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THE METHOD OF QUASILINEARIZATION AND A THREE-POINT BOUNDARY VALUE PROBLEM

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ABSTRACT. The method of quasilinearization generates a monotone iteration scheme whose iterates converge quadratically to a unique solution of the problem at hand. In this paper, we apply the method to two families of three-point boundary value problems for second order ordinary differential equations: Linear boundary conditions and nonlinear boundary conditions are addressed independently. For linear boundary conditions, an appropriate Green's function is constructed. For nonlinear boundary conditions, we show that these nonlinearities can be addressed similarly to the nonlinearities in the differential equation.

1. Introduction

In this paper, we shall first apply the method of quasilinearization to the two-point boundary value problem (BVP),

$$x''(t) = f(t, x(t)), \quad t \in [0, 1], \tag{1.1}$$

$$x(0) = a, \quad x(1) = x(1/2), \tag{1.2}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}^2$ is continuous. We shall obtain a Green's function associated to the BVP, (1.1), (1.2); we shall then indicate how the method of quasilinearization applies to the BVP, (1.1), (1.2).

We shall then apply the method of quasilinearization to a BVP with nonlinear boundary conditions,

$$x''(t) = f(t, x(t)), \quad t \in [0, 1], \tag{1.3}$$

$$x(0) = a, \quad x(1) = g(x(1/2)). \tag{1.4}$$

We shall assume the nonlinearity, g, is continuous.

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The method of quasilinearization has recently been studied and extended extensively. It is generating a rich history beginning with the works by Bellman and Bellman and Kalba [1, 2]. Lakshmikantham, Leela, Vatsala, and many co-authors have extensively developed the method and have applied the method to a wide range of problems. We refer the reader to the recent work by Lakshmikantham and Vatsala [14] and the extensive bibliography found there. The method has been applied to two-point boundary value problems for ordinary differential equations and we refer the reader to the papers, [19, 17, 18, 8, 6, 7]. We also point out the Cabada and Nieto and co-authors ([3, 4, 5], for example) have recently been providing new and interesting variations and applications of the method.

Multipoint boundary value problems for second order ordinary differential equations have also been receiving considerable attention lately. Kiguradze and Lomtatidze [13], and Lomtatidze [15, 16] have studied closely related problems. Gupta and co-authors ([9, 10, 11], for example) have been thoroughly studying problems related to the BVP, (1.1), (1.2). We point out that the three point boundary conditions considered here can be extended to other multipoint point conditions in many ways. Under suitable conditions, the methods we develop here will apply. We consider the specific conditions (1.2) or (1.4) for the sake of simplicity in exposition.

2. The BVP, (1.1), (1.2)

We begin by establishing the Green's function of the BVP, (1.1), (1.2). Define

$$G(t,s) = \begin{cases} G_1(t,s), & 0 < s \le 1/2, \\ G_2(t,s), & 1/2 < s < 1, \end{cases} \tag{2.1}$$

where

$$G_1(t,s) = \begin{cases} -t, & 0 \le t < s \le 1, \\ -s, & 0 \le s < t \le 1, \end{cases}$$

and

$$G_2(t,s) = \begin{cases} 2(s-1)t, & 0 \le t < s \le 1, \\ -s + (2s-1)t, & 0 \le s < t \le 1. \end{cases}$$

It is a straightforward calculation to show that $x(t)$ is solution of (1.1), (1.2), if and only if

$$x \in C[0,1] \text{ and } x(t) = a + \int_0^1 G(t,s)f(s,x(s))\,ds.$$

THEOREM 2.1. *Assume that* $f : [0,1] \times \mathbb{R} \to \mathbb{R}^2$ *is continuous and bounded. Then the BVP,* (1.1), (1.2), *has a solution.*

Proof. The proof is a standard application of Schauder's fixed point theorem. □

We now indicate how to employ the quasilinearization method. We shall leave the details to the next section. We only point out here that since $G(t,s) < 0$ on $(0,1) \times (0,1)$, the details are completely standard.

DEFINITION 2.1. Let $\alpha, \beta \in C^2[0,1]$. We say α is a lower solution of the BVP, (1.1), (1.2), if

$$\alpha''(t) \geq f(t,\alpha(t)), \quad t \in [0,1],$$
$$\alpha(0) \leq a, \quad \alpha(1) \leq \alpha(1/2).$$

We say β is an upper solution of the BVP, (1.1), (1.2), if

$$\beta''(t) \leq f(t,\beta(t)), \quad t \in [0,1],$$
$$\beta(0) \geq a, \quad \beta(1) \geq \beta(1/2).$$

THEOREM 2.2. *Assume* f *is continuous and assume that* $f_x > 0$ *on* $[0,1] \times \mathbb{R}$. *Assume* β *and* α *are upper and lower solutions of the BVP,* (1.1), (1.2) *respectively. Then*

$$\alpha(t) \leq \beta(t), \quad t \in [0,1].$$

Proof. Define $h(t) = \alpha(t) - \beta(t)$. For the sake of contradiction, suppose $h(t) > 0$ for some $t \in (0,1]$. Note that $h(0) \leq 0$. Let

$$t_0 = \inf\{\tau \in [0,1] : h(\tau) \geq h(t), 0 \leq t \leq 1\}.$$

By continuity, $0 < t_0$. Moreover, by the definition of upper and lower solutions, $h(1/2) \geq h(1)$. So, $t_0 \in (0,1)$. Thus,

$$h''(t_0) = \alpha''(t_0) - \beta''(t_0) \geq f(t_0,\alpha(t_0)) - f(t_0,\beta(t_0)) > 0,$$

since $f_x > 0$. We obtain a contradiction since $h(t_0)$ is a local maximum and $h''(t_0) \leq 0$. Therefore,

$$\alpha(t) \leq \beta(t), \quad 0 \leq t \leq 1.$$ □

COROLLARY 2.1. *Assume f is continuous and assume that $f_x > 0$ on $[0,1] \times \mathbb{R}$. Then solutions of the BVP,* (1.1), (1.2), *are unique.*

Proof. A solution is an upper (lower) solution. □

THEOREM 2.3. *Assume f is continuous on $[0,1] \times \mathbb{R}$. Assume that there exist an upper solution, β, and a lower solution, α, of the BVP,* (1.1), (1.2). *Moreover, assume that*

$$\alpha(t) \leq \beta(t), \quad t \in [0,1].$$

Then, there exists a solution x, of the BVP, (1.1), (1.2), *satisfying*

$$\alpha(t) \leq x(t) \leq \beta(t), \quad t \in [0,1].$$

Proof. Truncate $f(t,x)$ as follows:

$$F(t,x) = \begin{cases} f(t,\beta) + \frac{x-\beta}{1+x-\beta} & \text{if } x(t) > \beta(t); \\ f(t,x) & \text{if } \alpha(t) \leq x(t) \leq \beta(t); \\ f(t,\alpha) + \frac{x-\alpha}{1+|x-\alpha|} & \text{if } x(t) < \alpha(t). \end{cases}$$

Since F is bounded and continuous on $[0,1] \times \mathbb{R}$, it follows from Theorem 2.1 that there exists a solution, x, of the BVP, $x''(t) = F(t,x(t))$, $0 \leq t \leq 1$, with the boundary conditions, (1.2). The proof is complete once we show

$$\alpha(t) \leq x(t) \leq \beta(t), \quad t \in [0,1].$$

This argument is analogous to the proof of Theorem 2.2. Set $h(t) = \alpha(t) - x(t)$. Assume the inequalitiy is false and define t_0 as in the proof of Theorem 2.2. Argue that $t_0 \in (0,1)$ and then argue that the sign of $h''(t_0)$ gives a contradiction. This argument is standard in the applications of upper and lower solutions and we refer you to [12]. □

The development of the quasilinearization method follows in a standard way. Since many of the standard details will be specifically addressed in the next section, we shall state the following theorem without proof.

THEOREM 2.4. *Assume f, f_x, f_{xx} are continuous on $[0,1] \times \mathbb{R}$. Assume $f_x > 0$ on $[0,1] \times \mathbb{R}$. Moreover, assume*

$$f_{xx}(t,x) \geq 0, \quad (t,x) \in [0,1] \times \mathbb{R}. \tag{2.2}$$

Assume α_0 *is a lower solution of the BVP,* (1.1), (1.2), *and assume that* β_0 *is an upper solution of the BVP,* (1.1), (1.2). *Then there exist monotone sequences,* $\{\alpha_n\}$, $\{\beta_n\}$, *that converge monotonically and quadratically in the space of continuous functions on* $[0,1]$ *to the unique solution,* x, *of the BVP,* (1.1), (1.2).

3. The BVP, (1.3), (1.4)

We shall model the development of Section 3 by the development of Section 2. Here, we supply the details.

THEOREM 3.1. *Assume that* $f : [0,1] \times \mathbb{R} \to \mathbb{R}^2$ *is continuous and bounded. Also, assume that* $g : \mathbb{R} \to \mathbb{R}$ *is continuous and bounded. Then the BVP,* (1.3), (1.4) *has a solution.*

Proof. Again, the proof is a standard application of the Schauder fixed point theorem once an appropriate fixed point operator is developed. Let $G(t,s)$ denote the usual Green's function for the conjugate or Dirichlet BVP. In particular, set

$$G(t,s) = \begin{cases} t(s-1), & 0 \le t < s \le 1, \\ s(t-1), & 0 \le s < t \le 1. \end{cases} \tag{3.1}$$

It is a straightforward calculation to show that $x(t)$ is a solution of the BVP, (1.3), (1.4), if and only if $x \in C[0,1]$ and

$$x(t) = a(1-t) + g(x(1/2))t + \int_0^1 G(t,x) f(s,x(s))ds. \qquad \square$$

DEFINITION 3.1. Let α, β be such that α'', β'' are continuous. We say α is a lower solution of the BVP, (1.3), (1.4), if

$$\alpha''(t) \ge f(t,\alpha(t)), \quad t \in [0,1],$$

$$\alpha(0) \le a, \quad \alpha(1) \le g(\alpha(1/2)).$$

We say β is an upper solution of the BVP, (1.3), (1.4), if

$$\beta''(t) \le f(t,\beta(t)), \quad t \in [0,1],$$

$$\beta(0) \ge a, \quad \beta(1) \ge g(\beta(1/2)).$$

THEOREM 3.2. *Assume f is continuous and assume that $f_x > 0$ on $[0,1] \times \mathbb{R}$. Assume g is continuous and assume that $0 \le g' \le 1$ on $\mathbb{R}$. Assume β and α are upper and lower solutions of the BVP,* (1.3), (1.4) *respectively. Then*

$$\alpha(t) \le \beta(t), \quad t \in [0,1].$$

Proof. The proof for this theorem is analogous to the one in Theorem 2.2. The condition, $0 \le g' \le 1$ on $\mathbb{R}$, is employed to argue that $t_0 < 1$ where t_0 has been defined in the proof of Theorem 2.2. □

COROLLARY 3.1. *Assume f is continuous and assume that $f_x > 0$ on $[0,1] \times \mathbb{R}$. Assume g is continuous and assume that $0 \le g' \le 1$ on $\mathbb{R}$. Then solutions of the BVP,* (1.3), (1.4), *are unique.*

THEOREM 3.3. *Assume f is continuous on $[0,1] \times \mathbb{R}$. Assume g is continuous on $\mathbb{R}$ and assume that $0 \le g' \le 1$ on $\mathbb{R}$. Assume that there exist an upper solution, β, and a lower solution, α, of the BVP,* (1.3), (1.4). *Moreover, assume that*

$$\alpha(t) \le \beta(t), \quad t \in [0,1].$$

Then, there exists a solution x, of the BVP, (1.3), (1.4), *satisfying*

$$\alpha(t) \le x(t) \le \beta(t), \quad t \in [0,1].$$

Proof. Define F as in the proof of Theorem 2.3. Define a truncation, G, of g by

$$G(x) = \begin{cases} g(\beta(1/2)) & \text{if } x > \beta(1/2); \\ g(x) & \text{if } \alpha(1/2) \le x \le \beta(1/2); \\ g(\alpha(1/2)) & \text{if } x < \alpha(1/2). \end{cases}$$

By Theorem 3.1 there exists a solution, x of the BVP,

$$x''(t) = F(t, x(t)), \quad t \in [0,1],$$
$$x(0) = a, \quad x(1) = G(x(1/2)).$$

The proof is complete when we show

$$\alpha(t) \le x(t) \le \beta(t), \quad 0 \le t \le 1,$$

and the proof is completely analogous to the proof of Theorem 2.3 once the observation from the proof of Theorem 3.2 is employed. We point out that G satisfies $0 \le G' \le 1$ on $[\alpha(1/2), \beta(1/2)]$. □

THEOREM 3.4. *Assume f, f_x, f_{xx} are continuous on $[0,1]\times\mathbb{R}$. Assume $f_x > 0$ on $[0,1]\times\mathbb{R}$ and assume (2.2) is satisfied. Assume g, g', g'' are continuous on $\mathbb{R}$. Assume $0 \le g' < 1$ on $\mathbb{R}$ and assume*

$$g''(x) \le 0, \quad x \in \mathbb{R}. \tag{3.2}$$

Assume α_0 is a lower solution of the BVP, (1.3), (1.4), and assume that β_0 is an upper solution of the BVP, (1.3), (1.4). Then there exist monotone sequences, $\{\alpha_n\}$, $\{\beta_n\}$, that converge in the space of continuous functions on $[0,1]$ to the unique solution, x, of the BVP, (1.3), (1.4). Moreover, the convergence is quadratic.

Proof. First note that for any $t \in [0,1]$ and $x, y \in \mathbb{R}$,

$$f(t,x) \ge f(t,y) + f_x(t,y)(x-y) \tag{3.3}$$

follows from (2.2) and from (3.2) it follows that

$$g(x) \le g(y) + g'(y)(x-y). \tag{3.4}$$

Define two linearizations, F and G, of f.

$$F(t,x;\alpha_0,\beta_0) = f(t,\alpha_0) + f_x(t,\beta_0)(x-\alpha_0),$$
$$G(t,x;\beta_0) = f(t,\beta_0) + f_x(t,\beta_0)(x-\beta_0).$$

Define two linearizations, h and $\hat{h}$, of g.

$$h(x(1/2);\alpha_0,\beta_0) = g(\alpha_0(1/2)) + g'(\beta_0(1/2))(x(1/2)-\alpha_0(1/2)),$$
$$\hat{h}(x(1/2);\beta_0) = g(\beta_0(1/2)) + g'(\beta_0(1/2))(x(1/2)-\beta_0(1/2)).$$

We shall now consider two BVPs in addition to BVP, (1.3), (1.4). Consider the BVP,

$$x''(t) = F(t,x;\alpha_0,\beta_0), \quad t \in [0,1], \tag{3.5}$$

$$x(0) = a, \quad x(1) = h(x(1/2);\alpha_0,\beta_0). \tag{3.6}$$

In addition, consider the BVP,

$$x''(t) = G(t,x;\beta_0), \quad t \in [0,1], \tag{3.7}$$

$$x(0) = a, \quad x(1) = \hat{h}(x(1/2);\beta_0). \tag{3.8}$$

The outline of the proof is as follows: First, show α_0 is a lower solution of the BVP, (3.5), (3.6), and show β_0 is an upper solution of the BVP, (3.5), (3.6). Apply Theorem 3.3 and obtain a solution, α_1, of the BVP, (3.5), (3.6), satisfying

$$\alpha_0(t) \le \alpha_1(t) \le \beta_0(t), \quad t \in [0,1]. \tag{3.9}$$

Note that Theorem 3.3 applies since $h' = g'(\beta_0(1/2))$. Second, show α_0 is a lower solution of the BVP, (3.7), (3.8), and show β_0 is an upper solution of the BVP, (3.7), (3.8). Apply Theorem 3.3 to obtain a solution, β_1, of the BVP, (3.7), (3.8), satisfying

$$\alpha_0(t) \leq \beta_1(t) \leq \beta_0(t), \quad t \in [0,1]. \tag{3.10}$$

Finally, show α_1 is a lower solution of the BVP, (1.3), (1.4), and show β_1 is an upper solution of the BVP, (1.3), (1.4). Apply Theorem 3.2 and obtain the inequality,

$$\alpha_1(t) \leq \beta_1(t), \quad t \in [0,1]. \tag{3.11}$$

In particular,

$$\alpha_0(t) \leq \alpha_1(t) \leq \beta_1(t) \leq \beta_0(t), \quad t \in [0,1].$$

One continues this process by induction to obtain

$$\alpha_n(t) \leq \alpha_{n+1}(t) \leq \beta_{n+1}(t) \leq \beta_n(t), \quad t \in [0,1], \quad n = 0, 1, \ldots,$$

where α_{n+1} satisfies the BVP,

$$x''(t) = F(t, x; \alpha_n, \beta_n), \quad t \in [0,1],$$

$$x(0) = a, \quad x(1) = h(x(1/2); \alpha_n, \beta_n),$$

and β_{n+1} satisfies the BVP,

$$x''(t) = G(t, x; \beta_n), \quad t \in [0,1],$$

$$x(0) = a, \quad x(1) = \hat{h}(x(1/2); \beta_n).$$

$[0,1]$ is compact, and the convergence is monotone; it follows that that convergence of each sequence, $\{\alpha_n\}$ or $\{\beta_n\}$, is uniform. It is then straightforward to see that if x is a limit point of either sequence, then

$$x(t) = a(1-t) + g(x(1/2))t + \int_0^1 G(t,x) f(s, x(s)) ds.$$

So, as in Theorem 3.1, x is a solution of the BVP, (1.3), (1.4). Uniqueness of x follows by Corollary 3.1.

Here, we shall obtain each of (3.9), (3.10) and (3.11). Since

$$F(t, \alpha_0; \alpha_0, \beta_0) = f(t, \alpha_0)$$

and

$$\alpha_0(1) \leq g(\alpha_0(1/2)) = h(\alpha_0(1/2); \alpha_0, \beta_0),$$

α_0 is trivially a lower solution of the BVP, (3.5), (3.6). As for β_0, there exists $c_0 \in (\alpha_0(1/2), \beta_0(1/2))$ such that

$$\begin{aligned} & g(\beta_0(1/2)) - h(\beta_0(1/2); \alpha_0, \beta_0) \\ &= g(\beta_0(1/2)) - g(\alpha_0(1/2)) - g'(\beta_0(1/2))(\beta_0(1/2) - \alpha_0(1/2)) \\ &= (g'(c_0) - g'(\beta_0(1/2))(\beta_0(1/2) - \alpha_0(1/2)) \geq 0 \end{aligned}$$

since g' is decreasing. Moreover,

$$f(t, \beta_0) \leq f(t, \alpha_0) + f_x(t, \beta_0)(\beta_0 - \alpha_0)$$

by (3.3). Thus, β_0 is an upper solution of the BVP, (3.5), (3.6). By Theorem 3.3 there is a solution α_1 of the BVP, (3.5), (3.6), satisfying (3.9).

Similarly, β_0 is an upper solution of the BVP, (3.7), (3.8) since

$$G(t, \beta_0; \beta_0) = f(t, \beta_0), \quad g(\beta_0(1/2)) = \hat{h}(\beta_0(1/2); \beta_0).$$

As for α_0, there exists $c_0 \in (\alpha_0(1/2), \beta_0(1/2))$ such that

$$\begin{aligned} & \hat{h}(\alpha_0(1/2); \beta_0) - g(\alpha_0(1/2)) \\ &= g(\beta_0(1/2)) - g(\alpha_0(1/2)) + g'(\beta_0(1/2))(\alpha_0(1/2) - \beta_0(1/2)) \\ &= (g'(c_1) - g'(\beta_0(1/2))(\beta_0(1/2) - \alpha_0(1/2)) \geq 0, \end{aligned}$$

and $f(t, \alpha_0) \geq f(t, \beta_0) + f_x(t, \beta_0)(\alpha_0 - \beta_0)$ follows from (3.3). By Theorem 3.3 there exists a solution β_1 of the BVP, (3.7), (3.8), satisfying (3.10).

To obtain (3.11), apply the mean value theorem appropriately to obtain c_1, c_2 satisfying $\alpha_0 \leq c_1 \leq c_2 \leq \beta_0$ such that

$$\begin{aligned} & \alpha_1''(t) - f(t, \alpha_1) \\ &= f(t, \alpha_0) + f_x(t, \beta_0)(\alpha_1(t) - \alpha_0(t)) - f(t, \alpha_1) \\ &= f_x(t, c_1)(\alpha_0 - \alpha_1) + f_x(t, \beta_0)(\alpha_1 - \alpha_0) \\ &= f_{xx}(t, c_2)(\beta_0 - c_1)(\alpha_1 - \alpha_0) \\ &\geq 0. \end{aligned}$$

Also, there exists $c_3 \in (\alpha_0(1/2), \alpha_1(1/2))$ such that

$$\begin{aligned} & g(\alpha_1(1/2)) - \alpha_1(1) \\ &= g(\alpha_1(1/2)) - g(\alpha_0(1/2)) - g'(\beta_0(1/2))(\alpha_1(1/2) - \alpha_0(1/2)) \\ &= (g'(c_3) - g'(\beta_0(1/2)))(\alpha_1(1/2) - \alpha_0(1/2)) \\ &> 0 \end{aligned}$$

Hence, α_1 is a lower solution of the BVP, (1.3), (1.4). Similarly, β_1 is an upper solution of the BVP, (1.3), (1.4). Apply Theorem 3.2 and we have obtained (3.11). □

COROLLARY 3.2. *The convergence of each of the sequences,* $\{\alpha_n\}$ *and* $\{\beta_n\}$*, is quadratic.*

Proof. Set $q_n = \beta_n - x$, $p_n = x - \alpha_n$, where x denotes the unique solution of the BVP, (1.3), (1.4). Set

$$e_n = \max\{||q_n||, ||p_n||\},$$

where $||\cdot||$ denotes the supremum norm on $C[0,1]$.

We show the quadratic convergence with q_n. Details for the quadratic convergence with p_n are similar. Note, $q_n \geq 0$ follows from the monotone convergence. Apply the mean value theorem; there exist

$$x \leq c_1 \leq c_2 \leq \beta_n$$

such that

$$\begin{aligned}
&q''_{n+1}(t) \\
&= f(t,\beta_n) + f_x(t,\beta_n)(\beta_{n+1} - \beta_n) - f(t,x) \\
&= f_x(t,c_1)(\beta_n - x) + f_x(t,\beta_n)(\beta_{n+1} - \beta_n) \\
&= f_x(t,c_1)(\beta_n - x) + f_x(t,\beta_n)((\beta_{n+1} - x) + (x - \beta_n)) \\
&= (f_x(t,c_1) - f_x(t,\beta_n))q_n + f_x(t,\beta_n)q_{n+1} \\
&= f_{xx}(t,c_2)(c_1 - \beta_n)q_n + f_x(t,\beta_n)q_{n+1} \\
&\geq -f_{xx}(t,c_2)(q_n)^2 + f_x(t,\beta_n)q_{n+1} \\
&\geq -M||q_n||^2,
\end{aligned}$$

where M is a bound on $|f_{xx}|$ for $t \in [0,1]$, $x(t) \leq c_2 \leq \beta_0(t)$. Refer to (3.1) to note that $G(t,s) < 0$ on $(0,1) \times (0,1)$. Thus,

$$G(t,s)q''_{n+1}(s) \leq -MG(t,s)||q_n||^2 = M|G(t,s)|||q_n||^2$$

for $s \in [0,1]$. Then there exists c_0 such that $x(1/2) \leq c_0 \leq \beta_n(1/2)$ such that

$$\begin{aligned}
q_{n+1}(t) &= (\hat{h}(\beta_{n+1}(1/2);\beta_n) - g(x(1/2))t + \int_0^1 G(t,s)q''_{n+1}(s)ds \\
&\leq [g(\beta_n(1/2)) + g'(\beta_n(1/2))(\beta_{n+1}(1/2) - \beta_n(1/2)) - g(x(1/2))]t
\end{aligned}$$

$$
\begin{aligned}
&+M||q_n||^2 \int_0^1 |G(t,s)|ds \\
&= [(g'(c_0) - g'(\beta_n(1/2)))q_n(1/2) + g'(\beta_n(1/2))q_{n+1}(1/2)]t + M_1||q_n||^2 \\
&= [g''(c_1)(c_0 - \beta_n(1/2))q_n(1/2) + g'(\beta_n(1/2))q_{n+1}(1/2)]t + M_1||q_n||^2 \\
&\leq g'(\beta_n(1/2))||q_{n+1}|| + (M_1 + M_2)||q_n||^2 \\
&\leq \lambda||q_{n+1}|| + M_3||q_n||^2,
\end{aligned}
$$

where $||g'|| \leq \lambda < 1$, M_1 provides a bound on $M\int_0^1 |G(t,s)|ds$, M_2 provides a bound on $|g''|$ on $[x(1/2), \beta_0(1/2)]$ and $M_3 = M_1 + M_2$. Solve algebraically for $||q_{n+1}||^2$ to obtain

$$||q_{n+1}|| \leq \frac{M_3}{1-\lambda}||q_n||^2. \qquad \square$$

REMARK 3.1. The boundary conditions, (1.2), are a special case of the boundary conditions, (1.4), with $g(x) = x$. Since, $g' = 1$ and $g'' = 0$, the BVP, (1.1), (1.2), can be considered a limiting case for the family of BVPs addressed in Theorem 3.4; we solve algebraically for $||q_{n+1}||$ in the proof of Theorem 3.4, and so, the quadratic convergence in Theorem 2.4 is not obtained using the methods of Section 3. Thus, the construction of the Green's function in Section 2 is of interest.

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