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Quotient Structure of product BCK-algebra via fuzzy ideals

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Abstract

We investigate the quotient structure of a product BCK-algebra and fundamental homomorphism theorem and show their some properties.

Key words : Fuzzy ideal, product BCK-algebra, quotient structure

1. Introduction

BCK-algebras are an important class of logical algebra introduced by Iseki in 1996, and the concept of a fuzzy set was introduced by L.A.Zadeh [11].

Since then, this notion has been applied to many algebraic structure such as semigroups, groups, rings, modules, vector spaces, topologies, and measures. In 1991, O.G. Xi [10] applied the concept of fuzzy sets to BCK-algebras. In [4, 5], Y.B.Jun considered the notion of closed fuzzy ideals of BCI-algebras which is a generalization of BCK-algebras, and solved the problem of classifying fuzzy ideals by their family of level subsets in BCK/BCI-algebras. For the general development of BCK-algebras, the fuzzy ideal theory plays an important role. Also, the quotient structure by fuzzy ideals plays an important role.

By [9], we have shown that if μ is a fuzzy commutative ideal In X, then X/μ is a commutative BCK-algebra. For an ideal A of a BCK-algebra X, it is well known that the relation $\sim$ on X, defined by $x\sim y$ if and only if $x*y\in A$ and $y*x\in A$, is an equivalence relation on X, and we obtain the quotient structure of BCK-algebras via ideals. Y.H. Kim and H.S. Kim [7] investigated the quotient structure of BCI-algebra using fuzzy ideals.

In this paper, we introduce some basic definitions and theorems of BCK-algebra and fuzzy concepts. We dicuss the qutient structure of product BCK-algebra and fundamental homomorphism theorem. We show that if μ (resp. ν) is a fuzzy ideals in BCK-algebra X (resp. Y), then

$$\frac{X\times Y}{\mu\times\nu}\cong X/\mu\times Y/\nu.$$

Moreover, if J(resp. I) is an ideal in X (resp. Y), then

$$\frac{X\times Y/\mu\times\nu}{J\times I/\mu\times\nu}\cong X/J\times Y/I$$

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2. Preliminary

A BCK-algebra is a nonempty set X on which a binary operation $*$ been specified with following property ; For all $x, y, z\in X$.

(i) $((x*y)*(x*z))*(z*y)=0$
(ii) $(x*(x*y))*y=0$
(iii) $x*x=0$
(iv) $0*x=0$
(v) $x*y=0$ and $y*x=0$ imply $x=y$

where 0 is a fixed element of X.

In a BCK-algebra X, one can define a partially ordered relation $\leq$ by letting $x\leq y$ if and only if $x*y=0$.

A BCK-algebra X enjoys the following properties;

(1) $(x*y)*z=(x*z)*y$
(2) $x*y\leq x$
(3) $x*0=x$
(4) $x*(x*(x*y))=x*y$
(5) $(x*z)*(y*z)\leq x*y$
(6) $x\leq y$ implies $x*z\leq y*z$ and $z*y\leq z*x$ for all $x, y, z\in X$

Throughout this paper, X and Y always mean BCK-algebras without any specification.

Definition 2.1 [9] A fuzzy set μ in a BCK-algebra X is said to be a fuzzy ideal in X if it satisfies

(F1) $\mu(0)\geq\mu(x)$ for all $x\in X$
(F2) $\mu(x)\geq\min\{\mu(x*y),\mu(y)\}$ for all $x,y\in X$

An ideal of X is a subset A of X containing 0 such that If $x*y\in A$ and $y\in A$, then $x\in A$. If A is an ideal of X and $x\in A$, $y\leq x$, then $y\in A$. A map $f:X\rightarrow Y$ of BCK-algebras is called a homomorphism if

$$f(x*y)=f(x)*f(y)$$

for all $x,y\in X$.

Now we review some fuzzy logic concepts. A fuzzy set in X is a function $\mu;X\rightarrow[0,1]$. We shall use the notation μ_t, called a level subset of μ, for

$\{x \in X \mid \mu(x) \geq t\}$ where $t \in [0, 1]$.

Let A be an ideal of X. We define a relation $\sim^A$ on X as follows; $x \sim^A y$ if and only if $x*y \in A$ and $y*x \in A$. Then $\sim^A$ is an equivalence relation on X. We denote by A_x the equivalence class containing x and by X/A the set of all equivalence classes of X with respect to $\sim^A$, i.e.,

$A_x := \{y \in X \mid x \sim^A y\}$ and

$X/A := \{A_x \mid x \in X\}$.

If we define binary operation $*_A$ on X/A by

$A_x *_A A_y = A_{x*y}$ for any A_x,

$A_y \in X/A$ then $(X/A; *_A, A_0)$

is a BCK-algebra.

Definition 2.2 [9] Let μ be a non-constant fuzzy ideal of X and define a binary relation $\sim^\mu$ on X by putting $x \sim^\mu y$ if and only if

$\mu(x*y) > 0$ and $\mu(y*x) > 0$.

By [9], $\sim^\mu$ is an equivalence relation on X. And also we have known that $x \sim^\mu y$ and $u \sim^\mu v$ imply

$x*u \sim^\mu y*v$, for all $x, y, u, v \in X$.

Definition 2.3 [8] Let X be a BCK-algebra and $\sim$ an equivalence relation on X. Then $\sim$ is called a congruence relation on X if

$x \sim y$ and $u \sim v$ implies $x*u \sim y*v$.

Theorem 2.4 [9] $\sim^\mu$ is a congruence relation on X.

We denote by μ_x the equivalence class containing x and by X/μ the set of all equivalence classes of X with respect to $\sim^\mu$, that is,

$\mu_x := \{y \in X \mid y \sim^\mu x\}$ and

$X/\mu := \{\mu_x \mid x \in X\}$.

And we define a binary operation $*$on X/μ by putting $\mu_x * \mu_y := \mu_{x*y}$.

Then Theorem 2.3 guarantees that this is well-define.

Next, we verify that $(X/\mu; *, \mu_0)$ is a BCK-algebra.

Let $\mu_x, \mu_y, \mu_z \in X/\mu$, then

$((\mu_x * \mu_y) * (\mu_x * \mu_z)) * (\mu_z * \mu_y)$

$= \mu_{((x*y)*(x*z))*(z*y)} = \mu_0$;

$(\mu_x * (\mu_x * \mu_y)) * \mu_y$

$= \mu_0$; $\mu_x * \mu_x = \mu_{x*x} = \mu_0$;

$\mu_0 * \mu_x = \mu_0$.

Hence, $(X/\mu; *, \mu_0)$ satisfies the conditions (i) - (v).

If

$\mu_x * \mu_y = \mu_0$ and $\mu_y * \mu_x = \mu_0$, then

$\mu_{x*y} = \mu_{y*x} = \mu_0$, so $x*y \sim^\mu 0$ and $y*x \sim^\mu 0$.

It follows that

$\mu_{(x*y)} = \mu_{((y*x)*0)} > 0$, $\mu_{(y*x)} = \mu_{((y*x)*0)} > 0$

that is,

$x \sim^\mu y$. Thus $\mu_x = \mu_y$.

Therefore,

$(X/\mu; *, \mu_0)$ is a BCK- algebra.

Thoerem 2.5 [9] Let μ be a fuzzy ideal in a BCK-algebra X. Then

$(X/\mu; *, \mu_0)$ is a BCK-algebra.

This algebra is called the quotient BCK-algebra induced by a fuzzy ideal μ.

Definition 2.6 [9] Let μ be a fuzzy ideal in X.

(1) μ is called a fuzzy positive implicative ideal if it satisfies

(F1) and $\mu(x*y) \geq \min\{\mu((x*y)*z), \mu(y*z)\}$

for all $x, y, z \in X$.

(2) μ is called a fuzzy implicative ideal if it satisfies (F1) and

$\mu(x) \geq \min\{\mu((x*(y*x))*z)\ \mu(z)\}$

for all $x, y, z \in X$.

(3) μ is called a fuzzy commutative ideal if it satisfies (F1) and

$\mu(x*(y*(y*x))) \geq \min\{\mu((x*y)*z), \mu(z)\}$

for all $x, y, z \in X$.

3. Quotient structure of product BCK-algebra via fuzzy ideals.

In general, the BCK-homomorphism

$\pi : X \to X/\mu$, given by

$\pi(x) = \mu_x$,

is called the natural (or canonical) homomorphism of X onto X/μ.

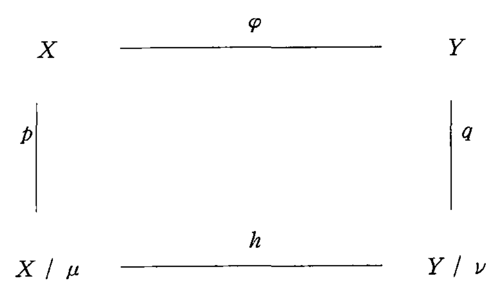
Let $\varphi : X \to Y$ be a BCK- epimorphism. By [7], we have known that if μ is a fuzzy ideal in X and if ν is a fuzzy ideal in Y such that $\nu \circ \varphi = \mu$, then

$X/\mu \cong Y/\nu$.

Therefore, if we define canonical homomorphisms

$p : X \to X/\mu$ and $q : Y \to Y/\nu$,

then it is easy to show that $h \circ p = q \circ \varphi$, i.e., the following diagram commutes:



Also, the fundamental homomorphism theorem for BCK-algebra is well known, i.e., if $\varphi : X \to Y$ is a BCK epimorphism, then

$X/Ker\varphi \cong Y$.

Definition 3.1 Let BCK-algebra X and Y, if we define

$(x_1, y_1) * (x_2, y_2) := (x_1 * x_2, y_1 * y_2)$

in $X \times Y$, then

$(X \times Y; *, (0, 0))$

is also a BCK-algebra and it is called product BCK-algebra.

Proposition 3.2 Let μ (resp., ν) be a fuzzy ideal in a BCK-algebra X (resp., Y).

If we define

$\mu \times \nu(x, y) := \min\{\mu(x), \nu(y)\}$

in $X \times Y$, then $\mu \times \nu$ is also a fuzzy ideal in $X \times Y$.

Proof. For any $(x, y) \in X \times Y$,

$\mu \times \nu(x, y) := \min\{\mu(x), \nu(y)\} \leq \min\{\mu(0), \nu(0)\} = \mu \times \nu(0, 0)$.

For any

$(x_1, x_2), (y_1, y_2) \in X \times Y$, $\mu \times \nu((x_1, x_2) * (y_1, y_2)) = \mu \times \nu\ (x_1 * y_1, x_2 * y_2) = \min\{\mu(x_1 * y_1), \nu(x_2 * y_2)\}$

and

$\mu \times \nu(y_1, y_2) := \min\{\mu(y_1), \nu(y_2)\}$.

Hence

$\min\{\mu \times \nu((x_1, x_2) * (y_1, y_2)), \mu \times \nu(y_1, y_2)\} = \min\{\min\{\mu(x_1 * y_1), \nu(x_2 * y_2)\}, \min\{\mu(y_1), \nu(y_2)\}\}$.

Since μ and ν are fuzzy ideals,

$\mu(x_1) \geq \min\{\mu(x_1 * y_1), \mu(y_1)\}$ and $\nu(x_2) \geq \min\{\nu(x_2 * y_2), \nu(y_2)\}$.

Hence

$\mu \times \nu(x_1, x_2) := \min\{\mu(x_1), \nu(x_2)\} \geq \min\{\mu \times \nu((x_1, x_2) * (y_1, y_2)), \mu \times \nu(y_1, y_2)\}$.

This is proves the proposition. As is well known, the characteristic function of a set is a special fuzzy set. Suppose I is a subset of a BCK-algebra X, denoted by μ^I the characteristic function of I, that is,

$$\mu^I(x) = \begin{cases} 1, & \text{if } x \in I \\ 0, & \text{if } x \notin I \end{cases}$$

Then μ^I is a fuzzy ideal in X if and only if I is an ideal of X.

Lemma 3.3 Suppose I is an ideal of X and assume $x, y \in X$. Then $x \sim^I y$ if and only if $x \sim^{\mu^I} y$.

Proof. In fact,

$x \sim^I y$ if and only if

$x * y \in I$ and $y * x \in I$

if and only if $\mu^I(x * y) = 1$ and $\mu^I(y * x) = 1$

if and only if

$\mu^I(x * y) > 0$ and $\mu^I(y * x) > 0$

if and only if $x \sim^{\mu^I} y$.

Thoerem 3.4 If μ (resp., ν) is a fuzzy ideal in a BCK-algebra X(resp., Y), then

$$\frac{X \times Y}{\mu \times \nu} \cong X/\mu \times Y/\nu.$$

Proof. If we define a map

$\Psi: X \times Y \to X/\mu \times Y/\nu$ by $\Psi(x, y) := (\mu_x, \nu_y)$,

then it is easy to show that Ψ is a BCK-epimorphism. By the fundamental homomorphism theorem, we obtain

$$\frac{X \times Y}{Ker \Psi} \cong X/\mu \times Y/\nu.$$

Let $K := Ker \Psi$.

Then

$$\frac{X \times Y}{Ker \Psi} = \frac{X \times Y}{K} = \{K_{(x,y)} \mid (x, y) \in X \times Y\}.$$

Similarly,

$$\frac{X \times Y}{\mu \times \nu} = \{\mu \times \nu_{(x,y)} \mid (x, y) \in X \times Y\}.$$

We claim that $K_{(x,y)} \cong \mu \times \nu_{(x,y)}$.

For $(\alpha, \beta) \in K_{(x,y)}$, $(\alpha, \beta) \in K_{(x,y)}$

$\Leftrightarrow (\alpha, \beta) * (x, y), (x, y) * (\alpha, \beta) \in K$

$\Leftrightarrow \Psi(\alpha * x, \beta * y) = (\mu_0, \nu_0) = \Psi(x * \alpha, y * \beta)$

where μ_0 and ν_0 : fixed element in X/μ and Y/ν, resp.

$\Leftrightarrow \mu_{\alpha * x} = \mu_{x * \alpha} = \mu_0$, $\nu_{\beta * y} = \nu_{y * \beta} = \nu_0$

$\Leftrightarrow \mu_\alpha = \mu_x$, $\nu_\beta = \nu_y$.

· · · · · · · · · · (1)

And, for $(\alpha, \beta) \in \mu \times \nu_{(x,y)}$, $(\alpha, \beta) \in \mu \times \nu_{(x,y)}$

$\Leftrightarrow (\alpha, \beta) \sim^{\mu \times \nu} (x, y)$

$\Leftrightarrow \mu \times \nu_{((\alpha,\beta)*(x,y))} > 0$ and $\mu \times \nu_{((x,y)*(\alpha,\beta))} > 0$

$\Leftrightarrow \mu \times \nu_{(\alpha * x, \beta * y)} > 0$ and $\mu \times \nu_{(x * \alpha, y * \beta)} > 0$

$\Leftrightarrow \min\{\mu_{(\alpha * x)}, \nu_{(\beta * y)}\} > 0$ and $\min\{\mu_{(x * \alpha)}, \nu_{(y * \beta)}\} > 0$

$\Leftrightarrow \mu_{(\alpha * x)} > 0$, $\nu_{(\beta * y)} > 0$ and $\Leftrightarrow \mu_{(x * \alpha)} > 0$, $\nu_{(y * \beta)} > 0$

$\Leftrightarrow \alpha \sim^\mu x$, $\beta \sim^\nu y$

$\Leftrightarrow \mu_\alpha = \mu_x$, $\nu_\beta = \nu_y$.

· · · · · · · · (2)

By (1) and (2), we obtain that

$(\alpha, \beta) \in K_{(x,y)}$

$\Leftrightarrow (\alpha, \beta) \in \mu \times \nu_{(x,y)}$.

Hence

$$\frac{X \times Y}{\mu \times \nu} \cong \frac{X \times Y}{Ker \Psi} \cong X/\mu \times Y/\nu,$$

proving the theorem.

Corollary 3.5 [9] Let μ (resp. ν) be a fuzzy ideal in a BCK-algebra X. (resp. Y)

1. If $\mu \times \nu$ be a fuzzy commutative ideal in $X \times Y$, then $X \times Y/\mu \times \nu$ is a commutative BCK-algebra.
2. If $\mu \times \nu$ be a fuzzy positive implicative ideal in $X \times Y$, then $X \times Y/\mu \times \nu$ is a positive implicative BCK-algebra.
3. If $\mu \times \nu$ be a fuzzy implicative ideal in $X \times Y$, then $X \times Y/\mu \times \nu$ is a implicative BCK-algebra.

Proposition 3.6 Let μ be a fuzzy ideal in a BCK-algebra X. If J is an ideal of X, then J/μ is an ideal of X/μ.

Proof. First, we claim that every element of J/μ is also an element of X/μ. To avoid the ambiguity, we denote the element of J/μ containing x by μ_x^J, i.e.,

$J/\mu = \{\mu_x^J \mid x \in J\}$.

Let $x \in J$. If $y \in \mu_x^J$ then

$x \sim^\mu y$ and hence $y \in \mu_x$.

If $y \in \mu_x$, then $x \sim^\mu y$ and

hence $y \in \mu_x^J$, since $x \in J$. Thus $\mu_x^J = \mu_x$.

Theorem 3.7 If J^* is an ideal of X/μ, then there exist an ideal

$J = \bigcup\{\mu_x \mid \mu_x \in J^*\}$ in X such that

$J/\mu = J^*$.

Proof. Since J^* is an ideal of

X/μ, $\mu_0 \in J^*$ and $0 \in J$.

Assume $x*y \in J$, $y \in J$.

Then $x*y \sim^\mu \alpha$, $y \sim^\mu \beta$ for some $\mu_\alpha, \mu_\beta \in J^*$.

This mean that

$\mu_x * \mu_y = \mu_{x*y} = \mu_\alpha$, $\mu_y = \mu_\beta$.

Since J^* is an ideal of X/μ, $\mu_x \in J^*$ and hence $x \in J$. Hence J is an ideal of X. Moreover,

$$
\begin{aligned}
J/\mu &= \{\mu_j \mid j \in J\} \\
&= \{\mu_j \mid \exists\, \mu_x \in J^* \textit{such that } j \in \mu_x\} \\
&= \{\mu_j \mid \exists\, \mu_x \in J^* \textit{such that } \mu_j = \mu_x\} \\
&= \{\mu_j \mid \mu_j \in J^*\} = J^*,
\end{aligned}
$$

proving the theorem.

Theorem 3.8 Let μ be a fuzzy ideal in a BCK-algebra X.

If J is an ideal in X, then $\frac{X/\mu}{J/\mu} \cong X/J$.

Proof. Note $\frac{X/\mu}{J/\mu} = \{(J/\mu)_{\mu_x} \mid \mu_x \in X/\mu\}$.

If we define

$\varphi: \frac{X/\mu}{J/\mu} \to X/J$ by $\varphi((J/\mu)_{\mu_x}) = J_x$,

then it is well-defined. In fact, suppose

$(J/\mu)_{\mu_x} = (J/\mu)_{\mu_y}$. Then $\mu_x \sim^{J/\mu} \mu_y$

and hence

$\mu_{x*y} = \mu_x * \mu_y \in J/\mu$, $\mu_{y*x} = \mu_y * \mu_x \in J/\mu$.

This means that

$x*y \in J$, and $y*x \in J$. i.e., $x \sim^J y$. Thus

$\varphi((J/\mu)_{\mu_x}) = J_x = J_y = \varphi((J/\mu)_{\mu_y})$. Now

$\varphi((J/\mu)_{\mu_x} * (J/\mu)_{\mu_y}) = J_{x*y} = J_x * J_y = \varphi((J/\mu)_{\mu_x}) * \varphi((J/\mu)_{\mu_y})$, for any

$(J/\mu)_{\mu_x}, (J/\mu)_{\mu_y} \in \frac{X/\mu}{J/\mu}$.

Hence φ is a homomorphism.

Obviously, φ is a onto. Finally, we show that φ is a one-one.

If $\varphi((J/\mu)_{\mu_x}) = \varphi((J/\mu)_{\mu_y})$,

then $J_x = J_y$

and hence $x \sim^J y$.

If $\mu_a \in (J/\mu)_{\mu_x}$ then $\mu_a \sim^{J/\mu} \mu_x$ and

hence $\mu_{a*x} \in J/\mu$, and $\mu_{x*a} \in J/\mu$.

It follows that $a*x \in J$, and $x*a \in J$, i.e., $a \sim^J x$.

Since $\sim^J$ is an equivalence relation, $a \sim^J y$. Hence

$\mu_a \in (J/\mu)_{\mu_y}$.

Thus $(J/\mu)_{\mu_x} \subseteq (J/\mu)_{\mu_y}$.

Similarly, we obtain

$(J/\mu)_{\mu_y} \subseteq (J/\mu)_{\mu_x}$.

Therefore $\frac{X/\mu}{J/\mu} \cong X/J$,

proving the theorem.

Theorem 3.9 Let μ (resp. ν) be a fuzzy ideal in a BCK- algebra X (resp. Y). If J(resp. I) is an ideal in X (resp. Y), then $\frac{X \times Y/\mu \times \nu}{J \times I/\mu \times \nu} \cong X/J \times Y/I$.

proof. Since J(resp. I) is an ideal in X (resp. Y), so $J \times I$ is an ideal in $X \times Y$. By proposition 3.2, since $\mu \times \nu$ is a fuzzy ideal in $X \times Y$, so is in $J \times I$.

Hence we obtain

$\frac{X \times Y}{\mu \times \nu} \cong X/\mu \times Y/\nu$ and $\frac{J \times I}{\mu \times \nu} \cong J/\mu \times I/\nu$,

by Theorem 3.4. And also,

$$\frac{X \times Y/\mu \times \nu}{J \times I/\mu \times \nu} \cong \frac{X/\mu \times Y/\nu}{J/\mu \times I/\nu} \cong \frac{X/\mu}{J/\mu} \times \frac{Y/\nu}{I/\nu}$$

$\cong X/J \times Y/I$

proving the theorem, by Thoerem3.8.

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