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Fast Iterative Solving Method of Fuzzy Relational Equation and its Application to Image Compression/Reconstruction

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Abstract

A fast iterative solving method of fuzzy relational equation is proposed. It is derived by eliminating a redundant comparison process in the conventional iterative solving method (Pedrycz, 1983). The proposed method is applied to image reconstruction, and confirmed that the computation time is decreased to 1 / 40 with the compression rate of 0.0625. Furthermore, in order to make any initial solution converge on a reconstructed image with a good quality, a new cost function is proposed. Under the condition that the compression rate is 0.0625, it is confirmed that the root mean square error of the proposed method decreases to 27.34% and 86.27% compared with those of the conventional iterative method and a non iterative image reconstruction method, respectively.

Key words : Fuzzy relational equation, Image compression, Optimization, Gradient method

I. Introduction

An iterative solving method of fuzzy relational equation has been proposed in [5], followed by its generalized methods [4][6][7]. Approximate solutions obtained by these conventional methods [4][6][7] have been evaluated in some small numerical experiments, but they have not been done in specific application fields. In this paper, an iterative solving method of fuzzy relational equation is formalized in the framework of image compression/reconstruction [1][2][3]. Moreover, a fast iterative solving method of fuzzy relational equation is proposed. In the image compression/ reconstruction experiments, it is pointed out that not every initial solution is converged on a reconstructed image with a good image quality by minimizing the conventional cost function. In order to make any initial solution converge on a reconstruction image with a good image quality, a new cost function is also proposed and its effectiveness is shown through experiments. Furthermore, it is confirmed that the image quality of the proposed iterative method is better than those of the conventional iterative method [5]-[7] and a non iterative reconstructed method [3], respectively.

II. Fast iterative solving method of fuzzy relational equation

In Image Compression and reconstruction method based on Fuzzy relational equation (ICF) [1][2], a image of the sizepixels is expressed as fuzzy relation

$$R \in F(\mathbf{X}\times\mathbf{Y}),\ \mathbf{X}=\{x_1, x_2, ..., x_M\},\ \mathbf{Y}=\{y_1, y_2, ..., y_N\}$$

by normalizing the intensity range of each pixel into [0,1]. The fuzzy relation is compressed into $G \in F(\mathbf{I}\times\mathbf{J})$ by the max- continuous t-norm composition of R,

$$\mathbf{A}=\{A_i \in F(\mathbf{X}) \mid i \in \mathbf{I}=\{1,2,...,I\}\}$$

$$\mathbf{B}=\{B_j \in F(\mathbf{Y}) \mid j \in J=\{1,2,...,J\}\}$$

that is,

$$G(i,j)=\max_{y \in Y}\left\{\left\{\max_{x \in X}\left(R(x,y)\,t\,A_i(x)\right)\right\}t\,B_j(y)\right\} \quad (\in[0,1]). \tag{1}$$

Where, t denotes a continuous t-norm, and the families of fuzzy sets **A** and **B** are called coder. In this paper, we take algebraic product, that is,

$$a t b = a \cdot b \tag{2}$$

as the continuous t-norm, and the coder **A** and **B** are defined as

$$\mathbf{A}=\{A_1, A_2, ..., A_I\} \tag{3}$$

$$A_i\{X_m\}=\exp-\left(-Sh\left(\frac{iM}{I}-m\right)^2\right), \tag{4}$$
$$(m=1,2,...,M).$$

$$\mathbf{B}=\{B_1, B_2, ..., B_J\} \tag{5}$$

$$B_j(y_n)=\exp\left(-Sh\left(\frac{iN}{J}-n\right)^2\right), \tag{6}$$
$$(n=1,2,...,N)$$

where, denotes the sharpness of the fuzzy sets of the

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coder, and is appropriately adjusted by the compression rate. Image reconstruction problem deals with a problem of finding a fuzzy relation $\overline{R} \in F(\mathbf{X} \times \mathbf{Y})$ leading to the minimum value of the cost function

$$\rho_1(G, \overline{G}) = \sum_{(i,j) \in I \times J} \{G(i,j) - \tilde{G}(i,j)\}^2, \tag{7}$$

$$\tilde{G}(i,j) = \max\{\{\max(\overline{R}(x,y) t A_i(x))\} t B_j(y)\}. \tag{8}$$

An iterative solving method of the above problem is

$$\tilde{R}^{(iter+1)} = \overline{R}^{(iter)} - \alpha_{(iter)} f(\tilde{R}^{(iter)}), \tag{9}$$

$$f(\overline{R}) = \begin{bmatrix} \dfrac{\partial \rho_1(G, \tilde{G})}{\partial \overline{R}(1,1)} \\ \vdots \\ \dfrac{\partial \rho_1(G, \overline{G})}{\partial \overline{R}(k,1)} \\ \vdots \\ \dfrac{\partial \rho_1(G, \tilde{G})}{\partial \overline{R}(M,N)} \end{bmatrix} = 2 \begin{bmatrix} \sum_{(i,j) \in I \times J} \{G(i,j) - \tilde{G}(i,j)\} P^{(1,1)}_{(i,j)} \\ \vdots \\ \sum_{(i,j) \in I \times J} \{G(i,j) - \overline{G}(i,j)\} P^{(k,1)}_{(i,j)} \\ \vdots \\ \sum_{(i,j) \in I \times J} \{G(i,j) - \tilde{G}(i,j)\} P^{(M,N)}_{(i,j)} \end{bmatrix} \tag{10}$$

$$P^{(k,1)}_{(i,j)} = \begin{cases} \dfrac{\partial R(k,1) t A_i(k) t B_j(1)}{\partial \tilde{R}(k,1)} & \text{if } \Phi^{(k,1)}_{(i,j)} \text{ is true}, \\ 0 & \text{otherwise}, \end{cases} \tag{11}$$

$$\Phi^{(k,1)}_{(i,j)} : \tilde{R}(k,1) t A_i(k) t B_j(1) \geq \max_{(x,y) \subset X \times Y} \{\tilde{R}(x,y) t A_i(x) t B_j(y)\}, \tag{12}$$

which is shown in [5]. Where, denotes the number of iterations, and $\alpha_{(iter)}$ denotes the learning rate, which is defined by

$$\alpha_{(iter)} = \alpha_{(0)} (1 - 10^{-3})^{(iter)} \tag{13}$$

In this paper, a fast iterative method of fuzzy relational equation is proposed. First, Eq. (12) is transformed into

$$\Phi^{(k,j)}_{(i,j)} : R(k,1) t A_i(k) t B_j(1) \geq \max_{y \subset Y} \{\max_{x \subset X} \{\overline{R}(x,y) t A_i(x)\} t B_j(y)\} \tag{14}$$

Then, a redundant computation of the conventional method is eliminated, that is,

$$\max_{x \subset X} \{\tilde{R}(x,y) t A_i(x)\} \tag{15}$$

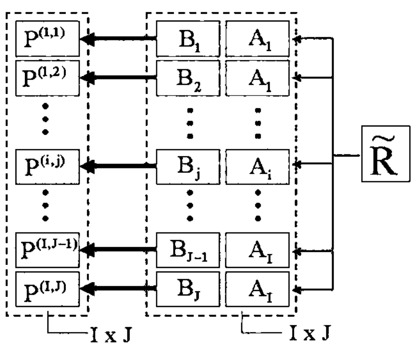


Fig. 1. Calculation Process (Conventional Method)

which is obtained during the calculation of $P^{(k,1)}_{(i,j)}$, (i, j') $\in$ I × J, can be reused to the calculation of $P^{(k,1)}_{(i,j'')}$, (i, j'') $\in$ I × J {(i, j')}. The calculation process of the conventional method and proposed method are shown in Fig. 1 and Fig. 2, respectively.

The computational times of the conventional method and the proposed one are

$$\text{Conventional Method} = IJMN(T+C), \tag{16}$$

$$\text{Proposed Method} = IN(M+J)(T+C), \tag{17}$$

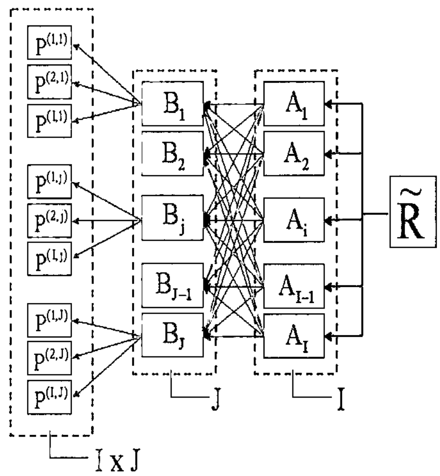


Fig. 2. Calculation Process (Proposed Method)

where, M × N and I × J denote the size of the original image and the compressed image, respectively. T and C denote the computation time of t-norm and max operations, respectively. This iterative solving method elimination of fuzzy relational equations can be applied to other types of fuzzy relational equation, e.g., adjoint max t-norm composition type, min s-norm composition type, and adjoint s-norm composition type of fuzzy relational equations [4]. The original image of the size M × N = 256 × 256 pixels(Fig. 3) is compressed into the size of I × J = 64 × 64(Compression rate = 0.0625), where is 0.05. The computation time of the image reconstruction with the proposed iterative solving method and that with the conventional method are measured on the workstation Sun Ultra 10 (440MHz). These results are shown in Table 1.

Table 1. Computation Time Comparison of the Proposed and Conventional Methods (per iteration)

	Computation Time	Ratio
Proposed	6.42(s)	1.0
Conventional	248.03(s)	38.67

Under the condition that Sh = 0.05 and $\alpha_{(0)}$=1.0, the results of image reconstruction are shown in figures 4, where, initial fuzzy relation $\tilde{R}^{(0)}$ is defined as

$$\forall (x,y) \in \mathbf{X} \times \mathbf{Y}, \overline{R}^{(0)}(x,y) = \text{random value of } [0,1] \tag{18}$$

Fig. 3. Original Image

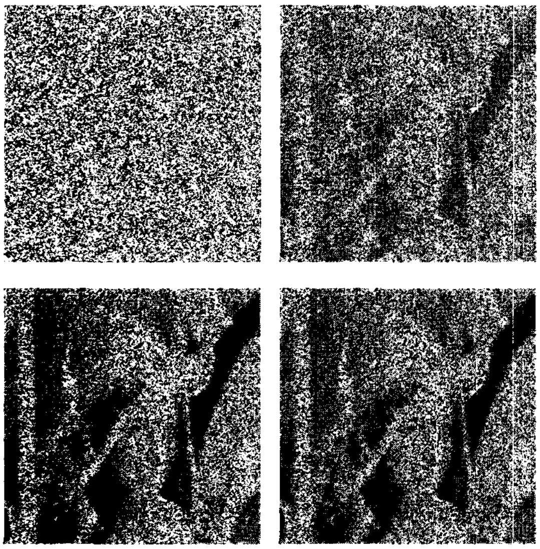


Fig. 4. Reconstructed Images (upper left : iter = 0, upper right : iter = 10^2, lower left : iter = 10^3, lower right : iter = 10^4)

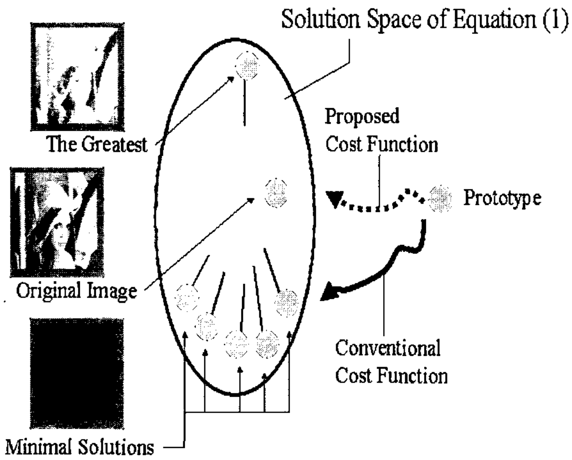
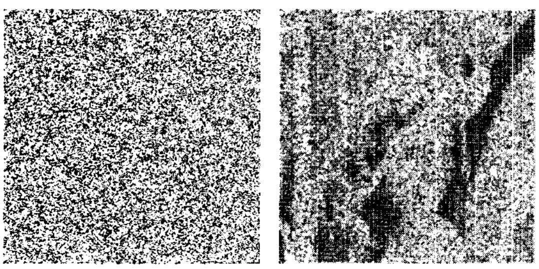


Fig. 5. Convergence of the Prototype Fuzzy Relation



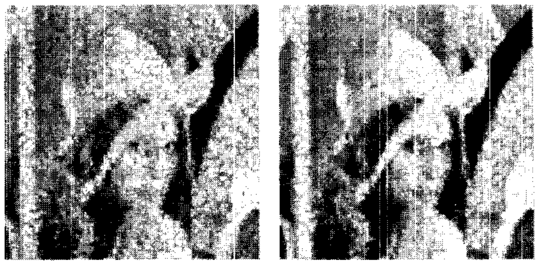


Fig. 6. Reconstructed Images (Condition 2-(a))(upper left : iter =0, upper right : iter=10^2, lower left : iter=10^3, lower right : iter = 10^4)

Fig. 7. Reconstructed Images (Condition 2-(b)) (upper left : iter = 0, upper right : iter = 10^2, lower left : iter = 10^3, lower right : iter = 10^4)

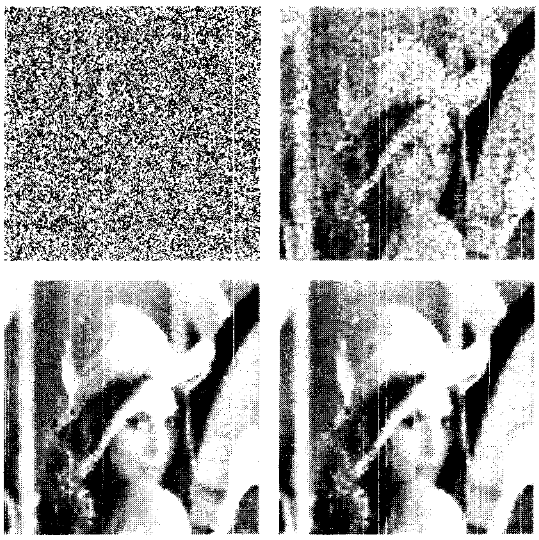


Fig. 8. Reconstructed Images (Condition 2-(c)) (upper left : iter = 0, upper right : iter=10^2, lower left : iter=10^3, lower right : iter = 10^4)

The root mean square errors of the reconstructed image are shown in table 2. The initial fuzzy relation does not converge on the reconstructed image with a good image quality. This is because the cost function $\rho_1$does not reflect the smoothness of the reconstructed image. From the point of view of fuzzy relational equation theory, minimizing the conventional cost

Fig. 9. Reconstructed Images Comparison (left : Condition 2-(c)(iter = 10^4,RMSE = 18.35), right : non-iterative reconstruction method [3] (RMSE = 21.27))

function leads to the convergence of the prototype fuzzy relation on meaningless solutions (minimal solutions) as reconstructed image (See Figure 5). In this paper, in order to make any initial fuzzy relation converge on a reconstruction image with a good quality, a new cost function is proposed.

Table 2. RMSE Comparison

Iteration Number	0	10^2	10^3	10^4
RMSE	86.57	71.21	66.25	66.10

III. Improvement on Cost Function

As stated in section 2, the conventional cost function has several weaknesses. Here the new cost function is proposed :

$$\rho_2(G, \tilde{G}) = \sum_{(i,j)\in I\times J}\{G(i,j) - \tilde{G}(i,j)\}^2$$

$$+\beta\sum_{(k,l)\in X\times Y}\left\{\overline{R}(k, l) - \frac{1}{|W^w_{(k,l)}|}\sum_{(k',l')\in W^w_{(k,l)}}\overline{R}(k, l)\right\}^2 \quad (19)$$

Where $W^w_{(k,l)}$ denotes the local window with size and is defined as,

$$W^w_{(k,l)} = \left\{\begin{matrix}(k'', l'')\in X\times Y \mid k-w\le k''\le k+w, \\ l-w\le l''\le l+w\end{matrix}\right\}$$

The cost function ρ_2 is composed of ρ_1 and a local variance term reflecting the smoothness of the reconstructed image $\tilde{R}$. If the cost function ρ_2 is influenced considerably by the local variance term, the reconstructed image will be a flat image, leading to a poor quality. Therefore, the weight β of the local variance term must be properly adjusted.

Under the conditions given in table 3, the results of image reconstruction are shown in figures 6 to 8, and the root mean square errors of the reconstructed images are shown in table 4.

Table 3. Image Reconstruction Conditions,
(F. R. = Fuzzy Relation, L. R. = Learning Rate)

	Initial F. R.	Initial L. R.	Sh	w
Condition 2-(a)	Random	$\alpha_{(0)} = 1.0$ $\beta = 0.01$	0.15	3
Condition 2-(b)	Random	$\alpha_{(0)} = 2.0$ $\beta = 0.03$	0.15	3
Condition 2-(c)	Random	$\alpha_{(0)} = 3.0$ $\beta = 0.03$	0.25	3

Table 4. RMSE Comparison

Iteration Num.	0	10^2	10^3	10^4
Condition 2-(a)	86.36	45.49	29.31	25.03
Condition 2-(b)	86.17	30.41	19.09	19.09
Condition 2-(c)	86.42	30.00	18.42	18.35

By applying the proposed cost function, any initial fuzzy relation converge on the reconstructed image with a good quality. In the case of condition 2-(a), a low quality image is reconstructed due to the small value of parameter . In the case where the initial learning rate and are high, the quality of the reconstructed image tend to be good. As shown in Table 4, the root mean square of the proposed method (condition 2-(c)) decreases to 27.34% compared with that of the conventional one.

A comparison of the proposed method (condition 2-(c)) with the non iterative image reconstruction method [3] from a viewpoint of image quality (RMSE) is shown in Figure 9. The root mean square error of the proposed method decreases to 86.27% compared with that of the non iterative method. In particular, a white slit noise that is observed on the reconstructed image of the non iterative method, can be eliminated successfully from that of the proposed one.

IV. Conclusion

A fast iterative solving method of fuzzy relational equation is proposed. It is applied to image reconstruction, and confirmed that the computation time is decreased to 1 / 40 with the compression rate of 0.0625. It is pointed out that not every initial solution is converged on a reconstructed image with a good quality by minimizing the conventional cost function. A new cost function is proposed in order to obtain the reconstruction image with a good quality, and experimental results show that any initial solution converge on a reconstructed image with a good quality. In the image reconstruction experiments with the compression rate being 0.0625, the root mean square error of the proposed method decreases to 27.34% and 86.27% compared with those of the conventional iterative method and a non iterative image reconstruction method [3], respectively.

References

[1] K. Hirota, W. Pedrycz, "Fuzzy Relational Compression," *IEEE Transactions on Systems, Man, and Cybernetics, Part B*, vol. 29, no. 3, pp. 407-415, 1999.

[2] H. Nobuhara, W. Pedrycz, and K. Hirota, "Fast Solving Method of Fuzzy Relational Equation and its Application to Image Compression/Reconstruction," *IEEE Transaction on Fuzzy Systems*, vol. 8, no. 3, pp. 325-334, 2000.

[3] H. Nobuhara, Y. Takama, and K. Hirota, "Image Compression/Reconstruction Based on Various Types of Fuzzy Relational Equations," *The Transactions of The institute of Electrical Engineers of Japan*, vol. 121-C, no. 6, pp. 1102-1113, 2001.

[4] J.V. Oliveira, "Neuron Inspired Learning Rules for Fuzzy Relational Structure," *Fuzzy Sets and Systems*, vol. 57, no.1, pp. 41-54, 1993.

[5] W. Pedrycz, "Numerical and Applicational Aspects of Fuzzy Relational Equations," *Fuzzy Sets and Systems*, vol. 11, no. 1, pp. 1-18, 1983.

[6] W. Pedrycz, "Neurocomputations in Relational Systems," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 13, no. 3, pp. 289-296, 1991.

[7] W. Pedrycz and J.V.Oliveira, "Optimization of Fuzzy Models," *IEEE Transactions on Systems, Man, and Cybernetics, Part B*, vol. 26, no. 4, pp. 627-636, 1996.