A Study on the Effective Hydraulic Conductivity of an Anisotropic Porous Medium

Kwanjae Seong
Associate Professor, Dept. of Mechanical Engineering Dong Guk University, Seoul 100-715, Korea

Effective hydraulic conductivity of a statistically anisotropic heterogeneous medium is obtained for steady two-dimensional flows employing stochastic analysis. Flow equations are solved up to second order and the effective conductivity is obtained in a semi-analytic form depending only on the spatial correlation function and the anisotropy ratio of the hydraulic conductivity field, hence becoming a true intrinsic property independent of the flow field. Results are obtained using a statistically anisotropic Gaussian correlation function where the anisotropy is defined as the ratio of integral scales normal and parallel to the mean flow direction. Second order results indicate that the effective conductivity of an anisotropic medium is greater than that of an isotropic one when the anisotropy ratio is less than one and vice versa. It is also found that the effective conductivity has upper and lower bounds of the arithmetic and the harmonic mean conductivities.

Key Words : Effective Hydraulic Conductivity, Groundwater Flow, Heterogeneous Anisotropic Porous medium, Stochastic Analysis

Nomenclature

C_Y	: Covariance function of Y
H	: Hydraulic head
I_{Yi}	: Integral scale of Y in i-direction
J	: Mean head gradient
K	: Hydraulic conductivity
K_{eff}	: Effective hydraulic conductivity
m_Y	: Expected value of Y
n	: Porosity
q	: Specific discharge
$Y=\ln K$	: Log-conductivity
$\lambda=I_{Y2}/I_{Y1}$	: Anisotropy ratio
ρ_Y	: Correlation function of Y
σ_Y^2	: Variance of Y
$\langle\ \rangle$	: Expected value operator
$\hat{}$	: Fourier transform

* Corresponding Author,
E-mail : seongkj@cakra.dongguk.ac.kr
TEL : +82-2-260-3703; **FAX** : +82-2-263-9379
Associate Professor, Dept. of Mechanical Engineering Dong Guk University, Seoul 100-715, Korea. (Manuscript **Received** July 27, 2001; **Revised** May 27, 2002)

1. Introduction

It is now generally accepted that hydrogeologic properties such as hydraulic conductivity of aquifers are highly heterogeneous, i.e., they vary significantly over a wide range of spatial scales, and hence flow and transport by groundwater are governed by this large-scale spatial heterogeneity of natural formations (Dagan, 1989; Gelhar, 1993). It is practically impossible to map the exact spatial distribution of the hydraulic conductivity field because this would require an enormous amount of data acquisition through such methods as drilling and well tests entailing in alteration of the aquifer properties. It is this highly heterogeneous nature that led to the application of stochastic approach in groundwater flow and solute transport analysis (Dagan, 1987). However, in many applications where knowledge of an average response is sufficient for management and decision making such as in contaminated soil remediation (Joo et al., 1998), this costly effort is unnecessary.

The average response of flow in aquifers can be characterized by the effective hydraulic conductivity, K_{eff}. To qualify as an effective property with general applicability, K_{eff} must be an intrinsic property, i.e., it should be a function of the hydraulic properties of the aquifer itself and not of the hydraulic head gradient. The effective conductivity provides primary statistical information and in cases where the flux is stationary, the average discharge can be used to estimate the space average of the flux. Computation of the effective hydraulic conductivity in heterogenous media requires knowledge of the statistical moments of the hydraulic conductivity and its spatial covariances.

In many cases porous media such as sedimentary rocks, soils and manufactured material such as catalysts are anisotropic in their resistance to the flow of fluids and solutes. The anisotropy stems from the process by which the porous medium is formed. For example, sedimentation process in the case of sedimentary rocks or soils and extrusion or pelleting process in the case of catalyst pellets (Friedman and Seaton, 1996). Flow and transport through anisotropic porous media have many engineering applications in such fields as subsurface hydrology, petroleum engineering and chemical engineering. The purpose of this study is to derive an effective hydraulic conductivity of an anisotropic heterogeneous porous medium using the method of small perturbations in the framework of stochastic analysis.

2. Mathematical Statement of the Problem

We consider a case of steady, two-dimensional flow in the horizontal saturated porous formation without recharge. Our starting point is the mass conservation equation without source or sink terms expressed as the following ;

$$\nabla \cdot \boldsymbol{q}(\boldsymbol{x}) = 0 \qquad (1)$$

and the Darcy's law ;

$$\boldsymbol{q}(\boldsymbol{x}) = -K(\boldsymbol{x}) \nabla H(\boldsymbol{x}) \qquad (2)$$

where $\boldsymbol{q}$ is the specific discharge, K is the hydraulic conductivity, H is the hydraulic head and $\boldsymbol{x} = x_i$ is the Cartesian coordinate. The specific discharge is related to the flow velocity through the following ;

$$U(\boldsymbol{x}) = \frac{\boldsymbol{q}(\boldsymbol{x})}{n} \qquad (3)$$

where n is the porosity of the medium. Here and subsequently, boldface letters denote vectors.

Natural formations are highly heterogeneous and it is common to model the hydraulic conductivity and its natural logarithm $Y = \ln K$ as a spatial random function (SRF) in order to account for their irregular spatial variations and for the uncertainty of their distribution. Following numerous findings from field studies (e.g., Freeze, 1975 ; Hoeksema and Kitanidis, 1984), the hydraulic log-conductivity Y is treated as a normal SRF and is modeled as the sum of its expected value, m_Y, and a small scale local fluctuation $Y'(\boldsymbol{x})$;

$$Y(\boldsymbol{x}) = \langle Y(x) \rangle + Y'(\boldsymbol{x}) = m_Y + Y'(\boldsymbol{x}) \qquad (4)$$

and for simplicity it is assumed to be stationary, i.e., of a constant m_Y.

Hence the log-conductivity is characterized by its mean m_Y (angled brackets denote expected-value operator) and the spatial covariance function,

$$\begin{aligned} C_Y(\boldsymbol{x}, \boldsymbol{y}) &= \langle Y'(\boldsymbol{x}) Y'(\boldsymbol{y}) \rangle = C_Y(\boldsymbol{r}) \\ &= \sigma_Y^2 \rho_Y(\boldsymbol{r}) \end{aligned} \qquad (5)$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$ is the separation vector, σ_Y^2 is the variance of Y and ρ_Y is its correlation function. Under the assumption of stationarity m_Y and σ_Y^2 are constants while ρ_Y is a function of the separation vector. The effects of porosity variations have been found to be secondary compared to those of hydraulic conductivity variations (Naff, 1978) and therefore is treated as a constant in this analysis. Combining Eqs. (1) ~ (4) leads to the following equation for the flow field ;

$$\nabla^2 H(\boldsymbol{x}) = -\nabla Y'(\boldsymbol{x}) \cdot \nabla H(\boldsymbol{x}) \qquad (6)$$

which is a stochastic partial differential equation due to the randomness of $Y'(\boldsymbol{x})$.

The problem is to solve Eq. (6) for $H(\boldsymbol{x})$ with different structures of $Y'(\boldsymbol{x})$. Equation (6) depends on the standard deviation of the log -conductivity σ_Y and tends to a deterministic solution as $\sigma_Y \rightarrow 0$, since the geologic formation becomes homogeneous at this limit. The hydraulic head field $H(\boldsymbol{x})$ is expanded in series up to order σ_Y^2 as the following ;

$$H(\boldsymbol{x}) = H_0(\boldsymbol{x})[1] + H_1(x)[\sigma_Y] + H_2(\boldsymbol{x})[\sigma_Y^2] + O[\sigma_Y^3] \quad (7)$$

where the term in brackets represents the order of magnitude and the solution to Eq. (6) is sought using the method of small perturbations. We seek here the solution up to order σ_Y^2.

Substitution of Eq. (7) into Eq. (6) leads to the following sequence of equations stated in the ascending order of magnitude up to $O[\sigma_Y^2]$.

$$O[1] \ : \nabla^2 H_0(\boldsymbol{x}) = 0 \quad (8a)$$
$$O[\sigma_Y] : \nabla^2 H_1(\boldsymbol{x}) = -\nabla Y'(\boldsymbol{x}) \cdot H_0(\boldsymbol{x}) \quad (8b)$$
$$O[\sigma_Y^2] : \nabla^2 H_2(\boldsymbol{x}) = -\nabla Y'(\boldsymbol{x}) \cdot H_1(\boldsymbol{x}) \quad (8c)$$

For the boundary condition we take the head gradient at some point in the flow domain given as ;

$$\langle \nabla H(\zeta) \rangle = -\boldsymbol{J} = -(J_1, 0) \quad (9)$$

where for convenience the coordinate axis x_1 is aligned in the direction of the mean flow. In this study we assume the flow domain to be unbounded. Although actual aquifers are obviously bounded, our analysis and solution based on the assumption of infinite domain are applicable to situations of finite domain as long as the domains under consideration are sufficiently removed from the boundaries (Rubin and Dagan, 1988). When we limit our solution of the flow field in heterogeneous formations up to order σ_Y^2, Eqs. (8a) ~ (8c) and (9) constitute the entire set of equations that need to be solved. Our model can be summed up as an unbounded horizontal plane of saturated porous formation and we seek a steady, first-order solution in the variance of Y under the assumption of stationary and anisotropic conductivity field in the framework of stochastic analysis using the method of small perturbations

3. Solution Using the Method of Perturbations

3.1 Flow field solution and effective conductivity

Taking the expected value of the order one hydraulic head equation (Eq. (8a)) with boundary condition of Eq. (9) leads to the following ;

$$\langle H_0(x) \rangle = -\boldsymbol{J} \quad (10)$$

and subsequently the first order equation for H_1 becomes

$$\nabla^2 H_1 = J_1 \frac{\partial Y'}{\partial x_1} \quad (11)$$

with the solution given by

$$H_1(\boldsymbol{x}) = J_1 \int G(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial Y'(\boldsymbol{y})}{\partial y_1} d\boldsymbol{y} \quad (12)$$

where G is the Green's function for the Laplace equation and integration is performed over the entire flow domain. The second order equation for H_2 is solved and after much tedious calculation its mean is found to be identically zero, i.e., $\langle H_2(\boldsymbol{x}) \rangle = 0$.

Having solved the flow problem in terms of the hydraulic head, the mean velocity up to second order is obtained from a SRF for the velocity using Eqs. (2) and (3) as ;

$$\langle U \rangle = -\frac{e^{m_Y}}{n} \left\{ \langle H_0 \rangle + \sigma_Y^2 \left(\langle \nabla H_2 \rangle + \langle Y' \nabla H_1 \rangle + \frac{1}{2} \langle Y'^2 \nabla H_0 \rangle \right) \right\} \quad (13)$$

and when Eqs. (10), (12) and $\langle H_2(\boldsymbol{x}) \rangle = 0$ are substituted into Eq. (13), second order approximation of the mean velocity is obtained as the following employing the Fourier transform (FT) technique ;

$$\langle U(\boldsymbol{x}) \rangle = \frac{e^{m_Y}}{n} \left\{ 1 + \sigma_Y^2 \left(\frac{1}{2} - \eta \right) \right\} J_1 \quad (14)$$

with η given by

$$\eta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\rho}_Y(\boldsymbol{k}) \frac{k_1^2}{k^2} d\boldsymbol{k} \quad (15)$$

where $\hat{\rho}_Y(\boldsymbol{k})$ represents the FT of $\rho_Y(\boldsymbol{r})$ and $\boldsymbol{k}$ is the wave number vector with $k^2 = k_1^2 + k_2^2$ and $d\boldsymbol{k} = dk_1 dk_2$. Now Eq. (14) can be rearranged to read as ;

$$\langle U(\boldsymbol{x})\rangle=\frac{1}{n}K_{eff}J_1 \tag{16}$$

which becomes the defining expression for the effective conductivity K_{eff} and for our problem can be expressed as the following ;

$$K_{eff}=e^{m_Y}\left\{1+\sigma_Y^2\left(\frac{1}{2}-\eta\right)\right\} \tag{17}$$

with η expressed as Eq. (15).

3.2 Anisotropic structure of hydraulic conductivity

The hydraulic conductivity of isotropic aquifers are represented by isotropic correlation functions which depend on the distance $r=|\boldsymbol{r}|$ only and their FT depends on the wave number $k=|k|$ (Freeze, 1975). However, due mainly to stratification, aquifers may display anisotropy and in line with Gelhar and Axness (1983) we assume that the anisotropy of the hydraulic conductivity is manifested in the scaling of the log-conductivity correlation function. Thus an anisotropic correlation function can be reduced to an isotropic one by rescaling the space coordinates as following ;

$$r_1'=r_1/I_{Y1} \quad r_2'=r_2/I_{Y2} \quad \lambda=I_{Y2}/I_{Y1} \tag{18}$$

where I_{Y1} and I_{Y2} are the log-conductivity integral scales in the r_1 and r_2 directions respectively and their ratio λ is defined as the anisotropy ratio. For example, $\lambda=1$ represents a statistically isotropic heterogeneous porous medium.

In this study we adopt an anisotropic Gaussian correlation function which in physical space coordinates is expressed as the following ;

$$\rho_Y(r)=\exp\left(-\frac{\pi}{4}r'^2\right) ; \quad r'^2=(r_1/I_{Y1})^2+(r_2/I_{Y2})^2 \tag{19}$$

and its FT becomes

$$\hat{\rho}_Y(k)=I_{Y1}I_{Y2}\hat{\rho}_Y(k') =I_{Y1}I_{Y2}\frac{2}{\pi}\exp\left[-\frac{1}{\pi}\{(k_1I_{Y1})^2+(k_2I_{Y2})^2\}\right] \tag{20}$$

where k' is the dimensionless wave number, $\{(k_1I_{Y1})^2+(k_2I_{Y2})^2\}^{1/2}$. This expression is used in Eqs. (15) and (17) for the evaluation of the effective hydraulic conductivity.

4. Results and Conclusions

Using the method of small perturbations in the stochastic analysis we obtained a semi-analytic expression for the effective hydraulic conductivity of a statistically anisotropic heterogeneous porous medium. The effective conductivity is an intrinsic function independent of the flow field and depends only on the statistical spatial distribution characteristics. In this study an anisotropic Gaussian correlation function was used.

4.1 Effective hydraulic conductivity

The dependence of the function (η) defined in Eq. (15), on the anisotropy ratio (λ) is presented in Fig. 1. The result is obtained by numerical integration of Eq. (15) with FT of an anisotropic Gaussian correlation function. The function η is 0.5 at $\lambda=1$, which corresponds to the isotropic heterogeneous porous medium and hence the effective conductivity reduces to $K_{eff}=e^{m_Y}$, the geometric mean of K ($K_G=\exp[\langle Y(\boldsymbol{x})\rangle]=e^{m_Y}$) (Dagan, 1989). It can also be seen that η is less than 0.5 when $\lambda\leq 1.0$ and greater than 0.5 when $\lambda\geq 1.0$ with limits of zero as $\lambda\rightarrow 0$ and 1.0 as $\lambda\rightarrow\infty$. From these characteristics of η and the

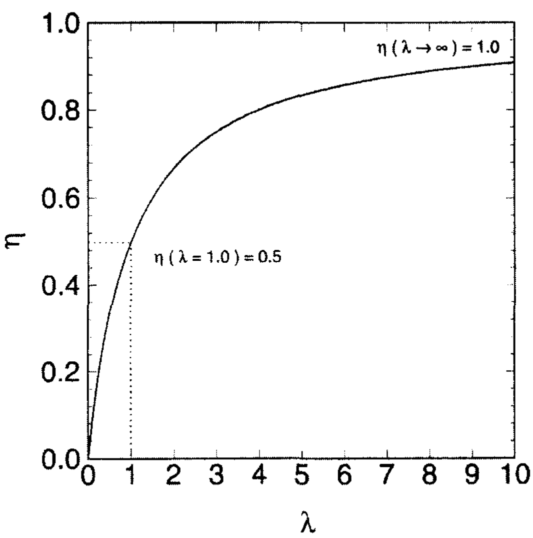
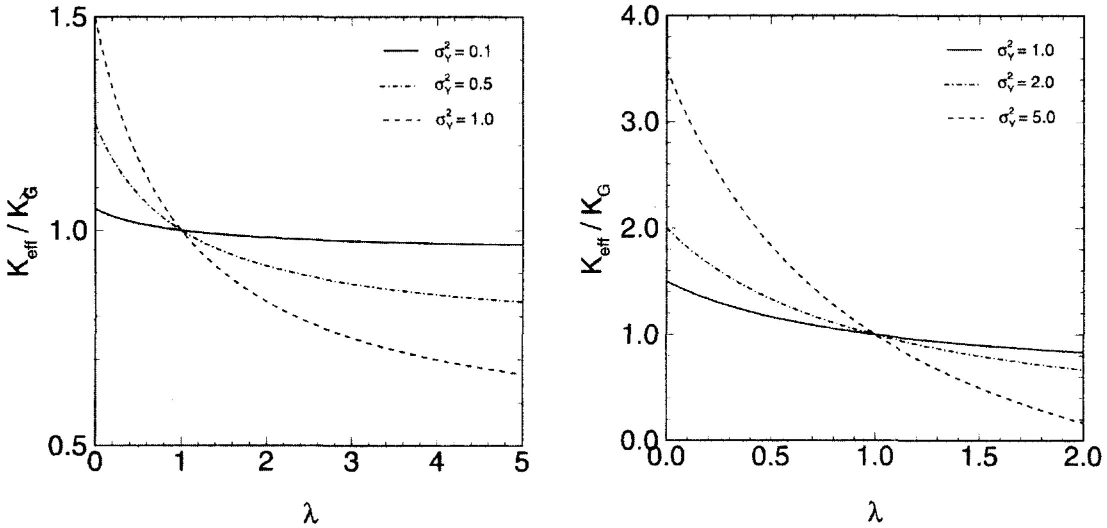


Fig. 1 Variation of η (Eq. (15)) as a function of anisotropy ratio λ



(a) $\sigma_Y^2 \leq 1.0$ (b) $\sigma_Y^2 \geq 1.0$

Fig. 2 Ratio K_{eff}/K_G as a function of anisotropy ratio λ

defining expression of the effective conductivity (Eq. (17)), we see that K_{eff} approaches the arithmetic mean as $\lambda \to 0$;

$$\lim_{\lambda \to 0} K_{eff} = K_G(1+\sigma_Y^2/2) = K_A \tag{21}$$

and the harmonic mean as $\lambda \to \infty$.

$$\lim_{\lambda \to \infty} K_{eff} = K_G(1-\sigma_Y^2/2) = K_H \tag{22}$$

Thus we can conclude that the limiting behaviors of the effective conductivity for an anisotropic medium are similar to those of a stratified formation. It has been shown by Dagan (1989) that for an isotropic bedding, the effective conductivity has the limits of arithmetic mean when the flow is parallel to bedding and harmonic mean when the flow is normal to bedding.

Figure 2 depicts the ratio of the effective conductivity K_{eff} to the geometric mean, K_G (effective conductivity of an isotropic medium), as a function of the anisotropy ratio for several log-conductivity variance values. In Fig. 2(a) are presented the ratio for variances less than one and in Fig. 2(b) for variances greater than one. Here we can verify that the effective conductivity of an anisotropic medium reduces to the geometric mean of an isotropic one when $\lambda = 1$. Also, K_{eff} is found to be greater than K_G for $\lambda \leq 1$ and less than K_G for $\lambda \geq 1$ as predicted from Fig. 1 and deviation from the geometric mean is found to be greater as the variance increases and as the anisotropy becomes more severe, i. e., the more λ deviates from isotropic case of one.

In Fig. 3 are presented the ratio K_{eff}/K_G as function of the variance for several values of the anisotropy ratio. It is clear that K_{eff} has an upper bound of arithmetic mean as $\lambda \to 0$, corresponding to a perfectly layered formation parallel to the mean flow and a lower bound of harmonic mean as $\lambda \to \infty$, corresponding to a perfectly layered formation normal to mean flow. This result is in agreement with the general theory which states that the arithmetic mean and the geometric mean are the best upper and lower bounds respectively for an anisotropic medium (Dagan, 1989). The approximate solution for K_{eff}, based on a perturbation method where we assume small standard deviation σ_Y as employed in our study, can be applied up to $\sigma_Y^2 = 2$. Beyond a variance of two it predicts a physically impossible negative hydraulic conductivity in the limit $\lambda \to \infty$. There are many evidences which suggest that the results of small variance of log-conductivity perturbation analysis can be applied to situations where the variances are not so small (e.g., Salandin and Rinaldo, 1990; Deng and Cushman, 1998).

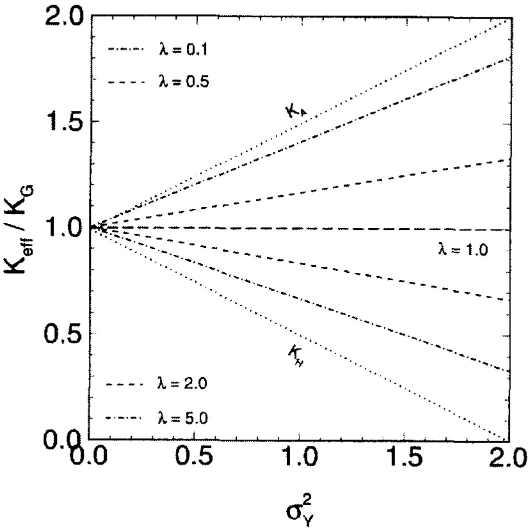


Fig. 3 Ratio K_{eff}/K_G as a function of variance σ_Y^2. Also shown are limiting cases of K_A as $\lambda \rightarrow 0$ and K_H as $\lambda \rightarrow \infty$

4.2 Conclusions

In this study an effective hydraulic conductivity of a statistically anisotropic heterogeneous medium is obtained for steady, two-dimensional flow through stochastic analysis using a perturbation method up to second order under the assumption of small log-conductivity variance. The effective conductivity is found as a semi-analytic form depending only on the anisotropy ratio, defined as the ratio of two integral scales, and the variance of the log-conductivity, both of which are hydrogeologic properties of the medium, making the effective conductivity a true intrinsic property.

Results with anisotropic Gaussian correlation function, where the anisotropy is manifested in different directional integral scales, are obtained by numerical integration. Second order solutions indicate that the effective conductivity of an anisotropic medium has upper and lower bounds of arithmetic and harmonic mean conductivities and may be applied for variances of less than two and the effective hydraulic conductivity results reduce to the classical case of an isotropic heterogeneity when the anisotropic ratio is one.

Effective hydraulic conductivity results obtained in this study can be used in estimating the representative block hydraulic conductivities of anisotropic media in numerical studies of groundwater flow and solute transport. Effective hydraulic conductivity characteristics can also be used in the inverse sense, in that measurements of bulk hydraulic properties may tell us something about the smaller scale heterogeneity. For example, anisotropy determined from aquifer tests can be used in conjunction with results such as that shown in Fig. 2 to estimate the ratio of the correlation scales.

The assumption of ergodicity is implicit in the stochastic approach used in this study. We are assuming that flows in an ensemble of aquifers with the assigned statistical properties approximate the real field situation, which involves flow in a single heterogeneous anisotropic aquifer. This assumption will be reasonable only if the scale of the flow system is large compared with the correlation scale of the aquifer (Lumley and Panofsky, 1964). Therefore the effective hydraulic conductivity developed above is meaningful only when the overall scale of the problem is large compared to the correlation scale of the hydraulic conductivity.

References

Dagan, G., 1987, "Theory of Solute Transport in Groundwater," *Ann. Rev. Fluid Mech*, Vol. 19, pp. 187~215.

Dagan, G., 1987, *Flow and Transport in Porous Formations*, p. 465, Springer-Verlag.

Deng, F. W. and Cushman, J. H., 1998, "Higher-order Corrections to the Flow Velocity Covariance Tensor, Revisited," *Water Resour. Res.*, Vol. 34, pp. 103~106.

Freeze, R. A., 1975, "A Stochastic Conceptual Analysis of One-dimensional Groundwater Flow in Nonuniform Homogeneous Media," *Water Resour. Res.*, Vol. 11, pp. 725~741.

Friedman, S. P. and Seaton, N. A., 1996, "On the Transport Properties of Anisotropic Network of Capillaries," *Water Resour. Res.*, Vol. 23, pp. 339~437.

Gelhar, L. W., 1993, *Stochastic subsurface hydrology*, p. 385, Prentice Hall.

Gelhar, L. W. and Axness, C. L., 1983, "Three-dimensional Stochastic Analysis of Macrodispersion in Aquifers," *Water Resour. Res.*, Vol. 19, pp. 161~180.

Hoeksema, R. J. and Kitanidis, P. K., 1984, "An Application of the Geostatistical Approach to the Inverse Problem in Two-dimensional Groundwater Modeling," *Water Resour. Res.*, Vol. 20, pp. 1021~1029.

Joo, Y., Fontinich, A. and Dhir, V. K., 1998, "Remediation of Contaminated Soil with Diesel by Soil Venting Technique," *KSME Int. J.*, Vol. 12, pp. 1174~1183.

Lumley, J. L. and Panofsky, H. A., 1964, *The Structure of Atmospheric Turbulence*, p. 239, John Wiley.

Naff, R. L., 1978, "A Continuum Approach to the Study and Determination of Field Longitudinal Dispersion Coefficients," *Ph. D. dissertation*, N. M. Inst. of Mining and Tech., p. 176, Socorro.

Rubin, Y. and Dagan, G., 1988, "Stochastic Analysis of Boundary Effects on Head Spatial Variability in Heterogeneous Aquifers, 1. Constant Head Boundaries," *Water Resour. Res.*, Vol. 24, pp. 1689~1697.

Salandin, P. and Rinaldo, A., 1990, "Numerical Experiment on Dispersion in Heterogeneous Porous Media," *Computational Methods in Subsurface Hydrology* (ed. G. Gambolati et al.) p. 575, Springer-Verlag.