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주문량에 종속적인 덤이 제공되는 상황하에서 퇴화성제품의 경제적 주문량 결정

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Economic order quantity of deteriorating items under free addition depending on the order quantity

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본 논문은 제품 공급자로부터 일정량의 덤이 제공되는 상황에서 퇴화성 제품의 경제적 주문량 결정에 대하여 연구한다. 공급자로부터 제공되는 덤의 크기는 주문량의 크기, 즉 거래단위에 따라 차별적으로 제공된다는 가정하에 최적주문 및 재고정책 결정을 위한 해법이 제시되며 예제를 통하여 덤의 크기에 따른 재고정책의 민감도를 계량적으로 분석한다.

Keywords: Economic order quantity, Deteriorating items, Free addition

1. Introduction

Faced with the problem of unexpected excess inventory, the supplier gives some incentives to the customer in order to increase the amount of the quantity purchased by the customer. In this regard, a considerable number of research papers have been studied on the subjects of inventory control involving lot sizing with quantity discount. The traditional quantity discount models have analyzed solely the unit purchasing price discount and considered two types of price discount, "all-units" and "incremental" discount. Abad [1, 2], Arcelus and Srinivasan[3], and Hadley and Within[14] analyzed the effect of quantity discount on the optimal inventory policy. Recognizing another type of discount structure, Burwell *et al.*[4] and Lee[11] formulated the classical EOQ model with set up cost including a fixed cost and freight cost, where the freight cost has a quantity discount. The common assumption of the above extended models is that the customer must pay for the items as soon as he receives them from a supplier. However, in some distribution systems, the supplier will allow a certain credit period for settling the amount the customer owes to him for the items supplied. Trade credit would play an important role in the conduct of business for many reasons. For a supplier who offers trade credit, it is an effective means of price discrimination which circumvents antitrust measures and is also an efficient method to stimulate the demand of the product. For a customer, it is an efficient method of bonding a supplier when the customer is at the risk of receiving inferior quality goods or service and is also an effective means of reducing the cost of holding stocks. Based upon the above observations, some research papers dealt with the inventory model under trade credit. Chung[6] and Goyal[9] examined the effects of trade credit on the optimal inventory policy. Also, Shinn *et al.*[12] introduced the joint price and lot size determination problem under conditions of trade credit and quantity discounts for freight cost.

However, in Korea, it is not uncommon that some supermarket or discount store managers offer a free addition related to the bundle size for reasons of marketing policy. Therefore, the customers can get some extra with no additive cost depending on the amount of the quantity purchased. The availability of opportunity to get some extra with no cost effectively reduces the customer's total purchasing cost and it enables the customer to choose an optimal ordering quantity from another options. In this regard, Shinn and Song[14] analyzed the problem of determining the optimal ordering quantity when the supplier permits a free addition for an order of a product by the customer. They assumed that the free addition rates suggested by the supplier are equal independently of the customer's order size.

All the research works mentioned above implicitly assume that inventory is depleted by customer's demand alone. This assumption is quite valid for nonperishable or nondeteriorating inventory items. However, there are numerous types of inventory whose utility does not remain constant over time. In this case, inventory is depleted not only by demand but also by deterioration. Ghare and Schrader[8], assuming exponential deterioration of the inventory in the face of constant demand, derived a revised form of the economic order quantity. Cohen[7] examined the joint price and lot size determination problem for an exponentially deteriorating product. Recently, Chu et al.[5], and Hwang and Shinn[10] analyzed the inventory model for deteriorating products under the condition of permissible delay in payments.

This paper is concerned with the problem of determining the ordering quantity for an exponentially deteriorating product when the supplier permits a free addition for an order of a product. Also, relaxing equal free addition rates assumed by Shinn and Song[13], we assume that the size of the free addition depends on the amount of the quantity purchased by the customer. In Section 2, we formulate a relevant mathematical model. A solution algorithm is developed in Section 3 based on the properties of an optimal solution. A numerical example is provided in Section 4, which is followed by concluding remarks.

2. Model Formulation

We develop the inventory model for a deteriorating product under order-size- dependent free addition. The objective of this model is to determine the optimal ordering quantity which minimizes the annual total cost for the customer.

The following assumptions will be used for our mathematical model.

1) Replenishments are instantaneous with a known and constant lead time.
2) The demand rate is known and constant.
3) Shortages are not allowed.
4) Inventory is depleted not only by demand but also by deterioration. And the time elapsed to deterioration follows exponential distribution with parameter λ.
5) The supplier allows a series of free addition depending on the amount of the quantity purchased.

And the following notations will be used.

D : annual demand rate
C_p : unit purchasing cost
Q : order size
T : replenishment cycle time
C_o : ordering cost per order
C_h : unit inventory holding cost per unit time
U : bundle size for a free addition
α_j : free addition rate for jth bundle as a percentage of U ($0<\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_n$)
λ : the inventory deteriorating rate ($0<\lambda\ll 1$)
$I(t)$: inventory level at time t

In this situation, the supplier suggests a constant bundle size(U) and a series of free addition rate(α_j) depending on the amount of the quantity purchased to stimulate the demand of the product. For the first U, as products are purchased to $(1-\alpha_1)U$, the unit purchasing cost C_p is charged for each unit. Therefore, the total purchasing cost becomes C_pQ for $Q<(1-\alpha_1)U$. When the order size Q becomes $(1-\alpha_1)U$, the products are sold in a bundle of size U as the total purchasing cost is $C_p(1-\alpha_1)U$. Namely, there is no additive purchasing cost for $\alpha_1 U$ units. Figure 1 illustrates the total purchasing cost to the order size. Note that the feasible quantities of Q are $Q\in[(j-1)U,\ (j-1)U+(1-\alpha_j)U\ )$, $j=1,\ 2,\ \cdots n$ and therefore, the total purchasing cost $C(Q)$, is

$$C(Q)=C_pU\sum_{i=1}^{j-1}(1-\alpha_i)+C_p(Q-(j-1)U)$$
$$=C_p(Q-U\sum_{i=1}^{j-1}\alpha_i),$$
$$Q\in[(j-1)U,\ (j-1)U+(1-\alpha_j)U\ ),\quad j=1,\ 2,\ \cdots n \tag{1}$$

Note that when all $\alpha_j=0$, the total purchasing cost reduce to C_pQ.

For the case of exponential deterioration, as stated by Ghare and Schrader[8], the rate at which inventory deteriorates will be proportional to on hand inventory, $I(t)$. Thus, the inventory level depletes as the time passes due to demand and deterioration. The differential equation representing the inventory level at time t is

$$\frac{dI(t)}{dt}=-\lambda I(t)-D,\quad 0\le t\le T. \tag{2}$$

The solution of equation (2) with the boundary condition $I(T)=0$, is

$$I(t)=\frac{D}{\lambda}(e^{\lambda(T-t)}-1),\ 0\le t\le T. \tag{3}$$

From the condition of $I(0)=Q$, we have

$$Q=\frac{D}{\lambda}(e^{\lambda T}-1). \tag{4}$$

Now, we formulate the annual total cost $TC(T)$. The annual total cost consists of the following three elements.

1) Annual ordering cost $=\frac{C_o}{T}$.
2) Annual inventory holding cost
$$=\frac{C_h}{T}\int_0^T I(t)dt\ =\frac{C_hD}{\lambda^2T}(e^{\lambda T}-\lambda T-1).$$
3) Annual purchasing cost
$$=\frac{C(Q)}{T}=\frac{C_p}{T}(Q-U\sum_{i=1}^{j-1}\alpha_i),$$
$$Q\in[(j-1)U,\ (j-1)U+(1-\alpha_j)U\ ).$$

By equation (4), the annual purchasing cost can be

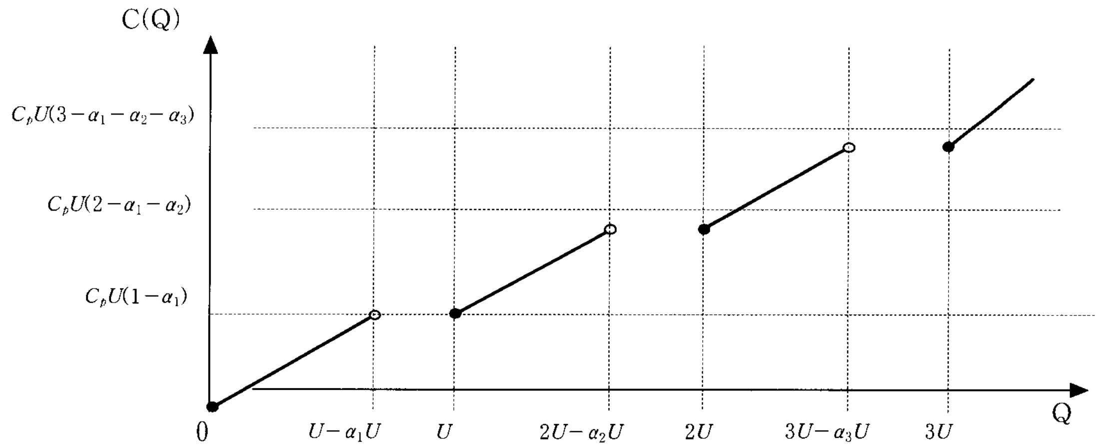

Figure 1. Total purchasing cost vs. Order quantity

rewritten as $\frac{C_p}{T}[\frac{D}{\lambda}(e^{\lambda T}-1)-U\sum_{i=1}^{j-1}\alpha_i]$.

And if let $I_j^L=\frac{1}{\lambda}\ln(\frac{\lambda}{D}(j-1)U+1)$ and $I_j^U=\frac{1}{\lambda}\ln(\frac{\lambda}{D}(jU-\alpha_j U)+1)$, then the annual purchasing cost becomes

$$\frac{C_p}{T}[\frac{D}{\lambda}(e^{\lambda T}-1)-U\sum_{i=1}^{j-1}\alpha_i],\quad T\in[I_j^L,\ I_j^U)\quad j=1,\ 2,\ \cdots n. \tag{5}$$

Then, the annual total cost can be expressed as

$TC(T)$ = Ordering cost + Inventory holding cost + Purchasing cost.

Depending on the relative size of T to I_j^L and I_j^U,

$$TC_j(T)=\frac{C_o}{T}+\frac{C_hD}{\lambda^2T}(e^{\lambda T}-\lambda T-1)+\frac{C_pD}{\lambda T}(e^{\lambda T}-1)-\frac{C_p}{T}U\sum_{i=1}^{j-1}\alpha_i,$$
$$T\in[I_j^L,\ I_j^U),\qquad j=1,\ 2,\ \cdots n. \tag{6}$$

Note that if all $\alpha_j=0$, then equation (6) reduce to the total cost function of the classical EOQ model for deteriorating product.

3. Analysis of the Model

The problem is to find an optimal replenishment cycle time T^* which minimizes $TC_j(T)$. Once T^* is found, an optimal ordering quantity Q^* can be obtained by equation (4). Although the objective function can be differentiated, it is impossible to find an optimal solution in explicit form. Thus the model will be solved approximately by using a truncated Taylor series expansion for the exponential term, i.e.,

$$e^{\lambda T}\cong 1+\lambda T+\frac{1}{2}\lambda^2T^2, \tag{7}$$

which is a valid approximation for smaller values of λT. With the above approximation, the annual total cost function can be rewritten as

$$TC_j(T)=\frac{C_o}{T}+\frac{1}{2}DT(C_h+\lambda C_p)+C_pD-\frac{C_p}{T}U\sum_{i=1}^{j-1}\alpha_i\quad T\in[I_j^L,\ I_j^U),$$
$$j=1,\ 2,\ \cdots n. \tag{8}$$

Note that equation (7) is exact when $\lambda=0$ so that equation (6) reduces to the exact formula equation (8) for nondeteriorating product.

From the annual total cost function by equation (8), it is self evident that for any fixed T, $TC_j(T) > TC_{j+1}(T)$, $j=1,\ 2,\ \cdots n$. So, $TC_j(T)$ is strictly decreasing for any fixed T as j increases.

For the index j which is satisfying the inequality, $C_o-C_pU\sum_{i=1}^{j-1}\alpha_i>0$, $TC_j(T)$ is a convex function and thus, there exists a unique value T_j, which minimizes and they are :

$$T_j=\sqrt{\frac{2(C_o-C_pU\sum_{i=1}^{j-1}\alpha_i)}{D(C_h+\lambda C_p)}}. \tag{9}$$

For the index j such that $C_o-C_pU\sum_{i=1}^{j-1}\alpha_i\le 0$, $TC_j(T)'>0$ and so $TC_j(T)$ is an increasing function of T.

Also, if T_j exists, then we have the following property.

Property 1.

$T_{j-1}>T_j$ holds for the index j such that

$$C_o-C_pU\sum_{i=1}^{j-1}\alpha_i>0.$$

Property 1 indicates that if T_j exists, then the value of T_j is strictly decreasing as j increases. From Property 1, we have the following useful property.

Property 2.

There exists at least one $T_j\ \ge\ I_j^L$.

Proof.

Because $C_o>0$, $TC_1(T)$ must be a convex function and thus, there exists at least one T_j. Also, from property 1, if all $T_j < I_j^L$ for every j, then $T_1 < 0$ holds, which contradicts the feasibility of T, i.e., $0 < T < \infty$. *Q.E.D.*

Now, from the above properties, we can make the following observation about the characteristics of $TC_j(T)$ for $T \in I_j^{LU} = \{ T \mid I_j^L \le T < I_j^U \}$, $j = 1, 2, \cdots n$. This observation simplifies our search process such that only a finite number of candidate values of T need to be considered to find an optimal value T^*. Let m be the largest index such that $T_m \ge I_m^L$ and $T^0 = I_m^U - \varepsilon$.

Observation.

(i) If $T_m \ge T^0$, then we only have to consider $T = T^0$ for $T \in I_m^{LU}$ as candidate for T^* and $T^* \ge T^0$.

(ii) Else if $T_m \ge I_m^L$, then we only have to consider $T = T^m$ for $T \in I_m^{LU}$ as candidate for T^* and $T^* \ge T_m$.

Proof.

(i) Because $T_m \ge T^0$, $TC_m(T)$ is a decreasing function for $T \in I_m^{LU}$ and so, we have $TC_m(T^0) \le TC_m(T)$ for $T \in I_m^{LU}$.

Therefore, if $T_m \ge T^0$, then $T = T^0$ yields the minimum total cost for $T \in I_m^{LU}$. Also, because $TC_j(T)$ is strictly decreasing for any fixed T as j increases, $TC_m(T) \le TC_j(T)$, $j < m$.

Hence,

$$T^* \ge T^0.$$

(ii) Since $I_m^L \le T_m < T^0$, $TC_m(T)$ is convex function for $T \in I_m^{LU}$. So, by definition of T_m, we have

$$TC_m(T_m) \le TC_m(T) \qquad \text{for } T \in I_m^{LU}.$$

Therefore, if $I_m^L \le T_m < T^0$, then the total cost becomes the minimum at $T = T_m$. Also, because $TC_j(T)$ is strictly decreasing for any fixed T as j increases,

$$TC_m(T) \le TC_j(T), \quad j < m.$$

Hence,

$$T^* \ge T_m \qquad Q.E.D.$$

Observation indicates that $TC_j(T)$ is increasing in $T \in I_j^{LU}$ for $j > m$ and so, we only need to consider $T = I_j^L$, $j > m$, in finding an optimal value T^* for $T \ge I_{m+1}^L$.

Based on the above proposition, we develop the following solution procedure to determine an optimal solution for the approximate model.

<Solution procedure>

Step 1. Compute T_1 by equation (9) and find the index k such that $T_1 \in [I_k^L, I_{k+1}^L)$.

Step 2. Find the largest index l such that $C_o - C_p U \sum_{i=1}^{l-1} \alpha_i > 0$.

Step 3. Compute T_j, $j \le \min\{k, l\}$ by equation (9) and find the largest index m such that $T_m \ge I_m^L$.

Step 4. If $T_m \ge I_m^U - \varepsilon$, then compute the annual total cost for $T = I_m^U - \varepsilon$.
Otherwise, compute the annual total cost for $T = T_m$.

Step 5. Compute the annual total cost for $T = I_j^L$, $j = m+1, m+2, \cdots, n+1$.

Step 6. Select the one that yields the minimum annual total cost as T^* and stop.

4. Numerical Example

In order to illustrate the proposed solution algorithm, let us consider the following problem.
D=2,000 units, C=\$2, A=\$300, H=\$0.12, U=400 units,

λ=0.1, $\alpha_1 = 0.05$(5%) and

$\alpha_j = \alpha_1 \times (1+0.05)^{j-1}$, $j=2, 3, \cdots, 10$.

The solution procedure generates an optimal solution for the approximate model through the following steps.

Step 1. Since T_1 = 0.97, k = 6.

Step 2. Since $C_o - C_p U \sum_{i=1}^{7-1} \alpha_i$ ($\fallingdotseq$ 27.92) > 0 and

$C_o - C_p U \sum_{i=1}^{8-1} \alpha_i$ ($\fallingdotseq$ −25.68) < 0, l = 7.

Step 3. Since min{6, 7} = 6, compute T_j for $j \le 6$. And since $T_5(= 0.63) < I_5^L(= 0.77)$ and $T_4(= 0.74) > I_4^L$ $(= 0.58)$, m = 4.

Step 4. Since $T_4 < I_4^U - \varepsilon$ ($\fallingdotseq$ 0.75), compute $TC_4(T_4)$.

Step 5. Since m = 4, compute $TC_j(I_j^L)$, $j = 5, 6, \cdots, 11$.

Step 6. Since $TC_7(I_7^L)$ = 4387.30 = min{ $TC_4(T_4)$ and $TC_j(I_j^L)$, $j=5,6,\cdots,$ 11}, an optimal replenishment cycle time becomes 1.13 with its minimum annual total cost of $4387.30 .

For the analysis, we evaluate the effect of the various values of λ and α_1. Four levels of λ are adopted, λ = 0.0, 0.05, 0.1 and 0.2. For each level of λ, four levels of α_1, ranging from 0.0 to 0.15 with an increment of 0.05 are tested. The results are shown in Table 1 and note that for the case with λ = 0 and α_1 = 0, the algorithm generates the same results as those by the classical EOQ model.

5. Conclusions

This paper dealt with the inventory model for an exponentially deteriorating product when the supplier allows the free addition for an order of the product by the customer. Recognizing that a major reason for the supplier to offer a free addition to the customers is to stimulate the order of the product, we formulated the size of the free addition depends on the amount of the quantity purchased by the customer. And so, we think that the model presented in this paper may be one of the realistic extensions for the classical EOQ model.

For the system presented, a mathematical model was developed. Recognizing that the model has a very complicated structure, a truncated Taylor series expansion is utilized to find a solution procedure. To illustrate the validity of the procedure, an example problem was solved and the effects of the values of λ and α_1 are examined on the annual total cost, replenishment cycle time and the customer's order size. The results show that the total cost seems quite

Table 1. Results with various values of α_1 and λ

		λ = 0.0	λ = 0.05	λ = 0.10	λ = 0.20
$\alpha_1 = 0.0$	T^*	1.58	1.17	0.97	0.76
	Q^*	3162	2405	2033	1641
	$TC(T^*)$	4379.47	4513.81	4619.68	4789.94
$\alpha_1 = 0.05$	T^*	1.80	1.35	1.13	0.74
	Q^*	3600	2800	2400	1600
	$TC(T^*)$	4137.63	4278.73	4387.30	4557.83
$\alpha_1 = 0.10$	T^*	2.00	1.54	1.13	0.91
	Q^*	4000	3200	2400	2000
	$TC(T^*)$	3886.89	4037.22	4147.22	4318.21
$\alpha_1 = 0.15$	T^*	2.00	1.91	1.31	0.91
	Q^*	4000	4000	2800	2000
	$TC(T^*)$	3635.33	3784.94	3902.59	4075.75

sensitive to λ and α_1.

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