Journal of the Korean Statistical Society (2002), **31**: *1, pp 17–31*

Noninformative Priors for the Power Law Process†

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ABSTRACT

This paper considers noninformative priors for the power law process under failure truncation. Jeffreys' priors as well as reference priors are found when one or both parameters are of interest. These priors are compared in the light of how accurately the coverage probabilities of Bayesian credible intervals match the corresponding frequentist coverage probabilities. It is found that the reference priors have a definite edge over Jeffreys' prior in this respect.

Keywords: Credible sets, frequentist coverage probabilities, Jeffreys' prior, power law process, probability matching priors, reference priors.

1. Introduction

In reliability analysis, a suitable model for the occurrence of failures in a complex system subject to corrective (instantaneous) actions, as for example in prototype development, is represented by the so-called power law process, that is a non-homogeneous Poisson process $\{X(t), t \geq 0\}$ with intensity function $\lambda(t) = \beta t^{\beta-1}/\alpha^{\beta}$, $\alpha > 0$, $\beta > 0$ and mean value function $\mu(t) = E(X(t)) = (t/\alpha)^{\beta}$. The popularity of the power law process is due to its flexibility in representing various situations like reliability growth ($\beta < 1$), constant reliability ($\beta = 1$) and reliability decay ($\beta > 1$).

Two sampling schemes are usually considered, namely failure truncation and time truncation. In failure truncation, a predetermined number n of successive failure times of the $X(t)$ process is obtained. In time truncation, the observation of the failures is restricted to a pre-fixed interval $[0, t_0]$, and the number of failures, $X(t_0)$, along with the failure times during this interval are recorded.

†This work was supported by Korea Research Foundation Grant (KRF-99-041-D00100)

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The classical inferential procedures for the power law process under both sampling schemes are well-documented in the review article by Rigdon and Basu (1989). Prediction problem for times of future failures was considered by Lee and Lee (1978) and Engelhardt and Bain (1978).

From a Bayesian perspective, Kyparisis and Singpurwalla (1985) analyzed both time and failure truncation data by employing proper priors on α and β, and derived prediction distributions of future failure times and the number of failures in some future time interval. Guida *et al.* (1989) discussed point and interval estimations for α and β assuming failure truncation data and using several different choices of priors. Bar-Lev *et al.* (1992) indicated how a Bayesian approach unifies both failure and time truncation data in estimation and prediction problems in contrast to the frequentist approach. Recently Lingham and Sivaganesan (1997) considered a multiple hypotheses test for β based on the intrinsic Bayes factors under failure truncation.

The above last three papers are concerned with noninformative priors for the power law process. Bar-Lev *et al.* (1992) considered two types of priors $(\alpha)^{-1}$ and $(\alpha\beta)^{-1}$ for (α, β) following Jeffreys' rule in the location-scale situation (see Box and Tiao, 1973, pp. 56-57). They derived the noninformative priors using the argument based on the location-scale properties of the transformed Weibull distribution. The prior $(\alpha\beta)^{-1}$ was also used by Guida *et al.* (1989). Lingham and Sivaganesan (1997) derived the reference priors for (α, β) using the algorithm of Berger and Bernardo (1989) when β is the parameter of interest and α is the nuisance parameter.

The primary objective of this paper is to develop some objective priors on the power law process under failure truncation. We find Jeffreys' prior and the reference priors for this problem. These priors are compared in the light of how accurately the coverage probabilities of Bayesian credible intervals match the corresponding frequentist coverage probabilities. The present paper makes a modest attempt to justify the reference priors under the probability matching criterion for the power law process.

Jeffreys' prior is well known to be proportional to the positive square root of the Fisher information matrix. The reference priors, introduced originally in Bernardo (1979), and generalized further in Berger and Bernardo (1989, 1992a, b) are obtained by maximizing a suitable entropy distance. On the other hand, the matching idea goes back to Welch and Peers (1963) and Peers (1965). Interest in such priors revived with the work of Stein (1985) and Tibshirani (1989), and the last few years witnessed a phenomenal growth of research in this general area.

Among others, we may cite the work of Ghosh and Mukerjee (1993), Mukerjee and Dey (1993), Datta and Ghosh (1995), Datta and Ghosh (1995, 1996), Datta (1996), and Mukerjee and Ghosh (1997).

The outline of the remaining sections is as follows. In Section 2, we derive the expected partial derivatives of log-density up to the third order for the power law process model under failure truncation. Some results for matching priors are also given. In Section 3, we derive Jeffreys' prior as well as the reference priors. It is shown that Jeffreys' prior is not a first order matching prior, but the reference priors are second order probability matching priors when one of the parameters is of interest. Furthermore, when both parameters are of interest, the reference prior is the unique joint probability matching prior. In Section 4, the propriety of posteriors under a class of priors including Jeffreys' prior and the reference priors is proven. In Section 5, frequentist coverage probabilities of the posterior credible sets based on Jeffreys' prior and the reference priors are compared when the sample sizes are small. The small sample comparisons indicate that the reference priors perform better than Jeffreys' prior in terms of meeting the target coverage probabilities.

The utility of noninformative priors has always been questioned by subjectivists. Nevertheless, one cannot deny their pragmatic appeal. Indeed, one reason for the increasing popularity of Bayesian methods in recent years is due to the fact that even with little or no prior information, noninformative priors can be used routinely for data analysis. Moreover, these priors are worthwhile for studying the robustness of subjective Bayesian procedures. Thus, not surprisingly, over the years, various proposals have been put forward for the development of noninformative priors.

2. Preliminaries

Let $X_1, X_2, \cdots, X_n$ be the first n successive failure times of power law process with observed values $x_1 < x_2 < \cdots < x_n$. Then the joint density is

$$f(x_1, \cdots, x_n|\alpha, \beta) = (\frac{\beta}{\alpha})^n (\prod_{i=1}^{n} \frac{x_i}{\alpha})^{\beta-1} \exp\{-(\frac{x_n}{\alpha})^\beta\},\ 0 < x_1 < \cdots < x_n < \infty, \tag{2.1}$$

where α is the scale parameter and β is the shape parameter. In developing the noninformative priors for (α, β), the following expected partial derivatives of the log-density will be used.

$$I_{ij} = E_{(\alpha,\beta)}[\frac{\partial^{i+j}}{\partial\alpha^i\partial\beta^j}\log f(\mathbf{X}|\alpha,\beta)],\ i,j \geq 0,\ i+j = 2,3. \tag{2.2}$$

A standard computation provides the following results.

Lemma 2.1.

$$I_{20} = -\frac{n\beta^2}{\alpha^2},\ I_{11} = \frac{c_1}{\alpha},\ I_{02} = -\frac{n+c_2}{\beta^2},\ I_{30} = \frac{3n\beta^2+n\beta^3}{\alpha^3},$$
$$I_{21} = -\frac{2n\beta+c_1(\beta+1)}{\alpha^2},\ I_{12} = \frac{2c_1+c_2}{\alpha\beta},\ I_{03} = \frac{2n-c_3}{\beta^3},$$

where $c_1 = E[Z\log Z]$, $c_2 = E[Z(\log Z)^2]$, *and* $c_3 = E[Z(\log Z)^3]$, *and* Z *is a gamma random variable with parameters* $(n, 1)$.

The proof of Lemma 2.1 is straightforward and is omitted.

Since we are interested in developing a second order matching prior for a two-parameter case, we follow Mukerjee and Dey (1993), where θ_1 is a parameter of interest and θ_2 is a nuisance parameter. Let

$$I(\theta_1,\theta_2) = \begin{pmatrix} a_{20} & a_{11} \\ a_{11} & a_{02} \end{pmatrix}$$

be the Fisher information matrix for (θ_1, θ_2) based on one observation. Peers (1965) showed that π is a first order matching prior for θ_1 if and only if π is the solution of the partial differential equation

$$\frac{\partial}{\partial\theta_2}(\frac{a_{11}\pi}{a_{02}\sqrt{B}}) - \frac{\partial}{\partial\theta_1}(\frac{\pi}{\sqrt{B}}) = 0, \tag{2.3}$$

where $B = a_{20} - a_{11}^2/a_{02}$. Mukerjee and Dey (1993) showed that π is a second order matching prior for θ_1 if it satisfies (2.3) and an additional differential equation

$$T_2(\pi;\theta_1,\theta_2) \equiv \sum_{i=1}^{4} L_i(\pi;\theta_1,\theta_2) = 0, \tag{2.4}$$

where

$$\begin{aligned}
L_1(\pi;\theta_1,\theta_2) &= \{D_1^2B^{-1} - 2D_1^1D_2^1(a_{11}a_{02}^{-1}B^{-1}) + D_2^2(a_{11}^2a_{02}^{-2}B^{-1})\}/2, \\
L_2(\pi;\theta_1,\theta_2) &= -(D_1^2\pi - 2a_{11}a_{02}^{-1}D_1^1D_2^1\pi + a_{11}^2a_{02}^{-2}D_2^2\pi)/(2\pi B), \\
L_3(\pi;\theta_1,\theta_2) &= -[D_2^1\{(a_{02}B)^{-1}(K_{21} - 2a_{11}a_{02}^{-1}K_{12} + a_{11}^2a_{02}^{-2}K_{03})\pi\}]/(2\pi), \\
L_4(\pi;\theta_1,\theta_2) &= -[D_1^1\psi(\pi;\theta_1,\theta_2) - D_2^1\{a_{11}a_{02}^{-1}\psi(\pi;\theta_1,\theta_2)\}]/\pi.
\end{aligned}$$

Here $D_i^k = \partial^k/\partial\theta_i^k$, $K_{ij} = E_{(\theta_1,\theta_2)}\{D_1^i D_2^j \log p(X_1;\theta_1,\theta_2)\}$, and

$$\psi(\pi;\theta_1,\theta_2) = \frac{(K_{30} - 3a_{11}a_{02}^{-1}K_{21} + 3a_{11}^2 a_{02}^{-2}K_{12} - a_{11}^3 a_{02}^{-3}K_{03})\pi}{6B^2} + \frac{D_1^1\pi - a_{11}a_{02}^{-1}D_2^1\pi}{B}.$$

Also we are interested in obtaining the joint probability matching priors for (θ_1,θ_2). Ghosh and Mukerjee (1993) showed that a first order matching prior π for θ_1 satisfies

$$P_{(\theta_1,\theta_2)}[\sqrt{n}(\theta_1 - \hat{\theta}_1) \le z] = P_\pi[\sqrt{n}(\theta_1 - \hat{\theta}_1) \le z|\mathbf{X}] + o(n^{-\frac{1}{2}}), \tag{2.5}$$

for all z and (θ_1,θ_2). Here $\hat{\theta}_i$ is the posterior mode or the maximum likelihood estimator of θ_i. When π is a first order matching prior for each component of (θ_1,θ_2), it is true that such a prior is a first order joint matching prior for (θ_1,θ_2) when θ_1 and θ_2 are orthogonal. That is,

$$\begin{aligned} &P_{(\theta_1,\theta_2)}[\sqrt{n}(\theta_1 - \hat{\theta}_1) \le z_1, \sqrt{n}(\theta_2 - \hat{\theta}_2) \le z_2] \\ &= P_\pi[\sqrt{n}(\theta_1 - \hat{\theta}_1) \le z_1, \sqrt{n}(\theta_2 - \hat{\theta}_2) \le z_2|\mathbf{X}] + o(n^{-\frac{1}{2}}), \end{aligned} \tag{2.6}$$

for all (z_1,z_2) and (θ_1,θ_2). When θ_1 and θ_2 are not orthogonal, however, Datta (1996) found that having the common first order matching prior for each component is not sufficient to have a first order joint matching prior. Datta (1996) showed that a first order joint matching prior should satisfy two partial differential equations and an additional equation involving the Fisher information matrix.

3. Noninformative Priors

The Fisher information matrix of (α,β) by Lemma 2.1 is given by

$$I = -\begin{pmatrix} I_{20} & I_{11} \\ I_{11} & I_{02} \end{pmatrix} = \begin{pmatrix} \frac{n\beta^2}{\alpha^2} & -\frac{c_1}{\alpha} \\ -\frac{c_1}{\alpha} & \frac{n+c_2}{\beta^2} \end{pmatrix},$$

and the determinent of I is

$$|I| = \frac{n^2 + nc_2 - c_1^2}{\alpha^2}.$$

Hence the Jeffreys' prior is given by $\pi_J(\alpha,\beta) \propto 1/\alpha$.

Following the algorithm of Berger and Bernardo (1989), we obtain the reference priors when either α or β is the parameter of interest. Not surprisingly, the reference priors are of the same in this case.

Theorem 3.1. *The reference priors are given by*

$$\pi_R(\alpha,\beta)=\frac{1}{\alpha\beta}. \tag{3.1}$$

Proof. We will derive the reference prior when α is of interest only. We choose a sequence of compact sets for (α,β) by $(l_{1i},l_{2i})\times(k_{1i},k_{2i})$, so that $l_{1i},k_{1i}\to 0$ and $l_{2i},k_{2i}\to\infty$ as $i\to\infty$. Let I_A be the indicator of a set A. The conditional prior of β is

$$\pi_i(\beta|\alpha)=\frac{-I_{02}I_{(k_{1i},k_{2i})}(\beta)}{\int_{k_{1i}}^{k_{2i}}-I_{02}d\beta}=\frac{I_{(k_{1i},k_{2i})}(\beta)}{\beta[\log(k_{2i})-\log(k_{1i})]},$$

and the marginal reference prior of α is

$$\begin{aligned}\pi_i(\alpha) &\propto \exp\left\{\frac{1}{2}\int_{k_{1i}}^{k_{2i}}\frac{1}{\beta[\log(k_{2i})-\log(k_{1i})]}\log\left[\frac{(n^2+nc_2-c_1^2)/\alpha^2}{(n+c_2)/\beta^2}\right]d\beta\right\}\\ &= \frac{1}{\alpha}\sqrt{\frac{n^2+nc_2-c_1^2}{n+c_2}}\exp\left\{\frac{1}{2}\int_{k_{1i}}^{k_{2i}}\frac{2\log\beta}{\beta}d\beta\frac{1}{\log(k_{2i})-\log(k_{1i})}\right\}\\ &= \frac{1}{\alpha}\sqrt{\frac{n^2+nc_2-c_1^2}{n+c_2}}\exp\{\log(k_{2i})+\log(k_{1i})\}.\end{aligned}$$

Therefore the reference prior for $\{\alpha,\beta\}$ is

$$\pi_R(\alpha,\beta)=\lim_{i\to\infty}\frac{\pi_i(\alpha)\pi_i(\beta|\alpha)}{\pi_i(1.0)\pi_i(1.0|1.0)}=\frac{1}{\alpha\beta},$$

which is the desired result. □

Notice that the above priors are the same as the priors used by Bar-Lev *et al.* (1992). But they obtained their priors using different motivation. Also Lingham and Sivaganesan (1997) derived the reference prior when β is the parameter of interest only. Here the Jeffreys' prior differs from the reference prior when β is unknown. Furthermore, both forward and backward reference priors, when either parameter is of interest, are the same.

Probability matching priors when one of the two parameters is of interest are obtained by the following two theorems. We first consider the case when α is of interest.

Theorem 3.2. *Assume that α is the parameter of interest and β is a nuisance parameter. (i) The class of first order matching priors is given by*

$$\pi_M^{(1)}(\alpha,\beta)=\frac{1}{\alpha\beta}g_1\left(\frac{c_1}{n+c_2}\log\alpha+\frac{1}{\beta}\right), \tag{3.2}$$

where g_1 is a continuously differentiable function. (ii) The reference prior $\pi_R(\alpha,\beta)$ is a second order matching prior for α.

Proof. The first order matching prior is the solution of differential equation (2.3), where $(\theta_1,\theta_2)=(\alpha,\beta)$, $a_{20}=-I_{20}$, $a_{11}=-I_{11}$, $a_{02}=-I_{02}$ and $B=(\beta^2/\alpha^2)(n^2+nc_2-c_1^2)/(n+c_2)$. Then equation (2.3) simplifies to

$$\frac{c_1}{n+c_2}\beta\frac{\partial}{\partial\beta}(\beta\pi)+\frac{\partial}{\partial\alpha}(\alpha\pi)=0. \tag{3.3}$$

Then it can be shown that the solution of (3.3) is of the form

$$\pi_M^{(1)}(\alpha,\beta)=\frac{1}{\alpha\beta}g_1\left(\frac{c_1}{n+c_2}\log\alpha+\frac{1}{\beta}\right),$$

where g_1 is a continuously differentiable function. Thus the result (i) follows.

For (ii), note that the reference prior π_R is a first order matching prior obtained by taking $g_1(\cdot)\equiv 1$. It is enough to prove that π_R satisfies differential equation (2.4). Computing L_i in (2.4) is not easy. But we have

$$\begin{aligned} L_1 &= \frac{1}{k_1\beta^2}+\frac{c_1^2}{(n+c_2)^2k_1}\\ L_2 &= -\frac{1}{k_1\beta^2}-\frac{c_1}{k_1(n+c_2)\beta}-\frac{c_1^2}{(n+c_2)^2k_1}. \end{aligned}$$

Since $K_{ij}=I_{ij}$,

$$L_3=-\frac{c_1}{2k_1(n+c_2)\beta}.$$

The function $\psi(\pi_R;\alpha,\beta)$ in the formula L_4 is

$$\begin{aligned} \psi(\pi_R;\alpha,\beta) &= \frac{1}{6k_1^2\beta^3}(3n+n\beta)-\frac{c_1}{2k_1^2(n+c_2)\beta^3}[2n\beta+c_1(\beta+1)]\\ &\quad+\frac{c_1^2}{2k_1^2(n+c_2)^2\beta^2}(2c_1+c_2)+\frac{c_1^3}{6k_1^2(n+c_2)^3\beta^2}(2n-c_3)\\ &\quad+\frac{1}{k_1\beta^3}+\frac{c_1}{k_1(n+c_2)\beta^2}. \end{aligned}$$

Then

$$L_4=\frac{3c_1}{2k_1(n+c_2)\beta}.$$

Therefore, we have $T_2=L_1+L_2+L_3+L_4=0$ for any $\alpha,\beta>0$. □

Similarly, we can obtain the case when β is of interest.

Theorem 3.3. *Assume that β is the parameter of interest and α is a nuisance parameter. (i) The class of first order matching priors is given by*

$$\pi_M^{(2)}(\alpha,\beta) = \frac{1}{\alpha\beta} g_2(\frac{n}{c_1}\log\alpha + \frac{1}{\beta}), \tag{3.4}$$

where g_2 is a continuously differentiable function. (ii) The reference prior $\pi_R(\alpha,\beta)$ is a second order matching prior for β.

Proof. In case (i), $a_{20} = -I_{02} = (n+c_2)/\beta^2$, $a_{11} = -I_{11} = -c_1/\alpha$, $a_{02} = -I_{20} = n\beta^2/\alpha^2$ and $B = (1/\beta^2)(n^2 + nc_2 - c_1^2)/n$. Then equation (2.3) simplifies to

$$\beta\frac{\partial}{\partial\beta}(\beta\pi) + \frac{c_1}{n}\frac{\partial}{\partial\alpha}(\alpha\pi) = 0. \tag{3.5}$$

Using the same argument as in the differential equation (3.3), we obtain the first order matching prior

$$\pi_M^{(2)}(\alpha,\beta) = \frac{1}{\alpha\beta} g_2(\frac{n}{c_1}\log\alpha + \frac{1}{\beta}),$$

where g_2 is some continuously differentiable function.

For case (ii), it is enough to verify that the reference prior π_R satisfies equaticn (2.4). For this,

$$\begin{aligned} L_1 &= \frac{1}{k_2} + \frac{c_1^2}{k_2 n^2\beta^2} \\ L_2 &= -\frac{1}{k_2} - \frac{c_1}{k_2 n\beta} - \frac{c_1^2}{k_2 n^2\beta^2}. \end{aligned}$$

Since $K_{ij} = I_{ji}$, it can be shown that $L_3 = 0$. The function $\psi(\pi_R;\alpha,\beta)$ in L_4 is

$$\begin{aligned} \psi(\pi_R;\alpha,\beta) &= \frac{1}{6k_2^2\alpha}(2n - c_3) + \frac{c_1}{2k_2^2 n\alpha}(2c_1 + c_2) - \frac{c_1^2}{2k_2^2 n^2\alpha\beta}[2n\beta + c_1(\beta+1)] \\ &+ \frac{c_1^3}{6k_2^2 n^3\alpha\beta}(3n + n\beta) + \frac{1}{k_2\alpha} + \frac{c_1}{k_2 n\alpha\beta}, \end{aligned}$$

where $k_2 = (n^2 + nc_2 - c_1^2)/n$. Then $L_4 = c_1/(k_2 n\beta)$. Hence, we have $T_2 = L_1 + L_2 + L_3 + L_4 = 0$ for any $\alpha, \beta > 0$. □

Note that the Jeffreys' prior is not a first order matching prior for either α or β from Theorem 3.2 and 3.3.

Now we derive a joint probability matching prior when both parameters are of interest. Let $I^{-1} = (b_{ij})$ be the inverse of the Fisher information matrix and define $Q = (\rho_{ij})$, where $\rho_{ij} = b_{ij}/\sqrt{b_{ii}b_{jj}}$. Datta (1996) proved that if the matrix Q does not depend on the parameters, then the additional condition for the joint matching holds.

When both α and β are of interest, the Fisher information matrix is given by (2.1). It can be shown that $\rho_{11} = \rho_{22} = 1$ and $\rho_{12} = c_1/\sqrt{n^2 + nc_2}$. Therefore, all entries in the matrix Q are constants and Q is independent of (α, β).

Theorem 3.4. *The reference prior is the unique joint probability matching prior.*

Proof. It is enough to show that the reference prior is the only solution of two differential equations (3.3) and (3.5). In fact, these equations are equivalent to the equations

$$\frac{\partial}{\partial \alpha}(\alpha\pi) = 0 \quad \text{and} \quad \frac{\partial}{\partial \beta}(\beta\pi) = 0.$$

Thus the result follows immediately. □

4. Posterior Analysis

In this section, we establish the propriety of the posterior distribution for (α, β), corresponding to certain priors from a suitable class. For notational convenience we consider a general class of noninformative priors which is as follows:

$$\pi_b(\alpha, \beta) = \frac{1}{\alpha\beta^b}, \ \alpha > 0, \ \beta > 0,$$

where $b \geq 0$. Note that the Jeffreys' prior and the reference prior are obtained by taking $b = 0$ and 1, respectively. We provide sufficient conditions under which the posterior is proper.

Theorem 4.1. *(i) If $n = 1$, then the posterior distribution under the prior π_b is improper. (ii) If $n \geq 2$, then the posterior distribution under the prior π_b is proper.*

Proof. For $b \geq 0$, if $n = 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x_1|\alpha,\beta)}{\alpha\beta^b} d\alpha d\beta = \int_0^\infty \int_0^\infty \frac{\beta^{1-b} x_1^{\beta-1}}{\alpha^{\beta+1}} \exp\{-(\frac{x_1}{\alpha})^\beta\} d\alpha d\beta = \int_0^\infty \frac{x_1}{\beta^b} d\beta = \infty.$$

But when $n \geq 2$, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty f(\mathbf{x}|\alpha,\beta)\pi_b(\alpha,\beta) d\alpha d\beta \\ &= \int_0^\infty \int_0^\infty (\frac{\beta^{n-b}}{\alpha^{n+1}})(\prod_{i=1}^n \frac{x_i}{\alpha})^{\beta-1} \exp\{-(\frac{x_n}{\alpha})^\beta\} d\alpha d\beta \\ &= \int_0^\infty \frac{\Gamma(n)\beta^{n-b-1}(\prod_{i=1}^n x_i)^{\beta-1}}{x_n^{n\beta}} d\beta \\ &= \frac{\Gamma(n)\Gamma(n-b)}{\prod_{i=1}^n x_i (n\log x_n - \sum_{i=1}^n \log x_i)^{n-b}} < \infty. \end{aligned}$$

This completes the proof. □

Marginal inference on β is often of particular interest especially for reliability growth models. When $\beta = 1$, the power law process becomes a homogeneous Poisson process. For $\beta > 1$, the frequency of failures increases with the time and thus the system is deteriorating in a reliability sense, while for $\beta < 1$ the failure frequency decreases with time and thus the system is improving. If the posterior density of β is heavily concentrated over (0, 1), this would imply that there is an improvement over time. The marginal posterior densities of α and β have been given in Bar-Lev *et al.* (1992). But we need the marginal posterior cumulative distribution function (cdf) of α and β for our simulation studies to compare the frequentist coverage probabilities based on our noninformative priors.

Proposition 4.1. *(i) Under the prior $\pi_b(\alpha,\beta)$, the marginal posterior cdf of α is given by*

$$F_b(\alpha|\mathbf{X}) = \frac{(n\log x_n - \sum_{i=1}^n \log x_i)^{n-b}}{\Gamma(n)\Gamma(n-b)} \int_0^\infty s^{n-b-1} \prod_{i=1}^n x_i^s x_n^{-ns} I_\Gamma(n, \frac{x_n^s}{\alpha^s}) ds, \quad (4.1)$$

where $I_\Gamma(n,y)$ denotes the complementary incomplete gamma function, $\int_y^\infty s^{n-1} e^{-s} ds$. (ii) Under the prior $\pi_b(\alpha,\beta)$, the marginal posterior cdf of β is given by

$$F_b(\beta|\mathbf{X}) = \frac{(n\log x_n - \sum_{i=1}^n \log x_i)^{n-b}}{\Gamma(n-b)} \int_0^\beta s^{n-b-1} \prod_{i=1}^n x_i^s x_n^{-ns} ds. \quad (4.2)$$

The proof of Proposition 4.1 is straightforward and is omitted.

5. Small Sample Comparisons: A Simulation Study

Now we compare the reference priors with Jeffreys' prior for the power law process. We accomplish this by calculating the frequentist coverage probabilities of the different Bayesian credible intervals having the same posterior coverage probabilities. That is to say, the frequentist coverage of a $(1-\eta)th$ posterior quantile should be close to $1-\eta$.

Let $\alpha^\pi(\eta|\mathbf{X})$ be the posterior η-quantile of α given $\mathbf{X} = (X_1, \cdots, X_n)$ under the prior π. That is, $F_b(\alpha^\pi(\eta|\mathbf{X})|\mathbf{X}) = \eta$. Then the frequentist coverage probability of one-sided credible interval for α is

$$P_{(\alpha,\beta)}(\eta;\alpha) = P_{(\alpha,\beta)}(\alpha \leq \alpha^\pi(\eta|\mathbf{X})). \tag{5.1}$$

Similarly, we define $\beta^\pi(\eta|\mathbf{X})$ and $P_{(\alpha,\beta)}(\eta;\beta)$ as the posterior η-quantile of β and the corresponding frequentist coverage probability, respectively. If the marginal posterior distributions yield quantiles so that $P_{(\alpha,\beta)}(\eta;\alpha)$ (or $P_{(\alpha,\beta)}(\eta;\beta)$) is close to η, even if the sample sizes are small, then we have evidence that the chosen prior performs well with respect to the probability matching criterion.

The estimated $P_{(\alpha,\beta)}(\eta;\alpha)$ and $P_{(\alpha,\beta)}(\eta;\beta)$ when $\eta = 0.05, 0.95$, $n = 2, 5, 10$ and $\beta = 0.5, 1, 5$ are shown in Table 5.1 and 5.2. Using Proposition 4.1, we easily see that $P_{(\alpha,\beta)}(\eta;\alpha)$ and $P_{(\alpha,\beta)}(\eta;\beta)$ do not depend on α under the prior π_b. So it is enough to consider a specific α. We choose $\alpha = 1$. Numerical values in Table 5.1 and 5.2 were computed in the following way. For fixed (α,β), 10,000 random samples $(X_1, \cdots, X_n)$ are generated from the power law process with parameters α and β. Then $F_b(\alpha|\mathbf{X})$ and $F_b(\beta|\mathbf{X})$ are computed for $b = 0, 1$ and each data $\mathbf{X} = (X_1, \cdots, X_n)$ based on Proposition 4.1. Here $b = 0, 1$ corresponds π_J and π_R, respectively. Note that under a prior π, for fixed $\mathbf{X}$, $\eta \leq \alpha^\pi(\eta|\mathbf{X})$ if and only if $F(\alpha^\pi(\eta|\mathbf{X})|\mathbf{X}) \leq \eta$. Then under a prior π, $P_{(\alpha,\beta)}(\eta;\alpha)$ can be estimated by the relative frequency of $F_b(\eta|\mathbf{X}) \leq \eta$.

For the cases presented in Table 5.1 and 5.2, we see that the estimated frequentist coverage probabilities of the upper 5% and 95% posterior quantiles under the reference priors π_R are much closer to the target coverage probabilities 0.05 and 0.95 than values under Jeffreys' prior π_J for different small sample sizes. These comparisons indicate that the reference prior performs better than the Jeffreys' prior in terms of matching the frequentist coverage probabilities. Clearly the reference priors should be recommended for a Bayesian analysis in the power law process whether one or both parameters are of interest.

TABLE 5.1 *Estimated frequentist coverage probabilities of posterior quantiles of α under Jeffreys' prior(π_J) and the reference prior (π_R) for different sample sizes*

		π_J		π_R	
β	n	$P(0.05;\alpha)$	$P(0.95;\alpha)$	$P(0.05;\alpha)$	$P(0.95;\alpha)$
0.05	2	0.2241	0.9239	0.0537	0.9522
0.05	5	0.1259	0.9645	0.0516	0.9482
0.05	10	0.0942	0.9715	0.0518	0.9507
1.0	2	0.2183	0.8966	0.0478	0.9573
1.0	5	0.1318	0.9687	0.0517	0.9512
1.0	10	0.1261	0.9606	0.0503	0.9510
5.0	2	0.2665	0.9994	0.0484	0.9777
5.0	5	0.1274	0.9716	0.0504	0.9460
5.0	10	0.0944	0.9697	0.0503	0.9505

TABLE 5.2 *Estimated frequentist coverage probabilities of posterior quantiles of β under Jeffreys' prior(π_J) and the reference prior (π_R) for different sample sizes*

		π_J		π_R	
β	n	$P(0.05;\beta)$	$P(0.95;\beta)$	$P(0.05;\beta)$	$P(0.95;\beta)$
0.05	2	0.2955	0.9916	0.0515	0.9464
0.05	5	0.1328	0.9821	0.0509	0.9516
0.05	10	0.0982	0.9718	0.0506	0.9498
1.0	2	0.3019	0.9899	0.0513	0.9536
1.0	5	0.1368	0.9802	0.0504	0.9514
1.0	10	0.1017	0.9768	0.0504	0.9500
5.0	2	0.2991	0.9909	0.0515	0.9464
5.0	5	0.1338	0.9807	0.0509	0.9516
5.0	10	0.1026	0.9726	0.0503	0.9508

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