An Analysis of Eigenvalues and Eigenvectors for V-notched Cracks in Pseudo-isotropic Dissimilar Materials

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ABSTRACT

The problem of eigenvalues and eigenvectors is obtained from a v-notched crack in pseudo-isotropic dissimilar materials by the traction free boundary and the perfect bonded conditions at interface. The complex stress function of the two-term William's type is used. The eigenvalues are solved by a commercial numerical program, MATHEMATICA. Stress singularities for v-notched cracks in pseudo-isotropic dissimilar materials are discussed. The RWCIM(Reciprocal Work Contour Integral Method) is applied to the determination of eigenvector coefficients associated with eigenvalues with egenvalues. The RWCIM algorithm is also coded by the MATHEMATICA.

Keywords : Eigenvalue, Eigenvector, Stress Singularity, V-notched Crack, Pseudo-isotropic Material, RWCIM(Reciprocal Work Contour Integral Method)

1. Introduction

Establishment of the estimating methods for bonding strength in fracture mechanics is of much importance so that cracks occur in the interface mainly due to stress singularity in the interface of bonded structures of dissimilar materials. But the study on this field is still insufficient.

Stress singularity and stress intensity factor, which are known as possible parameters of fracture mechanics, are referred to as important researches. With this basic theoretical research, strength estimation of fracture mechanics will be able to be established through experiments.

Bogy[1,2] commenced the research on v-notched crack problems in isotropic dissimilar materials. Hein and Erdogan[3] studied on stress singularity using the Mellin transform method as a mathematically difficult method. Carpenter and Byers[4] researched on evaluation of stress intensity factors using RWCIM and complex stress function of the two-term William's type as a simple method.

Researches by Ting and Chou[5], Zwiers et al[6], and Ting[7] were carried out on stress singularity for v-notched cracks in anisotropic dissimilar materials using complex stress function of the one-term William's type.

In the case of using the stress function for anisotropic materials, roots of characteristic equation are used and the characteristic roots are classified into the three following forms. The first one is that the characteristic roots are composed of real parts and imaginary ones. Second, they are all imaginary parts. The third is that characteristic roots are all imaginary and the same simultaneously.

This study is performed on eigenvalues and eigenvectors for v-notched cracks in pseudo-isotropic dissimilar materials of which characteristic roots are all imaginary and the same. Assumed is the form of stress complex of the two-term William's type needed for pseudo-isotropic dissimilar materials and evaluated are stress fields and displacement fields for v-notched cracks in pseudo-isotropic dissimilar materials. Eigenvalues are evaluated from the problems using the commercial numerical program, Mathematica and eigenvectors are evaluated applying RWCIM.

2. Basic Equations

The equilibrium equation for the elasticity problem of isotropic and anisotropic materials is as follows

$$\sigma_{ij,j} + b_i = 0 \qquad (1)$$

where b_i is body force. This study analyzes two-dimensional linear elasticity problems with body force neglected.

In two-dimensional linear elasticity problems, Airy's stress function satisfying the equilibrium equation is expressed with stress complex function $U(x, y)$ as follows;

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2},\ \sigma_{yy} = \frac{\partial^2 U}{\partial x^2},\ \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y} \qquad (2)$$

And, constitution equation(generalized Hooke's law) in plane stress state for anisotropic materials is:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{21} & a_{22} & a_{26} \\ a_{61} & a_{62} & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} \qquad (3)$$

where

$$\begin{aligned} & a_{11} = 1/E_{xx},\ a_{22} = 1/E_{yy}, \\ & a_{12} = -\nu_{xy}/E_{xx} = -\nu_{yx}/E_{yy} \\ & a_{21} = a_{12},\ a_{66} = 1/G_{xy}, \\ & a_{16} = \eta_{xy,x}/E_{xx} = \eta_{x,xy}/G_{xy},\ a_{16} = a_{61}, \\ & a_{26} = \eta_{xy,y}/E_{yy} = \eta_{y,xy}/G_{xy},\ a_{26} = a_{62} \end{aligned} \qquad (4)$$

in which E_{xx} and E_{yy} are Young's moduli, G_{xy} is shear modulus, ν_{xy} is Poisson's ratio, and $\eta_{xy,x}$ and $\eta_{xy,y}$ are called the coefficients of mutual influence of the first kind, and $\eta_{x,xy}$ and $\eta_{y,xy}$ are the coefficients of mutual influence of the second kind. For plane strain, a^*_{ij} is given as;

$$a^*_{ij} = a_{ij} - (a_{i3} a_{j3} / a_{33}) \qquad (5)$$

If the principal axes 1 and 2 of a material coincide with x and y axes as shown in Fig. 1, the material is orthotropic one which belongs to a special case of anisotropic materials and the constitution equation of equation (3) is;

$$\begin{aligned} & a_{11} = 1/E_{11},\ a_{22} = 1/E_{22}, \\ & a_{12} = -\nu_{12}/E_{11} = -\nu_{21}/E_{22} \\ & a_{66} = 1/G_{12},\ a_{16} = a_{61} = a_{26} = a_{62} = 0. \end{aligned} \qquad (6)$$

In addition, the compatibility equation which expresses continuity of displacement is;

$$\frac{\partial^2 \varepsilon_{yy}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad (7)$$

For the two-dimensional linear elastic problems of anisotropic materials, the following differential equation is obtained by substituting equation (3) of consitutional equation and equation (2) into equation (7);

$$a_{22}\frac{\partial^4 U}{\partial x^4} - 2a_{26}\frac{\partial^4 U}{\partial x^3 \partial y} + (2a_{12} + a_{66})\frac{\partial^4 U}{\partial x^2 \partial y^2} - 2a_{16}\frac{\partial^4 U}{\partial x \partial y^3} + a_{11}\frac{\partial^4 U}{\partial y^4} = 0 \qquad (8)$$

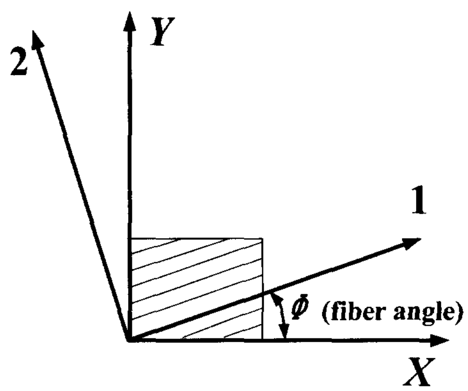


Fig. 1 Rotation of the coordinate system

3. Stress and displacement fields for v-notched cracks in pseudo-isotropic dissimilar materials.

In two-dimensional linear elastic problems, the stress complex function for anisotropic materials is assumed by two-term William's type stress function as;

$$U(z) = Az^{\lambda+1} + B\bar{z}^{\bar{\lambda}+1} \tag{9}$$

where $z = x + \mu y$ and μ, A and B are all complex. If the assumed solution of equation (9) is substituted into differential equation (8), the characteristic equation is obtained as follows;

$$a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0 \tag{10}$$

Root μ for characteristic equation (10) is a complex which has a real part and imaginary one and can be expressed as follows;

$$\begin{aligned} \mu_1 &= \alpha_1 + i\beta_1, & \mu_2 &= \alpha_2 + i\beta_2, \\ \mu_3 &= \overline{\mu_1}, & \mu_4 &= \overline{\mu_2} \end{aligned} \tag{11}$$

where α_j and β_j are real. Since $a_{16} = a_{26} = 0$ in the case of orthotropic materials, characteristic equation (1) becomes;

$$a_{11}\mu^4 + (2a_{12} + a_{66})\mu^2 + a_{22} = 0 \tag{12}$$

Characteristic equation (12) for orthotropic materials has only purely imaginary roots, so that α_1 and α_2 should be equal to zero in equation (11). A pseudo-isotropic materials which is a special case of anisotropic materials has roots of characteristic equation (12) as double roots with $\mu_1 = \mu_2$. Finally, they become purely imaginary roots. As the condition that roots of equation (12) become double roots, shear elasticity modulus should establish the following relation;

$$G_{12} = E_{11}/\{2 \times (\nu_{12} + \sqrt{E_{11}/E_{22}})\} \tag{13}$$

The case of $i\beta_1 = i\beta_2$ is a pseudo-isotropic material and moreover, if $\beta_1 = \beta_2 = 1$, it is an isotropic material. Since singularity causes analysis impossible in the case of using the traditional stress complex function for isotropic and pseudo-isotropic materials in analysis for those materials which coincide to special cases of pseudo-isotropic ones, the form of stress complex function for pseudo-isotropic materials should be expressed as;

$$U(z) = 2Re\{F_1(z_1) + \overline{z_1}F_2(z_1)\} \tag{14}$$

where

$$F_1(z_1) = A_1 z_1^{\lambda+1} + B_1 \bar{z}_1^{\bar{\lambda}+1} \tag{15}$$

$$F_2(z_1) = A_2 z_1^{\lambda+1} + B_2 \bar{z}_1^{\bar{\lambda}+1} \tag{16}$$

Substituting equation stress complex function of equation (14) into equation(2), the stresses for cartesian coordinates can be obtained as follows;

$$\begin{aligned} \sigma_{jxx} = 2Re\{ &\mu_{j1}{}^2 F_{j1}''(z_{j1}) - 2\,\mu_{j1}{}^2 F_{j2}'(z_{j1}) \\ &+ \mu_{j1}{}^2 \overline{z_{j1}} F_{j2}''(z_{j1})\} \end{aligned} \tag{17}$$

$$\sigma_{jyy} = 2Re\{F_{j1}''(z_{j1}) + 2F_{j2}'(z_{j1}) + \overline{z_{j1}}F_{j2}''(z_{j1})\} \tag{18}$$

$$\sigma_{jxy} = -2Re\{\mu_{j1}F_{j1}''(z_{j1}) + \mu_{j1}\overline{z_{j1}}F_{j2}''(z_{j1})\} \tag{19}$$

where j denotes material j and the upper symbols ' and " mean first order and second order differentiations, respectively. And, using equations (17), (18), and (19), strains can be obtained from the constitutional equation and by integrate the strains, displacement field can be obtained as follows;

$$\begin{aligned} u_{jx} = 2Re[&p_{j1}\{F_{j1}'(z_{j1}) + \overline{z_{j1}}F_{j2}'(z_{j1})\} \\ &+ p_{j2}F_{j2}(z_{j1})] - w_{j3}y + u_{j0x} \end{aligned} \tag{20}$$

$$\begin{aligned} u_{jy} = 2Re[&q_{j1}\{F_{j1}'(z_{j1}) + \overline{z_{j1}}F_{j2}'(z_{j1})\} \\ &+ q_{j2}F_{j2}(z_{j1})] + w_{j3}x + u_{j0y} \end{aligned} \tag{21}$$

where

$$p_{j1} = a_{j11}\mu_{j1}^2 + a_{j12} \tag{22}$$

$$p_{j2} = -3a_{j11}\mu_{j1}^2 + a_{j12} \tag{23}$$

$$q_{j1} = a_{j21}\mu_{j1} + \frac{a_{j22}}{\mu_{j1}} \tag{24}$$

$$q_{j2} = -a_{j21}\mu_{j1} + 3\frac{a_{j22}}{\mu_{j1}} \tag{25}$$

and w_{j3}, u_{jox} and u_{joy} are all constants and mean the terms expressing rigid displacement. Afterward, the rigid displacement is regarded as neglected.

For v-notched crack problems in pseudo-isotropic dissimilar materials, it is more convenient to express displacements and stresses in cylindrical coordinates than in Cartesian ones. For the reason, the following stress transformation is used;

$$u_i' = a_{ij} u_j \tag{26}$$

$$\sigma_{ij}' = a_{il} a_{jm} \sigma_{lm} \tag{27}$$

where

$$a_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{28}$$

4. Eigenvalues and eigenvectors for v-notched cracks in pseudo-isotropic dissimilar materials

4.1 Characteristic equation

From boundary conditions and continuity for v-notched cracks in pseudo-isotropic materials as shown in Fig. 2, the homogeneous simultaneous equation set is obtained as;

$$[D(\lambda)]\{A\} = \{0\} \tag{29}$$

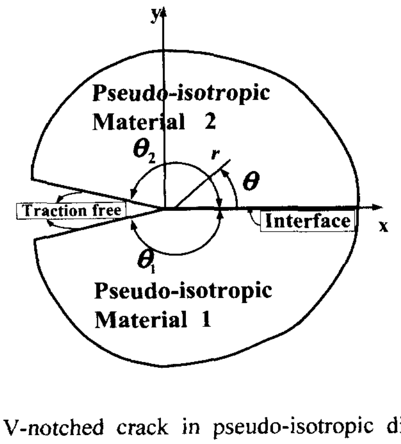


Fig. 2 V-notched crack in pseudo-isotropic dissimilar materials

where

$$\{A\} = [A_{11}\ \overline{B_{11}}\ A_{12}\ \overline{B_{12}}\ A_{21}\ \overline{B_{21}}\ A_{22}\ \overline{B_{22}}]^T \tag{30}$$

$$\{0\} = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]^T \tag{31}$$

$$[D(\lambda)] = \begin{bmatrix} s_{11} & 0 \\ s_{12} & s_{21} \\ u_{12} & u_{21} \\ 0 & s_{22} \end{bmatrix} \tag{32}$$

where

$$[s_{11}] = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \end{bmatrix}$$

$$d_{11} = (\lambda+1)\{\cos[-\theta_1] + \mu_{11}\sin[-\theta_1]\}^{\lambda+1}$$

$$d_{12} = (\lambda+1)\{\cos[-\theta_1] + \overline{\mu_{11}}\sin[-\theta_1]\}^{\lambda+1}$$

$$\begin{aligned} d_{13} &= \{2(\cos[-\theta_1] - \mu_{11}\sin[-\theta_1]) \\ &\quad + (\lambda-1)(\cos[-\theta_1] + \overline{\mu_{11}}\sin[-\theta_1])\} \\ &\quad \times \{\cos[-\theta_1] + \mu_{11}\sin[-\theta_1]\}^{\lambda} \end{aligned}$$

$$\begin{aligned} d_{14} &= \{2(\cos[-\theta_1] - \overline{\mu_{11}}\sin[-\theta_1]) \\ &\quad + (\lambda-1)(\cos[-\theta_1] + \mu_{11}\sin[-\theta_1])\} \\ &\quad \times \{\cos[-\theta_1] + \overline{\mu_{11}}\sin[-\theta_1]\}^{\lambda} \end{aligned}$$

$$\begin{aligned} d_{21} &= (\lambda+1)\{\sin[-\theta_1] - \mu_{11}\cos[-\theta_1]\} \\ &\quad \times \{\cos[-\theta_1] + \mu_{11}\sin[-\theta_1]\}^{\lambda} \end{aligned}$$

$$\begin{aligned} d_{22} &= (\lambda+1)\{\sin[-\theta_1] - \overline{\mu_{11}}\cos[-\theta_1]\} \\ &\quad \times \{\cos[-\theta_1] + \overline{\mu_{11}}\sin[-\theta_1]\}^{\lambda} \end{aligned}$$

$$\begin{aligned} d_{23} &= \{2(1+\mu_{11}^2)\cos[-\theta_1]\sin[-\theta_1] + (\lambda-1) \\ &\quad \times (\sin[-\theta_1] - \mu_{11}\cos[-\theta_1])(\cos[-\theta_1] + \overline{\mu_{11}} \\ &\quad \times \sin[-\theta_1])\} \times \{\cos[-\theta_1] + \mu_{11}\sin[-\theta_1]\}^{\lambda-1} \end{aligned}$$

$$\begin{aligned} d_{24} &= \{2(1+\overline{\mu_{11}}^2)\cos[-\theta_1]\sin[-\theta_1] + (\lambda-1) \\ &\quad \times (\sin[-\theta_1] - \overline{\mu_{11}}\cos[-\theta_1])(\cos[-\theta_1] + \mu_{11} \\ &\quad \times \sin[-\theta_1])\} \times \{\cos[-\theta_1] + \overline{\mu_{11}}\sin[-\theta_1]\}^{\lambda-1} \end{aligned}$$

$$\begin{aligned} [s_{12}] &= \begin{bmatrix} d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ (\lambda+1)\mu_{11} & (\lambda+1)\overline{\mu_{11}} & (\lambda-1)\mu_{11} & (\lambda-1)\overline{\mu_{11}} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} [s_{21}] &= \begin{bmatrix} d_{35} & d_{36} & d_{37} & d_{38} \\ d_{45} & d_{46} & d_{47} & d_{48} \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 & & \\ -(\lambda+1)\mu_{21} & -(\lambda+1)\overline{\mu_{21}} & & \\ & -1 & & -1 \\ & -(\lambda-1)\mu_{21} & & -(\lambda-1)\overline{\mu_{21}} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} [u_{21}] &= \begin{bmatrix} d_{51} & d_{52} & d_{53} & d_{54} \\ d_{61} & d_{62} & d_{63} & d_{64} \end{bmatrix} \\ &= \begin{bmatrix} (\lambda+1)p_{11} & (\lambda+1)\overline{p_{11}} & (\lambda p_{11} + p_{12}) & (\lambda\overline{p_{11}} + \overline{p_{12}}) \\ (\lambda+1)q_{11} & (\lambda+1)\overline{q_{11}} & (\lambda q_{11} + q_{12}) & (\lambda\overline{q_{11}} + \overline{q_{12}}) \end{bmatrix} \end{aligned}$$

$$[u_{22}] = \begin{bmatrix} d_{55} & d_{56} & d_{57} & d_{58} \\ d_{65} & d_{66} & d_{67} & d_{68} \end{bmatrix}$$

$$= \begin{bmatrix} -(\lambda+1)p_{21} & -(\lambda+1)\overline{p_{21}} & & \\ -(\lambda+1)q_{21} & -(\lambda+1)\overline{q_{21}} & & \\ & & -(\lambda p_{21}+p_{22}) & -(\lambda \overline{p_{21}}+\overline{p_{22}}) \\ & & -(\lambda q_{21}+q_{22}) & -(\lambda \overline{q_{21}}+\overline{q_{22}}) \end{bmatrix}$$

$$[s_{22}] = \begin{bmatrix} d_{75} & d_{76} & d_{77} & d_{78} \\ d_{85} & d_{86} & d_{87} & d_{88} \end{bmatrix}$$

$$d_{75} = (\lambda+1)\{\cos[\theta_2] + \mu_{21}\sin[\theta_2]\}^{\lambda+1}$$

$$d_{76} = (\lambda+1)\{\cos[\theta_2] + \overline{\mu_{21}}\sin[\theta_2]\}^{\lambda+1}$$

$$d_{77} = \{2(\cos[\theta_2] - \mu_{21}\sin[\theta_2]) + (\lambda-1)(\cos[\theta_2] + \overline{\mu_{21}}\sin[\theta_2])\} \times \{\cos[\theta_2] + \mu_{21}\sin[\theta_2]\}^{\lambda}$$

$$d_{78} = \{2(\cos[\theta_2] - \overline{\mu_{21}}\sin[\theta_2]) + (\lambda-1)(\cos[\theta_2] + \mu_{21}\sin[\theta_2])\} \times \{\cos[\theta_2] + \overline{\mu_{21}}\sin[\theta_2]\}^{\lambda}$$

$$d_{85} = (\lambda+1)\{\sin[\theta_2] - \mu_{21}\cos[\theta_2]\} \times \{\cos[\theta_2] + \mu_{21}\sin[\theta_2]\}^{\lambda}$$

$$d_{86} = (\lambda+1)\{\sin[\theta_2] - \overline{\mu_{21}}\cos[\theta_2]\} \times \{\cos[\theta_2] + \overline{\mu_{21}}\sin[\theta_2]\}^{\lambda}$$

$$d_{87} = \{2(1+\mu_{21}^2)\cos[\theta_2]\sin[\theta_2] + (\lambda-1)(\sin[\theta_2] - \mu_{21}\cos[\theta_2])(\cos[\theta_2] + \overline{\mu_{21}}\sin[\theta_2])\} \times \{\cos[\theta_2] + \mu_{21}\sin[\theta_2]\}^{\lambda-1}$$

$$d_{88} = \{2(1+\overline{\mu_{21}}^2)\cos[\theta_2]\sin[\theta_2] + (\lambda-1)(\sin[\theta_2] - \overline{\mu_{21}}\cos[\theta_2])(\cos[\theta_2] + \mu_{21}\sin[\theta_2])\} \times \{\cos[\theta_2] + \overline{\mu_{21}}\sin[\theta_2]\}^{\lambda-1}$$

The condition that homogeneous simultaneous equation (29) has non-trivial solution is that the determinant of the matrix, $|D(\lambda)|$, in equation (32) should be zero and this matrix is called a characteristic polynomial and expressed as follows;

$$|D(\lambda)| = 0 \tag{33}$$

The solutions to the polynomial become eigenvalues for a v-notched crack in pseudo-isotropic dissimilar materials and the non-trivial solution to each eigenvalue becomes each eigenvector with respect to the eigenvalue.

Besides, eigenvectors for since roots to characteristic polynomial (33) are composed of complex eigenvalues and real ones, eigenvectors for complex and real eigenvalues should be obtained. In this paper, the eigenvectors for these eigenvalues are evaluated from RWCIM which uses Betti's reciprocal theorem and finite element analysis.

4.2 eigenvectors with respect to complex eigenvalues

Explained is the method of evaluating an eigenvector in the case that eigenvalue λ as the root of characteristic polynomial (33) is complex. Since eigenvector, stress and displacement involved in complex eigenvalue are expressed as terms of complex coefficients, equation (32) should be defined as the following in order to express these coefficients with complex coefficient A_{11} of equation (30) which has a real part and imaginary one.

$$\begin{bmatrix} [D_{11}] & [D_{12}] \\ [D_{21}] & [D_{22}] \end{bmatrix} \begin{Bmatrix} A_{11} \\ \overline{B_{11}} \\ A_{12} \\ \overline{B_{12}} \\ A_{21} \\ \overline{B_{21}} \\ A_{22} \\ \overline{B_{22}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{34}$$

where

$$[D_{11}] = d_{11}$$

$$[D_{12}] = [d_{12}\ d_{13}\ d_{14}\ 0\ 0\ 0\ 0],$$

$$[D_{21}] = [d_{21}\ d_{31}\ d_{41}\ d_{51}\ d_{61}\ 0\ 0]^T,$$

$$[D_{22}] = \begin{bmatrix} d_{22} & d_{23} & d_{24} & 0 & 0 & 0 & 0 \\ d_{32} & d_{33} & d_{34} & d_{35} & d_{36} & d_{37} & d_{38} \\ d_{42} & d_{43} & d_{44} & d_{45} & d_{46} & d_{47} & d_{48} \\ d_{52} & d_{53} & d_{54} & d_{55} & d_{56} & d_{57} & d_{58} \\ d_{62} & d_{63} & d_{64} & d_{65} & d_{66} & d_{67} & d_{68} \\ 0 & 0 & 0 & d_{75} & d_{76} & d_{77} & d_{78} \\ 0 & 0 & 0 & d_{85} & d_{86} & d_{87} & d_{88} \end{bmatrix}$$

The following is obtained from Equation (34);

$$\begin{Bmatrix} \overline{B_{11}} \\ A_{12} \\ \overline{B_{12}} \\ A_{21} \\ \overline{B_{21}} \\ A_{22} \\ \overline{B_{22}} \end{Bmatrix} = -[D_{22}]^{-1}[D_{21}]A_{11} = \begin{Bmatrix} Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \\ Y(6) \\ Y(7) \end{Bmatrix} A_{11} \tag{35}$$

In equation (35), eigenvectors for materials 1 and 2 with respect to complex eigenvalues are defined as follows;

$$\begin{Bmatrix} A_{11} \\ \overline{B_{11}} \\ A_{12} \\ \overline{B_{12}} \end{Bmatrix} = \begin{bmatrix} 1 \\ Y(1) \\ Y(2) \\ Y(3) \end{bmatrix} \{A_{11}\} = [E_1]\{A_{11}\} \tag{36}$$

$$\begin{Bmatrix} A_{21} \\ \overline{B_{21}} \\ A_{22} \\ \overline{B_{22}} \end{Bmatrix} = \begin{bmatrix} Y(4) \\ Y(5) \\ Y(6) \\ Y(7) \end{bmatrix} \{A_{11}\} = [E_2]\{A_{11}\} \tag{37}$$

where

$$\{A_{11}\} = [\,1 \quad i\,]\begin{Bmatrix} Re[A_{11}] \\ Im[A_{11}] \end{Bmatrix} = [\,F\,]\begin{Bmatrix} Re[A_{11}] \\ Im[A_{11}] \end{Bmatrix} \tag{38}$$

From equations (36), (37) and (38), stresses and displacements for complex eigenvalue λ are expressed as follows;

$$\begin{aligned} \begin{Bmatrix} u_{jr} \\ u_{j\theta} \\ \sigma_{jrr} \\ \sigma_{jr\theta} \end{Bmatrix} &= 2Re\left[[G_j] \begin{Bmatrix} A_{j1} \\ \overline{B_{j1}} \\ A_{j2} \\ \overline{B_{j2}} \end{Bmatrix} \right] \\ &= 2Re\left[[G_j][E_j][F] \begin{Bmatrix} Re[A_{11}] \\ Im[A_{11}] \end{Bmatrix} \right] \\ &= 2Re\left[[J_j] \begin{Bmatrix} Re[A_{11}] \\ Im[A_{11}] \end{Bmatrix} \right] \\ &\equiv [M_j] \begin{Bmatrix} Re[A_{11}] \\ Im[A_{11}] \end{Bmatrix} \end{aligned} \tag{39}$$

where

$$[G_j] = \begin{bmatrix} G_{j11} & G_{j12} & G_{j13} & G_{j14} \\ G_{j21} & G_{j22} & G_{j23} & G_{j24} \\ G_{j31} & G_{j32} & G_{j33} & G_{j34} \\ G_{j41} & G_{j42} & G_{j43} & G_{j44} \end{bmatrix} \tag{40}$$

where

$$G_{j11} = r^{\lambda}(\lambda+1)(p_{j1}C+q_{j1}S)(C+\mu_{j1}S)^{\lambda}$$

$$G_{j12} = r^{\lambda}(\lambda+1)(\overline{p_{j1}}C+\overline{q_{j1}}S)(C+\overline{\mu_{j1}}S)^{\lambda}$$

$$\begin{aligned} G_{j13} &= r^{\lambda}\{\lambda(p_{j1}C+q_{j1}S)(C+\overline{\mu_{j1}}S) \\ &\quad + (p_{j2}C+q_{j2}S)(C+\mu_{j1}S)\}(C+\mu_{j1}S)^{\lambda-1} \end{aligned}$$

$$\begin{aligned} G_{j14} &= r^{\lambda}\{\lambda(\overline{p_{j1}}C+\overline{q_{j1}}S)(C+\mu_{j1}S) \\ &\quad + (\overline{p_{j2}}C+\overline{q_{j2}}S)(C+\overline{\mu_{j1}}S)\}(C+\overline{\mu_{j1}}S)^{\lambda-1} \end{aligned}$$

$$G_{j21} = r^{\lambda}(\lambda+1)(q_{j1}C-p_{j1}S)(C+\mu_{j1}S)^{\lambda}$$

$$G_{j22} = r^{\lambda}(\lambda+1)(\overline{q_{j1}}C-\overline{p_{j1}}S)(C+\overline{\mu_{j1}}S)^{\lambda}$$

$$\begin{aligned} G_{j23} &= r^{\lambda}\{\lambda(q_{j1}C-p_{j1}S)(C+\overline{\mu_{j1}}S) \\ &\quad + (q_{j2}C-p_{j2}S)(C+\mu_{j1}S)\}(C+\mu_{j1}S)^{\lambda-1} \end{aligned}$$

$$\begin{aligned} G_{j24} &= r^{\lambda}\{\lambda(\overline{q_{j1}}C-\overline{p_{j1}}S)(C+\mu_{j1}S) \\ &\quad + (\overline{q_{j2}}C-\overline{p_{j2}}S)(C+\overline{\mu_{j1}}S)\}(C+\overline{\mu_{j1}}S)^{\lambda-1} \end{aligned}$$

$$G_{j31} = r^{\lambda-1}(\lambda+1)\lambda(S-\mu_{j1}C)^2(C+\mu_{j1}S)^{\lambda-1}$$

$$G_{j32} = r^{\lambda-1}(\lambda+1)\lambda(S-\overline{\mu_{j1}}C)^2(C+\overline{\mu_{j1}}S)^{\lambda-1}$$

$$\begin{aligned} G_{j33} &= r^{\lambda-1}\lambda\{2(S^2-\mu_{j1}^2C^2)+(\lambda-1)(C+\overline{\mu_{j1}}S) \\ &\quad \times(S-\mu_{j1}C)^2(C+\mu_{j1}S)^{-1}\}(C+\mu_{j1}S)^{\lambda-1} \end{aligned}$$

$$\begin{aligned} G_{j34} &= r^{\lambda-1}\lambda\{2(S^2-\overline{\mu_{j1}^2}C^2)+(\lambda-1)(C+\mu_{j1}S) \\ &\quad \times(S-\overline{\mu_{j1}}C)^2(C+\overline{\mu_{j1}}S)^{-1}\}(C+\overline{\mu_{j1}}S)^{\lambda-1} \end{aligned}$$

$$G_{j41} = r^{\lambda-1}(\lambda+1)\lambda(S-\mu_{j1}C)(C+\mu_{j1}S)^{\lambda}$$

$$G_{j42} = r^{\lambda-1}(\lambda+1)\lambda(S-\overline{\mu_{j1}}C)(C+\overline{\mu_{j1}}S)^{\lambda}$$

$$\begin{aligned} G_{j43} &= r^{\lambda-1}\lambda\{2CS(1+\mu_{j1}^2) \\ &\quad + (\lambda-1)(C+\overline{\mu_{j1}}S)(S-\mu_{j1}C)\}(C+\mu_{j1}S)^{\lambda-1} \end{aligned}$$

$$\begin{aligned} G_{j44} &= r^{\lambda-1}\lambda\{2CS(1+\overline{\mu_{j1}^2}) \\ &\quad + (\lambda-1)(C+\mu_{j1}S)(S-\overline{\mu_{j1}}C)\}(C+\overline{\mu_{j1}}S)^{\lambda-1} \end{aligned}$$

in If $\theta_1 \le \theta \le 0$ in equations (39) and (40), j=1 and if $0 \le \theta \le \theta_2$, j=2. And $C=\cos[\theta]$and $S=\sin[\theta]$

As seen in equations (39) and (40), it can be known that eigenvalues, stresses and displacements with respect to each of eigenvalues are expressed only by a real term and imaginary one of complex coefficient A_{11}.

4.3 Eigenvectors with respect to real eigenvalues.

In the case that eigenvalue λ as the root of characteristic polynomial (29) is real, equation (29) becomes;

$$[D(\lambda)]\{Ar\} = \{0\} \tag{41}$$

where

$$\{Ar\} = [Ar_{11}\ \overline{Ar_{11}}\ Ar_{12}\ \overline{Ar_{12}}\ Ar_{21}\ \overline{Ar_{21}}\ Ar_{22}\ \overline{Ar_{22}}]^T \tag{42}$$

$[D(\lambda)]$ is equal to equation (39), $\{0\}$ is the same with equation (31) and equation (41) can be expressed as follows;

$$\begin{aligned} [D(\lambda)]\{Ar\} &= [D(\lambda)][UI]\{ar\} \\ &= [Dr(\lambda)]\{ar\} \\ &= \{0\} \end{aligned} \tag{43}$$

where

$$\{Ar\} = [UI]\{ar\} \quad (44)$$

$$\{ar\} = [Re[Ar_{11}]\ Im[Ar_{11}]\ Re[Ar_{12}]\ Im[Ar_{12}]\ Re[Ar_{21}]\ Im[Ar_{21}]\ Re[Ar_{22}]\ Im[Ar_{22}]\ ]^T \quad (45)$$

$$[UI] = \begin{bmatrix} 1 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i \end{bmatrix} \quad (46)$$

In the same method with the case of complex eigenvalues, stresses and displacements with respect to real eigenvalue λ can be obtained as;

$$\begin{Bmatrix} u_{jr} \\ u_{j\theta} \\ \sigma_{jrr} \\ \sigma_{jr\theta} \end{Bmatrix} = 2Re[\ [Gr_j][Fr][Er_j]\ ]Re[Ar_{11}] = 2Re[\ [Jr_j]\ ]Re[Ar_{11}] = [Mr_j]Re[Ar_{11}] \quad (47)$$

where

$$[Gr_j] = \begin{bmatrix} G_{j11} & G_{j13} \\ G_{j21} & G_{j23} \\ G_{j31} & G_{j33} \\ G_{j41} & G_{j43} \end{bmatrix} \quad (48)$$

$$[Er_j] = \begin{bmatrix} Er_{j1} \\ Er_{j2} \\ Er_{j3} \\ Er_{j4} \end{bmatrix} \quad (49)$$

$$[Fr] = \begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \end{bmatrix} \quad (50)$$

As seen in equation (47), it can be seen that eigenvectors, stresses and displacements with respect to real eigenvalues are expressed only by the term of $Re[Ar_{11}]$ which is the real part of complex coefficient Ar_{11}.

5. RWCIM

In the case that linear elastic body of thickness, t, in equilibrium is subject to two different load sets as in Fig. 3, Betti's reciprocal theorem is as follows,

$$\oint_c \sum_{i=1}^{2} (\sigma_i \hat{u}_i - \hat{\sigma}_i u_i) ds = \oint_c T ds = 0 \quad (51)$$

where 1 and 2 express components of the normal and tangential directions to the closed loops of system I and II.

If equation (51) which uses Betti's reciprocal theorem is applied to the problems of v-notch in Fig. 4, the closed loop, C, is;

$$C = C_i + C' + C_o + C'' \quad (52)$$

In addition, since C' and C'' have no traction in the loop, C, of equation (52) which includes v-notch, equation (51) becomes;

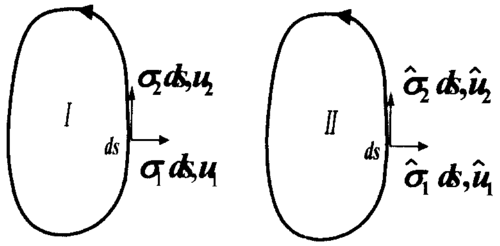


Fig. 3 Closed Contour

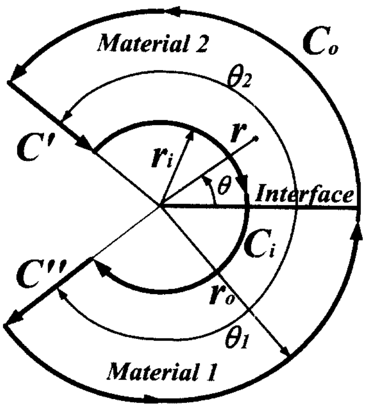


Fig. 4 V-notched configuration to which Betti's law is applied

$$\int_{C_i} Tds = -\int_{C_o} Tds \tag{53}$$

Stresses and displacements following the path of C_0 in system I are obtained from Finite Element Analysis and those following C_i are from the following equation.

$$\begin{Bmatrix} u_{jr} \\ u_{j\theta} \\ \sigma_{jr} \\ \sigma_{j\theta} \end{Bmatrix} = \left[\sum_{i=0}^{n} [M_j(\lambda_i, r, \theta)] \begin{Bmatrix} Re[A_{11}(\lambda_i)] \\ Im[A_{11}(\lambda_i)] \end{Bmatrix} + \sum_{i=n+1}^{m} [Mr_j(\lambda_i, r, \theta)]\{Re[Ar_{11}(\lambda_i)]\} \right] \equiv [K_j(\lambda_i)][A(\lambda_i)] \tag{54}$$

where n is the number of complex eigenvalue, m is the total number of eigenvalue and then (m-n) is the number of real eigenvalue. And, stresses and displacements following C_i and C_0 in system II are evaluated from eigenvalues and eigenvectors with a negative real part using equation (54).

The line integral following path C_i in Fig. 3 is;

$$\int_{C_i} Tds = -\int_{\theta=-\theta_1}^{\theta=\theta_2} Tr_i d\theta = -\{A(+\lambda)\}^T [CI] \{A(-\lambda)\} \tag{55}$$

where

$$[CI] = r_i \int_{\theta=-\theta_1}^{\theta=\theta_2} [K_j(+\lambda)]^T [II] [K_j(-\lambda)] d\theta \tag{56}$$

where

$$[II] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{57}$$

And the line integral following C_0 is;

$$\int_{C_o} Tds \equiv \int_{\theta=-\theta_1}^{\theta=\theta_2} Tr_o d\theta = [CO]\{A(-\lambda)\} \tag{58}$$

where

$$[CO] = r_o \int_{\theta=-\theta_1}^{\theta=\theta_2} [X]^T [II] [K_j(-\lambda)] d\theta \tag{59}$$

where as stresses and displacements $[X]$ is as follows;

$$[X] = \begin{Bmatrix} u_r \\ u_\theta \\ \sigma_{rr} \\ \sigma_{r\theta} \end{Bmatrix}_{FEA} \tag{60}$$

Substituting equation (55) of the value of the line integral following path C_i and equation (58) of that following C_0 into equation (53), the following equation is obtained;

$$\{A(\lambda)\}^T = [CO][CI]^{-1} \tag{61}$$

6. Numerical results and discussion.

6.1 Analysis of eigenvalues for pseudo-isotropic single materials.

For a pseudo-isotropic material which is a special case of orthogonal anisotropic materials, the root of characteristic equation (12) should have a pure imaginary root as a double one. When the modulus of shear elasticity satisfies equation (13), the root of characteristic equation (12) becomes a double root as a pure imaginary one.

As explained above, eigenvalues of v-notched crack between dissimilar materials into which two different pseudo-isotropic materials are bonded together can be obtained from the numerical analysis of characteristic polynomial expression (33) using the commercial numerical analysis program, Mathematica.

Fig. 6 shows the change of pure imaginary roots of characteristic equation (12) with respect to the ratio, $m=(E_{22}/E_{11})$ of the elasticity moduli of pseudo-isotropic single materials. It can result from Fig. 6 that the materials become isotropic materials in the case that the root of characteristic equation (12) for pseudo-isotropic materials is i as a double one.

Since isotropic materials are the special pseudo-isotropic ones, material properties satisfying isotropic materials are chosen and then, eigenvalues with stress singularities of plane stress state about v-notched crack in a single isotropic material are evaluated with respect to the change of the wedge angle. It is seen that the eigenvalues coincide with those of Dunn et al[9] and the result is shown in Fig. 5.

6.2 Eigenvalue analysis for v-notched cracks in pseudo-isotropic dissimilar materials.

For the numerical analysis of eigenvalues possessing stress singularity for the problems of v-notched cracks in pseudo-isotropic dissimilar materials, materials are selected as follows.

$$\begin{array}{lll} E_{111}=3.0GPa, & E_{122}=2.0GPa, & \nu_{112}=0.3 \\ E_{211}=2.0GPa, & E_{222}=1.0GPa, & \nu_{212}=0.29 \end{array} \quad (62)$$

Fig. 6 shows eigenvalues ($0.5 \le Re[\lambda] < 1$) having stress singularity with respect to the wedge angle from $\theta_1=\theta_2=90^\circ$ to 180°.

Eigenvalue λ has one real eigenvalue when $\theta_1=\theta_2$ exists between 90° and 131.795° , two real when $\theta_1=\theta_2$ varies from 131.795° to 168.91° and one complex when $\theta_1=\theta_2$ varies between 168.91° and 180°. These trends are shown to be similar to those of stress singularity for v-notched cracks in isotropic dissimilar and pseudo-isotropic dissimilar materials [10,11].

6.3 Analysis of eigenvectors

In the problem of v-notched cracks in pseudo-isotropic materials, the eigenvectors with respect to eigenvalues are evaluated by the numerical analysis using RWCIM as explained in chapter 7 and the validity of the numerical method and the program is investigated in the following manners.

First, an arbitrary eigenvector is determined and substituted into equation (54), and then stresses and displacements are evaluated. These are used for resultant values from finite element analysis of equation (60) and inversely, whether the same eigenvector is evaluated is examined. For an example, eigenvalue λ is 0.535686+i0.0575848 at 170° of the wedge angle in the dissimilar materials with two bonded materials where one pseudo-isotropic material has E_{111}=6.0GPa, E_{122}=2.5GPa, and ν_{112}=0.3 in area 1 and the other has E_{211}

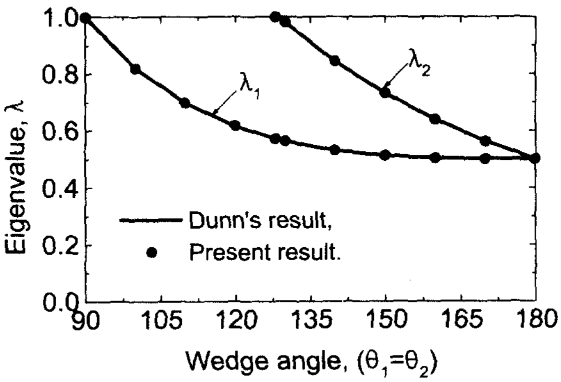


Fig. 5 Eigenvalue vs. various notch angles for a cracks in a isotropic material v-notched crack in an pseudo-isotropic material

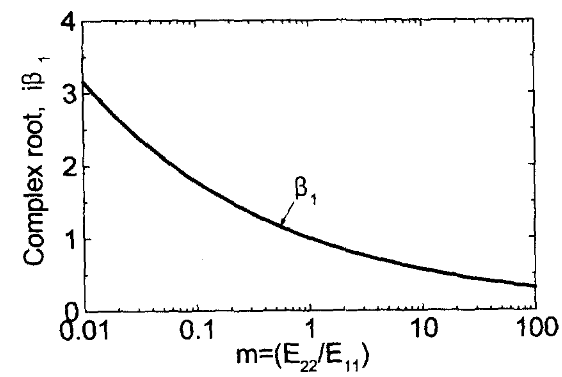


Fig. 6 Complex root for ratio of elastic modulus.

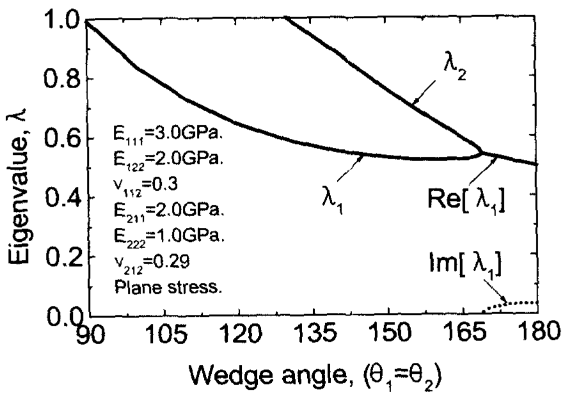


Fig. 7 Eigenvalue vs. wedge angle for a v-notched crack in an pseudo-isotropic dissimilar materials.

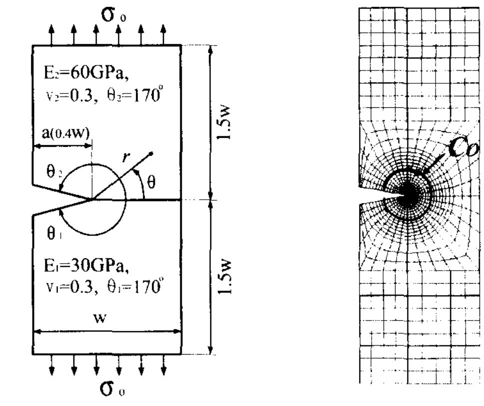
=1.5GPa, E_{222}=1.0GPa and ν_{112}=0.29 in area 2. An arbitrary eigenvector to the complex eigenvalue is assumed as $A_{11}=1.0+i\,1.0$ and the numerical analysis is performed using stress and displacement which are obtained from the eigenvalue and the eigenvector instead of the results from finite element analysis. As the consequence, $A_{11}=1.0+i\,1.0$ is accurately evaluated, inversely.

Second, stress and displacement of equation (60) are evaluated from the general finite element analysis program, ANSYS[12] and used in the program of RWCIM, so that an eigenvector to each eigenvalue is obtained and entered in equation (54). Then, whether the results are the identical with the stress and the displacement from finite element analysis is investigated. For this investigation by this method, an isotropic dissimilar material which is a special case of pseudo-isotropic dissimilar materials is selected as the one in Fig. 8 (a) and external contour C_o is selected at the position satisfying r_o/a=0.5. Eigenvalues and eigenvectors are evaluated for plane stress state of this case and are seen in Table. 1. These eigenvectors are substituted into equation (54) in order to obtain stresses. The stresses and those from Finite Element Analysis are made dimensionless and the result is seen in Fig. 9. It is known that there is a considerable difference of value between stresses of two different methods in the case of using only one complex eigenvalue and that roots converge well when more than several eigenvalues are determined.

Third, eigenvectors to the eigenvalues which have stress singularity for the problem of v-notched cracks including the one of cracks in pseudo-isotropic dissimilar

Table. 1 Coefficients associated with each eigenvetor

Eigenvector	Eigenvalue(λ)	Associated coefficient	Coefficient value
1a	0.5311+i0.02595	$Re[A_{11}(\lambda_1)]$	0.416103
1b	0.5311+i0.02595	$Im[A_{11}(\lambda_1)]$	0.062200
2	1.125371	$Re[Ar_{11}(\lambda_2)]$	0.114160
3	1.509138	$Re[Ar_{11}(\lambda_3)]$	0.036485
4	1.679738	$Re[Ar_{11}(\lambda_4)]$	-0.003773
5	1.992206	$Re[Ar_{11}(\lambda_5)]$	0.000321



a) Configuration of 2 materials b) FEA model

Fig. 8 V-notched crack in isotropic dissimilar materials

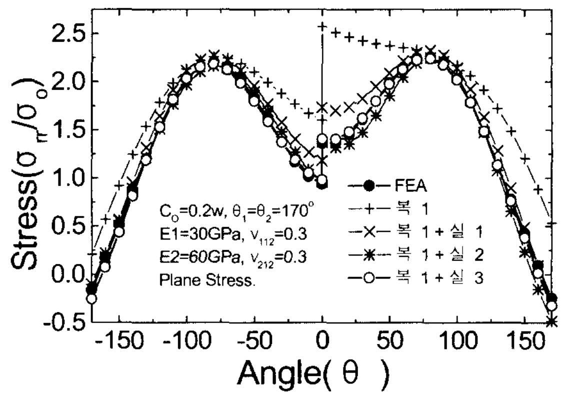


Fig. 9 Comparison of the state of stress for RWCIM and FEA

materials are evaluated by the numerical method of this paper, compared with the traditional numerical results and then examined for the verification of usefulness of the program of this paper. The eigenvectors to eigenvalues with stress singularity for the problem of interface cracks in isotropic dissimilar materials which are a special case of pseudo-isotropic dissimilar materials are obtained by the method in this paper and a certain relationship should be established between eigenvector A and stress intensity factors(K_1, K_2) which are used for the problem of interface cracks in dissimilar materials so that the eigenvector may be compared with the conventional result. Since first eigenvector λ_1=0.5+ $i\varepsilon$ for the problem of interface crack in dissimilar materia ls, the following related equation is drawn using the first eigenvalue and the corresponding eigenvector.

$$\begin{aligned} K_1 &= \sqrt{2\pi}(1+e^{2\pi\varepsilon})[(\varepsilon Re[A_{12}] - Im[A_{12}])\sin\{\varepsilon \ln(a)\}] \\ &+ (Re[A_{12}] - 2\varepsilon Im[A_{12}])\cos\{\varepsilon \ln(a)\} \\ K_2 &= \sqrt{2\pi}(1+e^{2\pi\varepsilon})[(\varepsilon Re[A_{12}] - Im[A_{12}])\cos\{\varepsilon \ln(a)\}] \\ &- (Re[A_{12}] - 2\varepsilon Im[A_{12}])\sin\{\varepsilon \ln(a)\} \end{aligned} \tag{63}$$

where a is edge crack length, and $Re[A_{12}]$ and $Im[A_{12}]$ are eigenvectors to eigenvalue λ_1.

Seen in Table 2 are the traditional result[13] and the result from the method of this paper for the problem of edge crack in isotropic dissimilar materials with $\theta_1 = \theta_2 = 180^o$ in Fig. 8(a). It can be known that the two results are almost the identical. $F_i = \sqrt{K_1^2 + K_2^2}/\sigma\sqrt{\pi a}$ and K_2/K_1 in Table 2 are dimensionless stress intensity factors.

Table. 2 Comparison of stress intensity factors
(Plane stress, ε =0.037306)

(a/W=0.4)	F_i (K_2/K_1)
Present Method	2.117 (-0.068)
BEM' Result[13]	2.117 (-0.094)

7. Conclusions

Contents evaluated are eigenvectors by the numerical method using RWCIM and eigenvalues with respect to the wedge angle and material properties of a dissimilar material for the problem of v-notched cracks between pseudo-isotropic dissimilar materials. The following conclusions are reached.

1) It can be seen that the numerical result for an isotropic material which is a special case of pseudo-isotropic dissimilar materials with a double root equal to $i\beta$ corresponds well to the conventional one and that one root exists when eigenvalue λ with stress singularity for pseudo-isotropic dissimilar materials is complex with $0.5 \leq Re[\lambda] \leq 1$ and one or two roots exist when it is real.

2) Since there is not much research on the problem of v-notched cracks, it is thought that the analysis of eigenvectors using RWCIM becomes a fundamental material for systematically analyzing the problem of cracks and strength of pseudo-isotropic dissimilar materials

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