Journal of the Korean Statistical Society (2002), **31**: *1, pp 129–140*

Strong Law of Large Numbers for Tight Fuzzy Random Variables†

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ABSTRACT

In this paper, we obtain a strong law of large numbers for convex tight random elements taking values in the space of fuzzy numbers in R.

Keywords: Fuzzy random variables, strong law of large numbers, convex tightness.

1. Introduction

In recent years, limit theorems for sums of fuzzy random variables have received much attentions because of its usefulness in several applied fields. Among others, strong laws of large numbers for independent fuzzy random variables have been studied by Klement *et al.* [12], Inoue [6], Molchanov [13], Kim [10], Joo and Kim [8], and so on. Joo and Kim [7] introduced a new metric d_s on the space $F(R)$ of fuzzy numbers in R so that $F(R)$ is separable and topologically complete and Ghil *et al.* [4] characterized compact subsets of $F(R)$. These results were used to study a strong law of large numbers for stationary fuzzy random variables by Joo *et al.* [9]. Also, Kim [11] proved that a fuzzy mapping is measurable if and only if it is measurable when considered as a function into the metric space $F(R)$ endowed with the metric d_s. Thus it is natural that we ask whether strong laws of large numbers for fuzzy random variables can also be obtained with respect to the metric d_s.

In this paper, motivated by the works of Joo and Kim [7], we establish a strong law of large numbers for convex tight fuzzy random variables. Section 2 is devoted to describe some preliminary results and the main results are given in Section 3.

†This paper is supported by Kangwon National University.
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2. Preliminary Results

Let R denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \to [0,1]$ with the following properties :

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R$ such that $\tilde{u}(x) = 1$.

(2) $\tilde{u}$ is upper semi-continuous.

(3) supp $\tilde{u} = cl\{x \in R : \tilde{u}(x) > 0\}$ is compact.

(4) $\tilde{u}$ is a convex fuzzy set, i.e., $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0,1]$.

We denote the family of all fuzzy numbers by $F(R)$. For a fuzzy set $\tilde{u}$, the α-level set of $\tilde{u}$ is defined by

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1, \\ supp\ \tilde{u}, & \alpha = 0. \end{cases}$$

Then it follows that $\tilde{u}$ is fuzzy number if and only if $L_1\tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0,1]$. From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is completely determined by the end points of the intervals $L_\alpha \tilde{u} = [u_\alpha^1, u_\alpha^2]$.

Theorem 2.1. *For $\tilde{u} \in F(R)$, we denote $u^1(\alpha) = u_\alpha^1$ and $u^2(\alpha) = u_\alpha^2$ by considering as a function of $\alpha \in [0,1]$. Then the followings hold.*

(1) *$u^1(\alpha)$ is a bounded increasing function on* [0,1].

(2) *$u^2(\alpha)$ is a bounded decreasing function on* [0,1].

(3) $u^1(1) \leq u^2(1)$.

(4) *$u^1(\alpha)$ and $u^2(\alpha)$ are left continuous on* (0,1] *and right continuous at 0.*

(5) *If $v^1(\alpha)$ and $v^2(\alpha)$ satisfy the above* (1)-(4), *then there exists a unique $\tilde{v} \in F(R)$ such that $L_\alpha \tilde{v} = [v^1(\alpha), v^2(\alpha)]$ for all $\alpha \in [0,1]$.*

Proof. See Theorem 1.1 of Goetschel and Voxman [5]. □

The above theorem implies that we can identify a fuzzy number $\tilde{u}$ with the parametrized representation $\{(u^1_\alpha, u^2_\alpha) \mid 0 \leq \alpha \leq 1\}$, where u^1 and u^2 satisfy (1)-(4) of Theorem 2.1. Suppose now that $\tilde{u}, \tilde{v} \in F(R)$ are fuzzy numbers whose representations are $\{(u^1_\alpha, u^2_\alpha) \mid 0 \leq \alpha \leq 1\}$ and $\{(v^1_\alpha, v^2_\alpha) \mid 0 \leq \alpha \leq 1\}$, respectively. If we define

$$(\tilde{u} + \tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)),$$

$$(\lambda \tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \lambda \neq 0, \\ \tilde{0}, & \lambda = 0, \end{cases}$$

where $\tilde{0} = I_{\{0\}}$ is the indicator function of $\{0\}$, then

$$\tilde{u} + \tilde{v} = \{(u^1_\alpha + v^1_\alpha,\ u^2_\alpha + v^2_\alpha) \mid 0 \leq \alpha \leq 1\},$$

$$\lambda \tilde{u} = \begin{cases} \{(\lambda u^1_\alpha, \lambda u^2_\alpha) \mid 0 \leq \alpha \leq 1\}, & \lambda \geq 0, \\ \{(\lambda u^2_\alpha, \lambda u^1_\alpha) \mid 0 \leq \alpha \leq 1\}, & \lambda < 0. \end{cases}$$

Now, we define the metric d_∞ on $F(R)$ by

$$d_\infty(\tilde{u}, \tilde{v}) = \sup_{0 \leq \alpha \leq 1} h(L_\alpha \tilde{u}, L_\alpha \tilde{v}), \tag{2.1}$$

where h is the Hausdorff metric defined as

$$h(L_\alpha \tilde{u}, L_\alpha \tilde{v}) = \max(|u^1_\alpha - v^1_\alpha|, |u^2_\alpha - v^2_\alpha|).$$

Also, the norm $||\tilde{u}||$ of fuzzy number $\tilde{u}$ will be defined as

$$||\tilde{u}|| = d_\infty(\tilde{u}, \tilde{0}) = \max(|u^1_0|, |u^2_0|).$$

Then it is well-known that $F(R)$ is complete but nonseparable with respect to the metric d_∞. Joo and Kim [7] introduced a metric d_s in $F(R)$ which makes it a separable metric space as follows.

Definition 2.1. *Let T denote the class of strictly increasing, continuous mapping of* [0,1] *onto itself. For $\tilde{u}, \tilde{v} \in F(R)$, we define*

$$d_s(\tilde{u}, \tilde{v}) = \inf\{\varepsilon : \text{there exists a } t \text{ in } T \text{ such that } \sup_{0 \leq \alpha \leq 1} |t(\alpha) - \alpha| \leq \varepsilon \text{ and } d_\infty(\tilde{u}, t \circ \tilde{v}) \leq \varepsilon\}, \tag{2.2}$$

where $t \circ \tilde{v}$ denotes the composition of $\tilde{v}$ and t.

Then it follows immediately that d_s is a metric on $F(R)$ and $d_s(\tilde{u},\tilde{v}) \leq d_\infty(\tilde{u},\tilde{v})$. The metric d_s will be called the Skorohod metric. Note that a sequence $\{\tilde{u}_n\}$ in $F(R)$ converges to a limit $\tilde{u}$ in the metric d_s if and only if there exists a sequence of functions $\{t_n\}$ in T such that

$$\lim_{n\to\infty} t_n(\alpha) = \alpha \text{ uniformly in } \alpha,$$

$$\lim_{n\to\infty} d_\infty(t_n(\tilde{u}_n),\tilde{u}) = 0.$$

If $d_\infty(\tilde{u}_n,\tilde{u}) \to 0$, then $d_s(\tilde{u}_n,\tilde{u}) \to 0$. But, the converse is not true. Now we define, for $\tilde{u} \in F(R)$ and $0 < \delta < 1,\ 0 \leq \alpha < \beta \leq 1$,

$$\begin{aligned} w_{\tilde{u}}(\alpha,\beta) &= h(L_{\alpha^+}\tilde{u},\ L_\beta\tilde{u}) \\ &= \max(u^1_\beta - u^1_{\alpha+},\ u^2_{\alpha+} - u^2_\beta), \end{aligned} \tag{2.3}$$

where $L_{\alpha^+}\tilde{u}$ denotes the closed interval $[u^1_{\alpha+}, u^2_{\alpha+}]$ with convention $u^i_{\alpha+}$ the right-limit of u^i at α. If we define

$$w'_{\tilde{u}}(\delta) = \inf \max_{1\leq i\leq r} w_{\tilde{u}}(\alpha_{i-1},\alpha_i), \tag{2.4}$$

where the infimum is taken over all partitions $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ of [0,1] satisfying $\alpha_i - \alpha_{i-1} > \delta$ for all i, then, Lemma 3.2 of Joo and Kim [7] implies that

$$\lim_{\delta\to 0} w'_{\tilde{u}}(\delta) = 0 \text{ for each } \tilde{u} \in F(R). \tag{2.5}$$

The following lemma which characterizes compact subsets of $F(R)$ is useful in proving the main result (for details, see Ghil, Joo and Kim [4]).

Theorem 2.2. *Let K be a subset of $F(R)$. Then K is relatively compact in the d_s-topology if and only if*

$$\sup\{||\tilde{u}|| : \tilde{u} \in K\} < \infty, \tag{2.6}$$

$$\lim_{\delta\to 0} \sup\{w'_{\tilde{u}}(\delta) : \tilde{u} \in K\} = 0. \tag{2.7}$$

3. Main Result

Throughout this section, we assume that the space $F(R)$ is considered as the metric space endowed with the metric d_s, unless otherwise stated. Also, we denote by $\mathcal{B}_s$ the Borel σ-field of $F(R)$ generated by the metric d_s.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A fuzzy number valued function $\tilde{X} : \Omega \to F(R)$ is called a fuzzy random variable if it is measurable, i.e.,

$$\tilde{X}^{-1}(B) = \{\omega : \tilde{X}(\omega) \in B\} \in \mathcal{A} \text{ for every } B \in \mathcal{B}_s.$$

If we denote $\tilde{X}(\omega) = \{(X_\alpha^1(\omega), X_\alpha^2(\omega)) \mid 0 \le \alpha \le 1\}$, then it is known that $\tilde{X}$ is a fuzzy random variable if and only if for each $\alpha \in [0,1]$, X_α^1 and X_α^2 are random variables in the usual sense (see Kim [11]). A fuzzy random variable $\tilde{X} = \{(X_\alpha^1, X_\alpha^2) \mid 0 \le \alpha \le 1\}$ is called integrable if for each $\alpha \in [0,1]$, X_α^1 and X_α^2 are integrable, equivalently, $\int ||\tilde{X}||dP < \infty$. In this case, the expectation of $\tilde{X}$ is the fuzzy number $E\tilde{X}$ defined by

$$E\tilde{X} = \{(EX_\alpha^1, EX_\alpha^2) \mid 0 \le \alpha \le 1\}. \tag{3.1}$$

Definition 3.1. *A sequence $\{\tilde{X}_n\}$ of fuzzy random variables is said to be convex tight if for each $\epsilon > 0$ there is a convex compact subset K of $F(R)$ such that*

$$P(\tilde{X}_n \notin K) < \epsilon \text{ for all } n.$$

Our main result is follows.

Theorem 3.1. *Let $\{\tilde{X}_n\}$ be a sequence of independent and convex tight fuzzy random variables. If*

$$\sup_n E||\tilde{X}_n||^p \le M < \infty \text{ for some } p > 1, \tag{3.2}$$

then

$$\lim_{n\to\infty} d_s\left(\frac{1}{n}\sum_{i=1}^n \tilde{X}_i, \frac{1}{n}\sum_{i=1}^n E\tilde{X}_i\right) = 0 \quad \text{a.s.}$$

To prove the above theorem, we need some lemmas.
First, for a positive integer m, if we define

$$\tilde{f}_m \;:\; F(R) \to F(R) \text{ by } \tilde{f}_m(\tilde{u})(x) = \sum_{k=1}^m \frac{k-1}{m} I_{A_{k-1}\setminus A_k}(x) + I_{A_m}(x),$$

where $A_k = L_{\frac{k}{m}}\tilde{u} = \left[u^1_{\frac{k}{m}}, u^2_{\frac{k}{m}}\right], k = 0, 1, \cdots, m,$
then it follows that

$$L_\alpha \tilde{f}_m(\tilde{u}) = \begin{cases} L_{\frac{1}{m}}\tilde{u} & \text{if } 0 \le \alpha \le \frac{1}{m} \\ L_{\frac{k}{m}}\tilde{u} & \text{if } \frac{k-1}{m} < \alpha \le \frac{k}{m},\ k = 2, \cdots, m. \end{cases} \tag{3.3}$$

From this fact, it is obvious that

$$\tilde{f}_m(\tilde{u} + \tilde{v}) = \tilde{f}_m(\tilde{u}) + \tilde{f}_m(\tilde{v}),$$

and

$$\tilde{f}_m(\lambda\tilde{u}) = \lambda\tilde{f}_m(\tilde{u}).$$

Lemma 3.1. *For each m, and $\tilde{u} \in F(R)$,*

$$d_s(\tilde{u}, \tilde{f}_m(\tilde{u})) \le w'_{\tilde{u}}\left(\frac{1}{m}\right) + \frac{1}{m}.$$

Proof. Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ be any partition of [0,1] such that $\alpha_{i+1} - \alpha_i > \frac{1}{m}$ for all $i = 1, 2, \cdots, r-1$. Then for each $i = 0, 1, \cdots, r-1$, there exists k such that

$$\alpha_i \le \frac{k}{m} < \alpha_{i+1}.$$

We define $i_m = \min\{k \mid \alpha_i \le \frac{k}{m}\}$ and take $t \in T$ to $\frac{i_m}{m}$ at the points α_i and be linear in between. Then

$$\sup_\alpha |\, t(\alpha) - \alpha| \le \frac{1}{m}. \tag{3.4}$$

If $\alpha_i < \alpha \le \alpha_{i+1}$ for some $i = 0, 1, \cdots, r-1$, then

$$\alpha_i \le t(\alpha_i) < t(\alpha) \le t(\alpha_{i+1})$$

and by (3.3), $L_{t(\alpha_i)}\tilde{f}_m(\tilde{u}) = L_{\alpha_i}\tilde{u}$. Hence

$$\sup_{\alpha_i < \alpha \le \alpha_{i+1}} h(L_\alpha\tilde{u},\ L_{t(\alpha)}\tilde{f}_m(\tilde{u})) \le w_{\tilde{u}}(\alpha_i, \alpha_{i+1}).$$

Therefore, we obtain

$$\begin{aligned} d_\infty(t \circ \tilde{u},\ \tilde{f}_m(\tilde{u})) &= \sup_\alpha h(L_\alpha\tilde{u},\ L_{t(\alpha)}\tilde{f}_m(\tilde{u})) \\ &\le \max_i w_{\tilde{u}}(\alpha_i, \alpha_{i+1}), \end{aligned}$$

which implies together with (3.4),

$$d_s(\tilde{u}, \tilde{f}_m(\tilde{u})) \leq \max_i w_{\tilde{u}}(\alpha_i, \alpha_{i+1}) + \frac{1}{m}.$$

By definition of $w'_{\tilde{u}}(\frac{1}{m})$, we conclude that

$$d_s(\tilde{u}, \tilde{f}_m(\tilde{u})) \leq w'_{\tilde{u}}(\frac{1}{m}) + \frac{1}{m}.$$

□

Lemma 3.2. *If K is a relatively compact subset of $F(R)$ in the d_s-topology, then*

$$\lim_{m\to\infty} \sup_{\tilde{u}\in K} d_s(\tilde{u}, \tilde{f}_m(\tilde{u})) = 0.$$

In particular, for each $\tilde{u} \in F(R)$,

$$\lim_{m\to\infty} d_s(\tilde{u}, \tilde{f}_m(\tilde{u})) = 0.$$

Proof. It follows immediately from (2.7) and Lemma 3.1. □

Lemma 3.3. *For each $\tilde{u}, \tilde{v}, \tilde{a}, \tilde{b} \in F(R)$, we have*

$$d_s(\tilde{u} + \tilde{a}, \tilde{v} + \tilde{b}) \leq d_s(\tilde{u}, \tilde{v}) + ||\tilde{a}|| + ||\tilde{b}||.$$

Proof. For a given $\epsilon > 0$, we find $t \in T$ such that

$$d_\infty(\tilde{u}, t \circ \tilde{v}) < d_s(\tilde{u}, \tilde{v}) + \epsilon, \tag{3.5}$$

$$\sup_\alpha |t(\alpha) - \alpha| < d_s(\tilde{u}, \tilde{v}) + \epsilon. \tag{3.6}$$

Then

$$\begin{aligned}
&d_\infty(\tilde{u} + \tilde{a},\ t \circ (\tilde{v} + \tilde{b})) \\
&\quad\leq\ d_\infty(\tilde{u} + \tilde{a},\ t \circ \tilde{v} + \tilde{a}) + d_\infty(t \circ \tilde{v} + \tilde{a},\ t \circ \tilde{v} + t \circ \tilde{b}) \\
&\quad=\ d_\infty(\tilde{u},\ t \circ \tilde{v}) + d_\infty(\tilde{a},\ t \circ \tilde{b}) \\
&\quad\leq\ d_s(\tilde{u}, \tilde{v}) + ||\tilde{a}|| + ||\tilde{b}|| + \epsilon \ \text{ by }\ (3.5)
\end{aligned}$$

This inequality and (3.6) implies that

$$d_s(\tilde{u} + \tilde{a}, \tilde{v} + \tilde{b}) \leq d_s(\tilde{u}, \tilde{v}) + ||\tilde{a}|| + ||\tilde{b}|| + \epsilon.$$

This completes the proof since $\epsilon > 0$ is arbitrary. □

Now we are in a position to prove the main theorem.

Proof of Theorem 3.1. Let $\epsilon > 0$ be given. By convex tightness of $\{\tilde{X}_n\}$, there exists a convex compact subset K of $F(R)$ such that

$$P(\tilde{X}_n \notin K) < \epsilon^{p/(p-1)} \text{ for all } n.$$

Without loss of generality, we can assume that $\tilde{0} \in K$.
Then,

$$\begin{aligned} E\|I_{\{\tilde{X}_n \notin K\}}\tilde{X}_n\| &\leq P(\tilde{X}_n \notin K)^{(p-1)/p}(E\|\tilde{X}_n\|^p)^{1/p} \\ &\leq M^{1/p}\epsilon. \end{aligned} \tag{3.7}$$

Now, if we denote

$$\tilde{Y}_n = I_{\{\tilde{X}_n \in K\}}\tilde{X}_n, \quad \tilde{Z}_n = I_{\{\tilde{X}_n \notin K\}}\tilde{X}_n,$$

then by Lemma 3.3,

$$\begin{aligned} & d_s\left(\frac{1}{n}\sum_{i=1}^n \tilde{X}_i, \ \frac{1}{n}\sum_{i=1}^n E\tilde{X}_i\right) \\ &= d_s\left(\frac{1}{n}\sum_{i=1}^n (\tilde{Y}_i + \tilde{Z}_i), \ \frac{1}{n}\sum_{i=1}^n (E\tilde{Y}_i + E\tilde{Z}_i)\right) \\ &\leq d_s\left(\frac{1}{n}\sum_{i=1}^n \tilde{Y}_i, \ \frac{1}{n}\sum_{i=1}^n \tilde{f}_m(\tilde{Y}_i)\right) && (3.8) \\ &\quad + d_s\left(\frac{1}{n}\sum_{i=1}^n \tilde{f}_m(\tilde{Y}_i), \ \frac{1}{n}\sum_{i=1}^n \tilde{f}_m(E\tilde{Y}_i)\right) && (3.9) \\ &\quad + d_s\left(\frac{1}{n}\sum_{i=1}^n \tilde{f}_m(E\tilde{Y}_i), \ \frac{1}{n}\sum_{i=1}^n E\tilde{Y}_i\right) && (3.10) \\ &\quad + \|\frac{1}{n}\sum_{i=1}^n \tilde{Z}_i\| + \|\frac{1}{n}\sum_{i=1}^n E\tilde{Z}_i\| \ . && (3.11) \end{aligned}$$

For (3.8), since K is convex and $\tilde{0} \in K$, we have

$$d_s\left(\frac{1}{n}\sum_{i=1}^n \tilde{Y}_i, \ \frac{1}{n}\sum_{i=1}^n \tilde{f}_m(\tilde{Y}_i)\right) \leq \sup_{\tilde{u} \in K} d_s(\tilde{u}, \tilde{f}_m(\tilde{u})).$$

Thus, by Lemma 3.2, there exists a m_0 such that

$$(3.8) < \epsilon \text{ for all } m \geq m_0.$$

Now, for (3.9), we have

$$\begin{aligned}(3.9) \;\; &\leq \;\; d_\infty\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{f}_m(\tilde{Y}_i),\; \frac{1}{n}\sum_{i=1}^{n}\tilde{f}_m(E\tilde{Y}_i)\right)\\ &\leq \;\; \sum_{k=1}^{m} h\left(\frac{1}{n}\sum_{i=1}^{n}L_{\frac{k}{m}}\tilde{Y}_i,\; \frac{1}{n}\sum_{i=1}^{n}L_{\frac{k}{m}}E\tilde{Y}_i\right).\end{aligned}$$

Since K is compact, we have, by (2.6),

$$\sup_{\tilde{u}\in K}||\tilde{u}|| \leq C < \infty \text{ for some constant } C.$$

If we denote

$$\tilde{X}_n = \{(X^1_{n,\alpha}, X^2_{n,\alpha}) \mid 0 \leq \alpha \leq 1\},$$

then

$$\tilde{Y}_n = \{(X^1_{n,\alpha}I_{\{\tilde{X}_n\in K\}},\; X^2_{n,\alpha}I_{\{\tilde{X}_n\in K\}} \mid 0 \leq \alpha \leq 1\},$$

and so, for each $\alpha \in [0,1]$,

$$|X^1_{n,\alpha}I_{\{\tilde{X}_n\in K\}}| \leq C, \quad |X^2_{n,\alpha}I_{\{\tilde{X}_n\in K\}}| \leq C.$$

Thus, by the strong law of large numbers for random variables,

$$\begin{aligned}&h\left(\frac{1}{n}\sum_{i=1}^{n}L_\alpha\tilde{Y}_i, \frac{1}{n}\sum_{i=1}^{n}L_\alpha E\tilde{Y}_i\right)\\ &\quad\leq \;\; \frac{1}{n}\left|\sum_{i=1}^{n}(X^1_{i,\alpha}I_{\{\tilde{X}_i\in K\}} - EX^1_{i,\alpha}I_{\{\tilde{X}_i\in K\}})\right|\\ &\qquad + \frac{1}{n}\left|\sum_{i=1}^{n}(X^2_{i,\alpha}I_{\{\tilde{X}_i\in K\}} - EX^2_{i,\alpha}I_{\{\tilde{X}_i\in K\}})\right|\\ &\quad\to \;\; 0 \text{ a.s. as } n \to \infty .\end{aligned}$$

Hence, for each given m, there exists a null set $A_m \in \mathcal{A}$ such that (3.9) converges to zero for all $\omega \notin A_m$. As for (3.10), by Lemma 3.2, there exists a m_1 such that

$$(3.10) \leq \sup_{\tilde{u}\in K} d_s(\tilde{u}, \tilde{f}_m(\tilde{u})) < \epsilon \text{ for all } m \geq m_1.$$

Now, for (3.11), we first note that

$$\begin{aligned}(3.11) \quad &\le \frac{1}{n}\sum_{i=1}^{n}||\tilde{Z}_i|| + \frac{1}{n}\sum_{i=1}^{n}E||\tilde{Z}_i|| \\ &= \frac{1}{n}\sum_{i=1}^{n}(||\tilde{Z}_i|| - E||\tilde{Z}_i||) + \frac{2}{n}\sum_{i=1}^{n}E||\tilde{Z}_i||.\end{aligned}$$

Since

$$\sum_{n=1}^{\infty}\frac{1}{n^p}E||\tilde{Z}_n|| \le \sum_{n=1}^{\infty}\frac{M}{n^p} < \infty,$$

by Chung's strong law of large numbers, there exists a null set $A \in \mathcal{A}$ such that the first term of the right-hand converges to zero for all $\omega \notin A$. For the second term, (3.7) yields

$$\frac{2}{n}\sum_{i=1}^{n}E||\tilde{Z}_i|| \le 2\epsilon M^{1/p}.$$

Now let $\Omega_0 = (\cup_{m=1}^{\infty}A_m) \cup A$. Then $P(\Omega_0) = \infty$ and for all $\omega \notin \Omega_0$ and $m \ge \max(m_0, m_1)$,

$$d_s\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i,\ \frac{1}{n}\sum_{i=1}^{n}E\tilde{X}_i\right) \le 2\epsilon(1 + M^{1/p}) \ \text{ as } \ n \to \infty\ .$$

This completes the proof.

Remark 1. In the above theorem, we assume that $\{\tilde{X}_n\}$ is convex tight. The need of convexity arises from the desired condition that a convex combination of elements $\{\tilde{u}\}$ of K, in particular, $\frac{1}{n}\sum_{i=1}^{n}\tilde{u}_i$, again belong to K. It remains an open problem whether the similar result holds if we replace convex tightness by tightness.

Remark 2. We note that not all *iid* sequence of fuzzy random variables are convex tight. For example, let $\tilde{u}_\lambda : R \to [0,1]$ be a fuzzy number defined by

$$\tilde{u}_\lambda(x) = \begin{cases} \lambda & \text{if } \ 0 \le x < 1, \\ 1 & \text{if } \ x = 1, \\ 0 & \text{elsewhere,} \end{cases}$$

where $\lambda \in [0,1]$, and let $\{\tilde{X}_n\}$ be *iid* with

$$P(\tilde{X}_1 \in \{\tilde{u}_\lambda : \lambda_1 \le \lambda \le \lambda_2\}) = \lambda_2 - \lambda_1.$$

Now suppose that there exists a convex compact set K such that $P(\tilde{X}_1 \notin K) < \epsilon$. Then K necessarily contains a set of the form

$$K_I = \{\tilde{u}_\lambda : \lambda \in I\},$$

where $I \subset [a, 1-a]$ and the Lebesque measure of I is larger than $1-2a-\epsilon$. Since K is convex and compact,

$$\overline{co}(K_I) \subset K,$$

where $\overline{co}(K_I)$ is the closed convex hull of K_I.
But by example 3.1 of Kim [12], $\overline{co}(K_I)$ is not compact. This is impossible. Hence $\{\tilde{X}_n\}$ is not convex tight. But while not being convex tight in this case, we can apply the result of Joo and Kim [8].

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