	n_0	N	a	n_0	N	a
1	69	2.1287	1	113	2.0444	1	163	1.9009
5	73	2.0039	5	117	1.9070	5	167	1.7492
10	78	1.8900	10	122	1.7548	10	172	1.5764
15	83	1.8171	15	127	1.6461	15	177	1.3291
20	88	1.7703	20	132	1.5745	20	182	1.1687

(10,11,5,4,5,5,6,2)			(12,13,4,3,4,4,9,1)			(14,35,7,2,3,7,28,1)		
1	197	2.8928	1	233	2.3635	1	309	1.7853
5	201	2.8254	5	237	2.2497	2	310	1.6608
10	206	2.7553	10	242	2.1110			
15	211	2.7009	15	247	1.9972			
20	216	2.6594	20	252	1.9154			

(15,20,5,3,4,5,15,1)			(16,28,6,4,3,12,16,1)					
1	351	2.2737	1	481	2.0264			
5	355	2.1280	2	482	1.9746			
10	360	1.8709	3	483	1.9092			
15	365	1.5740	4	484	1.8080			
20	370	1.3754						

NOTE : *Other results of SOSRDs constructed using Raghavarao (1962) are available with the author.*

REFERENCES

Das, M. N. and Narasimham, V. L. (1962). "Construction of rotatable designs through balanced incomplete block designs", *The Annals of Mathematical Statistics*, **33**, 1421–1439.

Hader, R. J. and Park, S. H. (1978). "Slope rotatable central composite designs", *Technometrics*, **20**, 413–417.

Jang, D. H. and Park, S. H. (1993). "A measure and graphical method for evaluating slope rotatability in response surface designs", *Communications in Statistics-Theory and Methods*, **22**, 1849–1863.

Park, S. H. and Kim, H. J. (1992). "A measure of slope rotatability for second order response surface experimental designs", *Journal of Applied Statistics*, **19**, 391–404.

Raghavarao, D. (1962), "Symmetrical unequal block arrangements with two unequal block sizes", *Annals of Mathematical Statistics*, **33**, 620–633.

Raghavarao, D. (1971). *Construction and Combinatorial Problems in Design of Experiments*, John Wiley, New York.

Victorbabu, B. Re. and Narasimham, V. L. (1991). "Construction of second order slope rotatable designs through balanced incomplete block designs", *Communications in Statistics-Theory and Methods*, **20**, 2468–2478.

Victorbabu, B. Re. and Narasimham, V. L. (1993). "Construction of second order slope rotatable designs using pairwise balanced designs", *Journal of the Indian Society of Agricultural Statistics*, **45**, 200–205.

Victorbabu. B. Re. and Narasimham, V. L. (1994). "A new type of slope rotatable central composite design", *Journal of the Indian Society of Agricultural Statistics*. **46**. 315–317.