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Smooth uniform spaces

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Abstract

We study some properties of smooth uniform spaces. We investigate the relationship between smooth topological spaces and smooth uniform spaces. In particular, we define a subspace of a smooth uniform space and a product of smooth uniform spaces.

Key words : Smooth topological spaces, Smooth uniform spaces, Product smooth uniform spaces

I. Introduction

Sostak [15] introduced the fuzzy topology as an extension of Chang's fuzzy topology [3]. Ramadan and his colleagues [2,4,13,14] called it smooth topology. Ramadan et al.[14] introduced the notion of smooth uniform spaces in slightly different ways from [2] and [10].

In this paper, we introduce some properties of smooth uniform spaces. We investigate the relationship between smooth topological spaces and smooth uniform spaces.

We define a subspace of a smooth uniform space and a product of smooth uniform spaces. In particular, the smooth topology induced by the product of smooth uniform spaces coincides with the product smooth topology of smooth topologies induced by them.

Throughout this paper, let X be a non-empty set, $I=[0,1]$ and $I_1=[0,1)$. The family I^X denotes the set of all fuzzy subsets of a given set X. For each $\alpha\in I$, let $\underline{\alpha}$ denote the constant fuzzy subset of X with value α. All the other notations and the other definitions are standard in fuzzy set theory.

II. Preliminaries

Definition 2.1 [13,15] A mapping $\tau: I^X\to I$ is called a *smooth topology* on X if it satisfies the following conditions:

(O1) $\tau(\underline{0})=\tau(\underline{1})=1$,

(O2) $\tau(\lambda_1\wedge\lambda_2)\geq\tau(\lambda_1)\wedge\tau(\lambda_2)$, for $\lambda_1,\lambda_2\in I^X$,

(O3) $\tau(\bigvee_{i\in\Gamma}\lambda_i)\geq\bigwedge_{i\in\Gamma}\tau(\lambda_i)$ for each $\{\lambda_i\mid i\in\Gamma\}\subset I^X$.

The pair (X,τ) is called a *smooth topological space.*

Let τ_1 and τ_2 be smooth topologies on X. We say τ_1 is finer than τ_2(or τ_2 is *coarser* than τ_1) if $\tau_2(\lambda)\leq\tau_1(\lambda)$ for all $\lambda\in I^X$. Let (X,τ_1) and (Y,τ_2) be smooth topological spaces. A map $f:(X,\tau_1)\to(Y,\tau_2)$ is called *smooth continuous* iff for every $\mu\in I^Y$, $\tau_1(f^{-1}(\mu))\geq\tau_2(\mu)$.

Definition 2.2 [9] Let $\underline{0}\notin\Theta$ be a subset of I^X. A map $\beta:\Theta\to I$ is called a base on X if it satisfies the following conditions:

(B1) $\beta(\underline{1})=1$,

(B2) $\beta(\lambda_1\wedge\lambda_2)\geq\beta(\lambda_1)\wedge\beta(\lambda_2)$, for each $\lambda_1,\lambda_2\in\Theta$.

Theorem 2.3 [9] Let β be a base on X. For each $\lambda\in I^X$, we define the function $\tau_\beta: I^X\to I$ as follows:

$$\tau_\beta(\lambda)=\begin{cases}\sup\{\bigwedge_{j\in\Lambda}\beta(\lambda_j)\}, & \text{if } \lambda=\bigvee_{j\in\Lambda}\lambda_j,\ \lambda_j\in\Theta\\ 1, & \text{if } \lambda=\underline{0},\\ 0, & \text{otherwise}.\end{cases}$$

Then (X,τ_β) is a smooth topological space.

Theorem 2.4 [9] Let $\{(X_i,\tau_i)\mid i\in\Gamma\}$ be a family of smooth topological spaces, X a set and for each $i\in\Gamma$, $f_i: X\to X_i$ a function. Let

$$\Theta=\{\underline{0}\neq\lambda=\bigwedge_{j=1}^{n}f_{k_j}^{-1}(\lambda_{k_j})\mid\tau_{k_j}(\lambda_{k_j})>0,\ k_j\in K\}$$

for every finite index set $K=\{k_1,\ldots,k_n\}\subset\Gamma$.

Define the function $\beta:\Theta\to I$ on X by

$$\beta(\lambda)=\sup\{\bigwedge_{j=1}^{n}\tau_{k_j}(\lambda_{k_j})\mid\lambda=\bigwedge_{j=1}^{n}f_{k_j}^{-1}(\lambda_{k_j})\},$$

for every finite index set $K=\{k_1,\ldots,k_n\}\subset\Gamma$. Then:

(1) β is a base on X.

(2) The smooth topology τ_β generated by β is the coarsest smooth topology on X which for each $i\in\Gamma$, f_i is smooth continuous.

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(3) A map $f:(Y,\tau')\to(X,\tau_\beta)$ is smooth continuous iff for each $i\in\Gamma$, $f_i\circ f:(Y,\tau')\to(X_i,\tau_i)$ is smooth continuous.

Definition 2.5 [10,14] (1) For each $\mu\in I^{X\times X}$ and for each $\lambda\in I^X$, the image $\mu[\lambda]$ with respect to μ is defined by

$$\mu[\lambda](x)=\sup_{y\in X}(\lambda(y)\wedge\mu(y,x)).$$

(2) For each $\nu,\mu\in I^{X\times X}$, the *composition* $\nu\circ\mu$ is defined by

$$\nu\circ\mu(x,y)=\sup_{z\in X}(\mu(x,z)\wedge\nu(z,y)).$$

(3) The *symmetric* μ^s of $\mu\in I^{X\times X}$ is defined by

$$\mu^s(x,y)=\mu(y,x).$$

(4) We denote $1_\Delta\in I^{X\times X}$ by

$$1_\Delta(x,y)=\begin{cases}1, & \text{if } x=y,\\ 0, & \text{if } ;x\neq y.\end{cases}$$

III. Smooth uniform spaces

Definition 3.1 [10] A function $U:I^{X\times X}\to I$ is called a *smooth uniformity* (s-uniformity, for short) on X if it satisfies the following conditions; for $\mu,\nu,\omega\in I^{X\times X}$,

(SU1) if $\mu\not\geq 1_\Delta$, then $U(\mu)=0$,
(SU2) $U(\mu\wedge\nu)=U(\mu)\wedge U(\nu)$,
(SU3) $U(\underline{1})=1$,
(SU4) $U(\mu)\leq U(\mu^s)$,
(SU5) $U(\mu)\leq\sup\{U(\omega)\mid\omega\circ\omega\leq\mu\}$.

The pair (X,U) is said to be a *smooth uniform space* (s-uniform space, for short).

Let U_1 and U_2 be s-uniformities on X. We say U_1 is *finer* than U_2(or U_2 is *coarser* than U_1) iff $U_2(\mu)\leq U_1(\mu)$ for all $\mu\in I^{X\times X}$.

Lemma 3.2 [10] Let (X,U) be a s-uniform space. For each $\mu,\mu_1,\mu_2\in I^{X\times X}$ and $\lambda,\lambda_1,\lambda_2\in I^X$, we have

(1) $\lambda\leq\mu[\lambda]$, for each $U(\mu)>0$,
(2) $\mu\leq\mu\circ\mu$, for each $U(\mu)>0$,
(3) $(\nu\circ\mu)[\lambda]=\nu[\mu[\lambda]]$,
(4) $(\mu_1\wedge\mu_2)[\lambda_1\vee\lambda_2]\leq\mu_1[\lambda_1]\vee\mu_2[\lambda_2]$,
(5) if $f:X\to Y$ is a function, then for each $\nu\in I^{Y\times Y}$,

$$f^{-1}(\nu[f(\lambda)])=(f\times f)^{-1}(\nu)[\lambda].$$

Theorem 3.3 Let (X,U) be a s-uniform space. Define a map $C_U:I^X\times I_1\to I^X$ by

$$C_U(\lambda,r)=\bigwedge\{\mu[\lambda]\mid U(\mu)>r\}.$$

For each $\lambda,\lambda_1,\lambda_2\in I^X$ and $r,r_1,r_2\in I_1$, we have the following properties:
(1) $C_U(\underline{0},r)=\underline{0}$,
(2) $\lambda\leq C_U(\lambda,r)$,
(3) if $\lambda_1\leq\lambda_2$, then $C_U(\lambda_1,r)\leq C_U(\lambda_2,r)$.
(4) $C_U(\lambda_1\vee\lambda_2,r)=C_U(\lambda_1,r)\vee C_U(\lambda_2,r)$,
(5) if $r_1\leq r_2$, then $C_U(\lambda,r_1)\leq C_U(\lambda,r_2)$,
(6) $C_U(C_U(\lambda,r),r)=C_U(\lambda,r)$.

Proof. (1) Since $\mu[\underline{0}]=\underline{0}$, $C_U(\underline{0},r)=\underline{0}$.
(2) For $U(\mu)>0$, by Lemma 3.2(1), $\lambda\leq\mu[\lambda]$ implies $\lambda\leq C_U(\lambda,r)$.
(3) and (5) are easily proved.
(4) From (3), we have

$$C_U(\lambda_1\vee\lambda_2,r)\geq C_U(\lambda_1,r)\vee C_U(\lambda_2,r).$$

Conversely, suppose there exist $\lambda_1,\lambda_2\in I^X$ and $r\in I$ such that

$$C_U(\lambda_1\vee\lambda_2,r)\not\leq C_U(\lambda_1,r)\vee C_U(\lambda_2,r).$$

There exist $x\in X$ and $t\in I_1$ such that

$$C_U(\lambda_1\vee\lambda_2,r)(x)>t>C_U(\lambda_1,r)(x)\vee C_U(\lambda_2,r)(x).$$

Since $C_U(\lambda_i,r)(x)<t$, for each $i\in\{1,2\}$, there exist $\mu_i\in I^{X\times X}$ with $U(\mu_i)>r$ such that

$$C_U(\lambda_i,r)(x)\leq\mu_i[\lambda_i](x)<t.$$

On the other hand, since $U(\mu_1\wedge\mu_2)>r$ and from Lemma 3.2(4), $(\mu_1\wedge\mu_2)[\lambda_1\vee\lambda_2]\leq\mu_1[\lambda_1]\vee\mu_2[\lambda_2]$, we have

$$C_U(\lambda_1\vee\lambda_2,r)(x)\leq(\mu_1\wedge\mu_2)[\lambda_1\vee\lambda_2](x)\leq\mu_1[\lambda_1](x)\vee\mu_2[\lambda_2](x)<t.$$

It is a contradiction.
(6) Suppose there exist $\lambda\in I^X$ and $r\in I_1$ such that

$$C_U(C_U(\lambda,r),r)\not\leq C_U(\lambda,r).$$

There exist $x\in X$ and $t\in I$ such that

$$C_U(C_U(\lambda,r),r)(x)>t>C_U(\lambda,r)(x).$$

Since $C_U(\lambda,r)(x)<t$, there exists $\mu\in I^{X\times X}$ with $U(\mu)>r$ such that

$$C_U(\lambda,r)(x)\leq\mu[\lambda](x)<t.$$

On the other hand, since $U(\mu)>r$, by (SU5), there exists $\mu_1\in I^{X\times X}$ such that

$$\mu_1\circ\mu_1\leq\mu,\quad U(\mu_1)>r.$$

Since $C_U(\lambda,r)\leq\mu_1[\lambda]$, by Lemma 3.2(3), we have

$$C_U(C_U(\lambda,r),r)\leq C_U(\mu_1[\lambda],r)\leq\mu_1[\mu_1[\lambda]]=(\mu_1\circ\mu_1)[\lambda]\leq\mu[\lambda].$$

It is a contradiction.

Theorem 3.4 Let (X,U) be a s-uniform space.

Define a map $\tau_U : I^X \to I$ by

$$\tau_U(\lambda) = \sup\{r \in I_1 \mid C_U(\underline{1}-\lambda, r) = \underline{1}-\lambda\}.$$

Then τ_U is a smooth topology induced by U.

Proof. (O1) Since $C_U(\underline{0}, r) = \underline{0}$ and $C_U(\underline{1}, r) = \underline{1}$ for all $r \in I_1$, $\tau_U(\underline{0}) = \tau_U(\underline{1}) = 1$.

(O2) Suppose there exist $\lambda_1, \lambda_2 \in I^X$ and $t \in I$ such that

$$\tau_U(\lambda_1 \wedge \lambda_2) < t < \tau_U(\lambda_1) \wedge \tau_U(\lambda_2).$$

Since $\tau_U(\lambda_i) > t$ for each $i \in \{1, 2\}$, there exist $r_i > t$ such that $\underline{1}-\lambda_i = C_U(\underline{1}-\lambda_i, r_i)$. Put $r = r_1 \wedge r_2$. By (4-5) of Theorem 3.3, we have

$$C_U(\underline{1}-(\lambda_1 \wedge \lambda_2), r) = \underline{1}-(\lambda_1 \wedge \lambda_2).$$

Thus, $\tau_U(\lambda_1 \wedge \lambda_2) \geq r > t$. It is a contradiction. Hence

$$\tau_U(\lambda_1 \wedge \lambda_2) \geq \tau_U(\lambda_1) \wedge \tau_U(\lambda_2).$$

(O3) Suppose there exists a family $\{\lambda_j \in I^X \mid j \in \Gamma\}$ and $t \in (0, 1)$ such that

$$\tau_U(\bigvee_{j \in \Gamma} \lambda_j) < t < \bigwedge_{j \in \Gamma} \tau_U(\lambda_j).$$

Since $\bigwedge_{j \in \Gamma} \tau_U(\lambda_j) < t$, for each $j \in \Gamma$, there exists $r_j > t$ such that $\underline{1}-\lambda_j = C_U(\underline{1}-\lambda_j, r_j)$.

Put $r = \bigwedge_{j \in \Gamma} r_j$. By (4-5) of Theorem 3.3, we have

$$C_U(\underline{1}-\bigvee_{j \in \Gamma} \lambda_j, r) = \underline{1}-\bigvee_{j \in \Gamma} \lambda_j.$$

Thus, $\tau_U(\bigvee_{j \in \Gamma} \lambda_j) \geq r > t$. It is a contradiction. Hence

$$\tau_U(\bigvee_{j \in \Gamma} \lambda_j) \geq \bigwedge_{j \in \Gamma} \tau_U(\lambda_j).$$

Definition 3.5 Let (X, U) and (Y, V) be s-uniform spaces. A function $f : (X, U) \to (Y, V)$ is said to be *smooth uniform continuous* if for each $\nu \in I^{Y \times Y}$,

$$V(\nu) \leq U((f \times f)^{-1}(\nu)).$$

Theorem 3.6 Let (X, U), (Y, V) and (Z, W) be s-uniform spaces. If $f : (X, U) \to (Y, V)$, $g : (Y, V) \to (Z, W)$ are smooth uniform continuous, then $g \circ f : (XU) \to (Z, W)$ is smooth uniform continuous.

Proof. For each $\omega \in I^{Z \times Z}$, we have

$$\begin{aligned} &U(((g \circ f) \times (g \circ f))^{-1}(\omega)) \\ &= U((f \times f)^{-1}((g \times g)^{-1}(\omega))) \\ &\geq V((g \times g)^{-1}(\omega)) \\ &\geq W(\omega). \end{aligned}$$

Theorem 3.7 Let (X, U) and (Y, V) be s-smooth uniform spaces. Let $f : (X, U) \to (Y, V)$ be smooth uniform continuous. Then:

(1) $f(C_U(\lambda, r)) \leq C_V(f(\lambda), r)$, for each $\lambda \in I^X$,

(2) $C_U(f^{-1}(\mu), r) \leq f^{-1}(C_V(\mu, r))$,for each $\mu \in I^Y$,

(3) $f : (X, \tau_U) \to (Y, \tau_V)$ is smooth continuous.

Proof. (1) Suppose there exist $\lambda \in I^X$ and $r \in I_1$ such that $f(C_U(\lambda, r)) \not\leq C_V(f(\lambda), r)$.
There exist $y \in Y$ and $t \in I_0$ such that

$$f(C_U(\lambda, r))(y) > t > C_V(f(\lambda), r)(y).$$

If $f^{-1}(\{y\}) = \emptyset$, it provides a contradiction that $f(C_U(\lambda, r))(y) = 0$. Hence $f^{-1}(\{y\}) \neq \emptyset$. Then there exists $x \in f^{-1}(\{y\})$ such that

$$f(C_U(\lambda, r))(y) \geq C_U(\lambda, r)(x) > t > C_V(f(\lambda), r)(f(x)).$$

Since $C_V(f(\lambda), r)(f(x)) < t$, there exists $\nu \in I^{Y \times Y}$ with $V(\nu) > r$ such that

$$C_V(f(\lambda), r)(f(x)) \leq \nu[f(\lambda)](f(x)) < t.$$

On the other hand, since f is smooth uniform continuous,

$$U((f \times f)^{-1}(\nu)) \geq V(\nu) > r.$$

It implies, by Lemma 3.2(5) and Theorem 3.3,

$$\begin{aligned} \nu[f(\lambda)](f(x)) &= (f \times f)^{-1}(\nu)[\lambda](x) \\ &\geq C_U(\lambda, r)(x). \end{aligned}$$

Thus $C_U(\lambda, r)(x) < t$. It is a contradiction.

(2) For $\mu \in I^Y$ and $r \in I_1$, put $\lambda = f^{-1}(\mu)$. From (1),

$$f(C_U(f^{-1}(\mu), r)) \leq C_V(f(f^{-1}(\mu)), r) \leq C_V(\mu, r).$$

It implies

$$\begin{aligned} C_U(f^{-1}(\mu), r) &\leq f^{-1}(f(C_U(f^{-1}(\mu), r))) \\ &\leq f^{-1}(C_V(\mu, r)). \end{aligned}$$

(3) From (2), $C_V(\mu, r) = \mu$ implies

$$C_U(f^{-1}(\mu), r) = f^{-1}(\mu).$$

It is easily proved.

IV. Product smooth uniform spaces

Theorem 4.1 Let $\{(X_k, V_k) \mid k \in \Gamma\}$ be a family of s-uniform spaces, X a set and for each $k \in \Gamma$, $f_k : X \to X_k$ a function. We define, for each $\mu \in I^{X \times X}$,

$$U(\mu) = \sup\{\bigwedge_{i=1}^{n} V_{k_i}(\mu_{k_i}) \mid \bigwedge_{i=1}^{n} (f_{k_i} \times f_{k_i})^{-1}(\mu_{k_i}) \leq \mu\}$$

where the supremum is taken over every finite index $K = \{k_1, \ldots, k_n\} \subset \Gamma$. Then:

(1) The structure U is the coarsest s-uniformity on X for which each f_k is smooth uniform continuous.

(2) A map $f : (Z, W) \to (X, U)$ is smooth uniform continuous iff for each $k \in \Gamma$, $f_k \circ f$ is smooth uniform continuous.

(3) $C_U(\lambda, r)$

$$= \bigwedge\{ \bigwedge_{i=1}^{n} (f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})[\lambda] \mid V_{k_i}(\nu_{k_i}) > r, k_i\in K\}$$

where the infimum is taken over every finite index $K=\{k_1,\dots,k_n\}\subset\Gamma$,

(4) The smooth topology τ_U induced by U coincides with the coarsest smooth topology τ_β on X for which each f_i: $(X,\tau_\beta)\to(X_i,\tau_{U_i})$ is smooth continuous.

Proof. (1) First, we will show that U is a s-uniformity on X.

(SU1) If $U(\mu)>0$, there exist finite indices $K=\{k_1,\dots,k_n\}\subset\Gamma$ such that

$$U(\mu)\geq \bigwedge_{i=1}^{n} V_{k_i}(\mu_{k_i})>0,\ \bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\mu_{k_i})\leq\mu$$

Since $V_{k_i}(\mu_{k_i})>0$ for each $k_i\in K$, by (SU1), there exists $1_\Delta\in I^{X_k\times X_k}$ with $1_\Delta\leq\mu_{k_i}$. Hence

$$1_\Delta\leq \bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(1_\Delta)$$
$$\leq \bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\mu_{k_i})\leq\mu.$$

(SU2) Since $U(\mu\wedge\nu)\leq U(\mu)$ and $U(\mu\wedge\nu)\leq U(\nu)$,

$$U(\mu\wedge\nu)\leq U(\mu)\wedge U(\nu).$$

For any $\mu,\nu\in I^{X\times X}$, we will show that

$$U(\mu\wedge\nu)\geq U(\mu)\wedge U(\nu).$$

If $U(\mu)=0$ or $U(\nu)=0$, it is trivial.

If $U(\mu)>0$ and $U(\nu)>0$, for $\varepsilon>0$ such that $U(\mu)\wedge U(\nu)>\varepsilon>0$, there exist finite indices $K=\{k_1,\dots,k_n\}$ and $L=\{l_1,\dots,l_m\}$ of Γ such that

$$\bigwedge_{i=1}^{n} V_{k_i}(\mu_{k_i})\geq U(\mu)-\varepsilon,\ \bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\mu_{k_i})\leq\mu,$$
$$\bigwedge_{j=1}^{m} V_{l_j}(\nu_{l_j})\geq U(\nu)-\varepsilon,\ \bigwedge_{j=1}^{m}(f_{l_j}\times f_{l_j})^{-1}(\nu_{l_j})\leq\nu.$$

Since $\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\mu_{k_i}))\wedge(\bigwedge_{j=1}^{m}(f_{l_j}\times f_{l_j})^{-1}(\nu_{l_j}))$ $\leq\mu\wedge\nu$, we have

$$U(\mu\wedge\nu)\geq(\bigwedge_{i=1}^{n} V_{k_i}(\mu_{k_i}))\wedge(\bigwedge_{j=1}^{m} V_{l_j}(\nu_{l_j}))$$
$$\geq U(\mu)\wedge U(\nu)-\varepsilon.$$

Since ε is arbitrary, this gives the desired result.

(SU3) From the definition of U, it is easy.

(SU4) Suppose that there exist $\mu\in I^{X\times X}$ and $r\in(0,1)$ such that $U(\mu^s)<r<U(\mu)$. Since $U(\mu)>r$, by the definition of U, there exists a finite index $K=\{k_1,\dots,k_n\}\subset\Gamma$ such that

$$U(\mu)\geq \bigwedge_{i=1}^{n} V_{k_i}(\mu_{k_i})>r,\ \bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\mu_{k_i})\leq\mu.$$

For each $k_i\in K$, since (X_{k_i}, V_{k_i}) is a s-uniform space and $V_{k_i}(\mu_{k_i})>r$, by (SU4),

$$V_{k_i}(\mu^s_{k_i})\geq V_{k_i}(\mu_{k_i})>r.$$

It follows that

$$\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\mu^s_{k_i})$$
$$=\bigwedge_{i=1}^{n}((f_{k_i}\times f_{k_i})^{-1}(\mu_{k_i}))^s\leq\mu^s.$$

Hence

$$U(\mu^s)\geq \bigwedge_{i=1}^{n} V_{k_i}(\mu^s_{k_i})>r.$$

It is a contradiction.

(SU5) For each $\mu\in I^{X\times X}$, we will show that

$$\sup\{U(\mu_1)\mid \mu_1\circ\mu_1\leq\mu\}\geq U(\mu).$$

Suppose there exist $\mu\in I^{X\times X}$ and $r\in(0,1)$ such that

$$\sup\{U(\mu_1)\mid\mu_1\circ\mu_1\leq\mu\}<r<U(\mu).$$

Since $U(\mu)>r$, by the definition of U, there exists a finite index $K=\{k_1,\dots,k_n\}\subset\Gamma$ such that

$$U(\mu)\geq \bigwedge_{i=1}^{n} V_{k_i}(\nu_{k_i})>r,\ \bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})\leq\mu.$$

For each $k_i\in K$, since (X_{k_i}, V_{k_i}) is a s-uniform space, by (SU5),

$$\sup\{V_{k_i}(\omega)\mid\omega\circ\omega\leq\nu_{k_i}\}\geq V_{k_i}(\nu_{k_i}).$$

Since $V_{k_i}(\nu_{k_i})>r$ for each $k_i\in K$, there exists $\omega_{k_i}\in I^{X_k\times X_{k_i}}$ and $r_i\in(0,1]$ such that

$$\omega_{k_i}\circ\omega_{k_i}\leq\nu_{k_i},\ V_{k_i}(\omega_{k_i})\geq r_i>r.$$

Put $\omega=\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\omega_{k_i})$. We have

$$\omega\circ\omega\leq\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\omega_{k_i})\circ(f_{k_i}\times f_{k_i})^{-1}(\omega_{k_i})$$
$$\leq\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\omega_{k_i}\circ\omega_{k_i})\quad(\omega_{k_i}\leq\omega_{k_i}\circ\omega_{k_i})$$
$$\leq\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})\leq\mu.$$

So, $\omega\circ\omega\leq\mu$ and $U(\omega)\geq\bigwedge_{i=1}^{n} V_{k_i}(\omega_{k_i})\geq\bigwedge_{i=1}^{n} r_i>r.$

Thus $\sup\{U(\mu_1)\mid\mu_1\circ\mu_1\leq\mu\}>r.$ It is a contradiction.

Second, by the definition of U, for all $k\in\Gamma$,

$$U((f_k\times f_k)^{-1}(\nu_k))\geq V_k(\nu_k),\ \text{for all}\ \nu_k\in I^{X_k}\times I^{X_k}.$$

Hence each f_k is smooth uniform continuous.

Finally, if $f_k: (X, U')\to(X, V_k)$ is smooth uniform continuous ,that is, $U'((f_k\times f_k)^{-1}(\nu))\geq V_k(\nu)$ for all $k\in\Gamma$, then it is proved from the following:

$$U(\mu)=\sup\{\bigwedge_{i=1}^{n} V_{k_i}(\nu_{k_i})\mid\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})\leq\mu\}$$

$$\leq \sup\{\bigwedge_{i=1}^{n} U'((f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i}))\}$$
$$= \sup\{U'(\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i}))\}$$
$$(\text{where } \bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})\leq\mu)$$
$$\leq U'(\mu),\ \text{for all } \mu\in I^{X\times X}.$$

(2) Let f be smooth uniform continuous. From Theorem 3.6 and (1), the composition of smooth uniform continuous maps is a smooth uniform continuous map.
Conversely, suppose that $f:(Z,W)\to(X,U)$ is not smooth uniform continuous. There exists $\mu\in I^{X\times X}$ such that

$$W((f\times f)^{-1}(\mu))<U(\mu).$$

By the definition of U, there exists a finite index $K=\{k_1,\ldots,k_n\}\subset\Gamma$ such that

$$W((f\times f)^{-1}(\mu))<\bigwedge_{i=1}^{n} V_{k_i}(\nu_{k_i})\leq U(\mu),$$
$$\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})\leq\mu.$$

On the other hand, since $f_{k_i}\circ f$ is smooth uniform continuous, we have

$$V_{k_i}(\nu_{k_i})\leq W(((f_{k_i}\circ f)\times(f_{k_i}\circ f))^{-1}(\nu_{k_i}))$$
$$= W((f\times f)^{-1}\circ(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})).$$

It follows that

$$\bigwedge_{i=1}^{n} V_{k_i}(\nu_{k_i})$$
$$\leq \bigwedge_{i=1}^{n} W(((f_{k_i}\circ f)\times(f_{k_i}\circ f))^{-1}(\nu_{k_i}))$$
$$\leq W(\bigwedge_{i=1}^{n}(f\times f)^{-1}\circ(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i}))$$
$$= W((f\times f)^{-1}(\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})))$$
$$\leq W((f\times f)^{-1}(\mu)).$$

It is a contradiction.

(3) From Theorem 3.3, we only show that

$$\bigwedge\{\mu[\lambda]\mid U(u)>r\}$$
$$=\bigwedge\{\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})[\lambda]\mid V_{k_i}(\nu_{k_i})>r,\ k_i\in K\},$$

where the infimum is taken over every finite index $K=\{k_1,\ldots,k_n\}\subset\Gamma$.
Since $U((f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i}))\geq V_{k_i}(\nu_{k_i})>r$, we have

$$\bigwedge\{\mu[\lambda]\mid U(u)>r\}$$
$$\leq\bigwedge\{\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})[\lambda]\mid V_{k_i}(\nu_{k_i})>r,\ k_i\in K\}.$$

Conversely, suppose there exists $x\in X$ such that

$$\bigwedge\{\mu[\lambda]\mid U(u)>r\}(x)$$
$$<\bigwedge\{\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})[\lambda]\mid V_{k_i}(\nu_{k_i})>r,\ k_i\in K\}(x).$$

There exists $\mu\in I^{X\times X}$ with $U(\mu)>r$ such that

$$\mu[\lambda](x)$$
$$<\bigwedge\{\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})[\lambda]\mid V_{k_i}(\nu_{k_i})>r,\ k_i\in K\}(x).$$

Since $U(\mu)>r$, there exists a finite indexnm $K=\{k_1,\ldots,k_n\}\subset\Gamma$ such that

$$U(\mu)\geq\bigwedge_{i=1}^{n} V_{k_i}(\nu_{k_i})>r,\ \bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})\leq\mu.$$

It implies

$$\bigwedge\{\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})[\lambda]\mid V_{k_i}(\nu_{k_i})>r\}\leq\mu[\lambda].$$

It is a contradiction.

(4) Suppose there exists $\lambda\in I^X$ such that $\tau_\beta(\lambda)<\tau_U(\lambda)$.
By the definition of τ_U from Theorem 3.4, there exists $r_0\in I_0$ such that $C_U(\underline{1}-\lambda,r_0)=\underline{1}-\lambda$ and

$$\tau_\beta(\lambda)<r_0\leq\tau_U(\lambda).$$

Since $C_U(\underline{1}-\lambda,r_0)=\underline{1}-\lambda$, by (3), we have

$$\underline{1}-\lambda$$
$$=C_U(\underline{1}-\lambda,r_0)$$
$$=\bigwedge\{\mu[\underline{1}-\lambda]\mid U(\mu)>r_0\}$$
$$=\bigwedge\{\bigwedge_{i=1}^{n}(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})[\underline{1}-\lambda]\mid U_{k_i}(\nu_{k_i})>r_0\}$$

where the infimum is taken over every finite index $K=\{k_1,\ldots,k_n\}\subset\Gamma$.
From Lemma 3.2(5), since

$$(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})[\underline{1}-\lambda]=f_{k_i}^{-1}(\nu_{k_i}[f_{k_i}(\underline{1}-\lambda)]),$$

for each $k_i\in K$, we have

$$\bigwedge\{(f_{k_i}\times f_{k_i})^{-1}(\nu_{k_i})[\underline{1}-\lambda]\mid U_{k_i}(\nu_{k_i})>r_0\}$$
$$=\bigwedge\{f_{k_i}^{-1}(\nu_{k_i}[f_{k_i}(\underline{1}-\lambda)])\mid U_{k_i}(\nu_{k_i})>r_0\}$$
$$=f_{k_i}^{-1}\{\bigwedge\nu_{k_i}[f_{k_i}(\underline{1}-\lambda)]\mid U_{k_i}(\nu_{k_i})>r_0\}$$
$$=f_{k_i}^{-1}(C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda),r_0)).$$

It follows that

$$\lambda=\underline{1}-\bigwedge\{\bigwedge_{i=1}^{n}f_{k_i}^{-1}(C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda),r_0))\}$$
$$=\bigvee\{\bigvee_{i=1}^{n}f_{k_i}^{-1}(\underline{1}-C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda),r_0))\}.$$

where the first $\bigvee$ is taken over every finite index $K=\{k_1,\ldots,k_n\}\subset\Gamma$. Since, by Theorem 3.3 (6),

$$C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda),r_0)=C_{U_{k_i}}(C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda),r_0),r_0),$$

by Theorem 3.4,

$$\tau_{U_{k_i}}(\underline{1}-C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda),r_0))\geq r_0.$$

Put $\mu_i=f_{k_i}^{-1}(\underline{1}-C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda),r_0))$. From Theorem 2.4, $\beta(\mu_i)\geq\tau_{U_{k_i}}(\underline{1}-C_{U_{k_i}}(f_{k_i}(\underline{1}-\lambda),r_0))\geq r_0$.

It implies $\tau_\beta(\bigvee_{i=1}^{n}\mu_i)\geq r_0$.

By the definition of τ_β from Theorem 2.4, we have

$\tau_\beta(\lambda) \geq \bigwedge \tau_\beta(\bigvee_{i=1}^{n} \mu_i) \geq r_0$. It is a contradiction. Therefore $\tau_\beta(\lambda) \geq \tau_U(\lambda)$ for all $\lambda \in I^X$.

We will show that $\tau_\beta(\lambda) \leq \tau_U(\lambda)$ for every $\lambda \in I^X$, equivalently, the identity function

$id_X: (X, \tau_U) \to (X, \tau_\beta)$ is smooth continuous.

We only show that $f_i \circ id_X: (X, \tau_U) \to (X_i, \tau_{U_i})$ is smooth continuous from Theorem 2.4(3). It is obvious from Theorem 4.1.

From Theorem 4.1, we can define subspaces and products in the obvious way.

Definition 4.2 Let (X, V) be a s-uniform space and A a subset of X. The pair (A, V_A) is said to be a

subspace of (X, V) if it is endowed with the coarsest s-uniformity structure induced by the inclusion map.

Definition 4.3 Let $\{(X_i, U_i) \mid i \in \Gamma\}$ be a family of s-uniform spaces. The coarsest s-uniformity structure $U = \otimes U_i$ on $X = \prod_{i \in \Gamma} X_i$ induced by the collection

$\{\pi_i: X \to X_i \mid i \in \Gamma\}$ of projections is called the *product smooth uniformity structure* of $\{U_i \mid i \in \Gamma\}$ and $(X, \otimes U_i)$ is called the *product smooth uniform space.*

Corollary 4.4 Let $(X_k, U_k)_{k \in \Gamma}$ be s-uniform spaces.

Let $X = \prod_{k \in \Gamma} X_k$ be a set andfor each $k \in \Gamma$, $\pi_k: X \to X_k$ a projection. The structure $U = \otimes U_k$ is defined by, for each $\mu \in I^{X \times X}$,

$$U(\mu) = \sup\{ \bigwedge_{i=1}^{n} U_{k_i}(\nu_{k_i}) \mid \bigwedge_{i=1}^{n} (\pi_{k_i} \times \pi_{k_i})^{-1}(\nu_{k_i}) \leq \mu \},$$

where the supremum is taken over every finite index $K = \{k_1, \ldots, k_n\} \subset \Gamma$. Then:

(1) U is the coarsest s-uniformity on X for which π_k is smooth uniform continuous.

(2) A map $f: (Z, W) \to (X, U)$ is smooth uniform continuous iff $\pi_k \circ f$ is smooth uniform continuous.

(3) The smooth topology τ_U induced by U coincides with the product smooth topology τ_β on X for which each $\pi_i: (X, \tau_\beta) \to (X_i, \tau_{U_i})$ is smooth continuous.

Corollary 4.5 Let $(X, U_i)_{i \in \Gamma}$ be s-uniform spaces.

We define, for $\mu \in I^{X \times X}$,

$$U(\mu) = \sup\{ \bigwedge_{i=1}^{n} U_{k_i}(\nu_{k_i}) \mid \bigwedge_{i=1}^{n} \nu_{k_i} \leq \mu \},$$

where the supremum is taken over every finite index $K = \{k_1, \ldots, k_n\} \subset \Gamma$. Then:

(1) The structure U is the coarsest s-uniformity on X finer than U_i.

(2) The smooth topology τ_U induced by U coincides with the product smooth topology τ_β on X of a family $\{(X_i, \tau_{U_i})\}_{i \in \Gamma}$.

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