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MATHIEU GROUP COVERINGS AND GOLAY CODES

IKKWON YIE

ABSTRACT. We associate binary codes to polynomials over fields of characteristic two and show that the binary Golay codes are associated to the Mathieu group polynomials in characteristic two. We give two more polynomials whose Galois group is M_{12} but different self-orthogonal binary codes are associated. Also, we find a family of M_{24}-coverings which includes previous ones.

1. Introduction

Let k be a field of characteristic two. In [1] Abhyankar proved that the Galois group $\mathrm{Gal}(F, k(X))$ of the polynomial $F = Y^{23} + XY^3 + 1$ over the rational function field $k(X)$ is isomorphic to M_{23}, where M_n is the Mathieu group of degree n. In [3] and [4] Abhyankar and Yie proved that for $\overline{F} = YF + T = Y^{24} + XY^4 + Y + T$, $f_{12} = f_{12}(Y) = Y^{12}+Y^6+Y^4+Y^2+Y$, and $f_{24} = f_{24}(Y) = Y^{24}+Y^8+Y^6+Y$, the Galois groups $\mathrm{Gal}(\overline{F}, k(X,T))$, $\mathrm{Gal}(f_{12}+X, k(X))$, and $\mathrm{Gal}(f_{24}+X, k(X))$ are isomorphic to M_{24}, M_{12}, and $\mathrm{Aut}(M_{12})$, respectively. The polynomials F, $f_{12}+X$, and $f_{24}+X$ give unramified coverings of the affine line over k, and $\overline{F}$ gives unramified covering of the affine line over $k(X)$. In order to get a suitable upper bound for the Galois groups, they use so called the "Linearization Process" to obtain the following.

$$\begin{aligned}\Lambda_F &= Y^{2048} + X^{16}Y^{256} + X^{96}Y^{128} + X^{64}Y^{32} + X^{10}Y^8 \\ &\quad + XY^4 + X^8Y^2 + Y \\ &\equiv 0 \pmod{F},\end{aligned}$$

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$$\begin{aligned}\Lambda_{\overline{F}} &= Y^{2048} + X^{16}Y^{256} + X^{96}Y^{128} + X^{64}Y^{32} + X^{10}Y^{8} \\ &\quad + XY^4 + X^8Y^2 + Y \\ &\quad + [T^{64}Y^{512} + X^8T^{16}Y^{32} + T^8Y^{16} + T^{16}Y^8 \\ &\qquad + X^{64}T^{32} + T^{24} + X^8T^2 + T] \\ &\equiv 0 \pmod{\overline{F}},\end{aligned}$$

$$\begin{aligned}\Lambda_{f_{24}+X} &= Y^{2048} + [(X+1)^{64} + (X+1)^{12}]Y^{512} + X^{32}Y^{256} \\ &\quad + [(X+1)^{32} + (X+1)^{28}]Y^{128} + X^8(X+1)^{12}Y^{64} \\ &\quad + [(X+1)^{20} + 1]Y^{32} + X^8(X+1)^8Y^{16} \\ &\quad + [(X+1)^{16} + (X+1)^{12}]Y^8 + [(X+1)^{14} + 1]Y^4 \\ &\quad + (X+1)^{12}Y + X^{24} + X^{13} + X^{12} + X^9 + X^5 + X \\ &\equiv 0 \pmod{f_{24} + X}.\end{aligned}$$

In the first page summary of [1], it was briefly mentioned that the binary Golay code $\mathcal{G}_{23}$ can be constructed from the cyclotomic polynomial $Y^{23} - 1$ and that such a construction together with the linearization process of F tells us that Galois group $\mathrm{Gal}(F, k(X))$ is a subgroup of the group of automorphisms of $\mathcal{G}_{23}$, which is isomorphic to M_{23}. In Section 2, we set up the notation and extend this idea further to associate linear codes to polynomials over fields of positive characteristic. Then we prove that the codes related to F and $\overline{F}$ are $\mathcal{G}_{23}$ and $\mathcal{G}_{24}$ (the extended binary Golay code), respectively.

In Section 3, we give a sufficient condition under which the Golay code $\mathcal{G}_{23}$ is associated to a certain type of degree 23 polynomials over $\mathbb{F}_2$. Then we prove that the code related to $f_{24} + X$ is again $\mathcal{G}_{24}$. We used the complete factorization of f_{24} over $\mathbb{F}_{2^{11}}$. The factorization is done by the algebra package MAPLE V.

In Section 4, we consider two polynomials $Y^{12}+Y^{10}+Y^6+Y^4+Y+X$ and $Y^{12}+Y^{10}+Y^8+Y^6+Y^4+Y^2+Y+X$ in $k(X)[Y]$ and prove that their splitting fields over $k(X)$ coincide and that their Galois group is M_{12}.

In Section 5, we consider the family of degree 24 polynomials

$$Y^{24} + U^2Y^{16} + (U^4 + V^6)Y^8 + VY^6 + XY^4 + UVY^2 + Y + T.$$

We apply the linearization process to this family and display the result in Appendix. Note that this family includes all so far known Mathieu group polynomials of degree 24 (except the one appears in Section 4) as special cases. We prove that some subfamilies of this has Galois group M_{24}.

Though Abhyankar's conjecture has been proved, the structure of the algebraic fundamental group of the affine line is still a mystery. Thus, as a beginning, it would be interesting to find certain family of unramified coverings of the affine line and study how these coverings are intertwined together.

2. Codes associated to polynomials

LEMMA 2.1. *Let ζ be a primitive 23rd root of 1 in $\mathbb{F}_{2^{11}}$. Let $\phi : \mathbb{F}_2^{23} \to \mathbb{F}_{2^{11}}$ be the linear transform defined by mapping the standard basis vector $\mathbf{e}_i$ to ζ^i. Then the nullspace N of ϕ is the Golay code $\mathcal{G}_{23}$.*

Proof. Since $\{\zeta^i \mid i = 0, 1, \dots, 22\}$ generates $\mathbb{F}_{2^{11}}$ as an $\mathbb{F}_2$ vector space, the nullspace N is of dimension 12. It is obvious that N is invariant under the cyclic shift $\mathbf{e}_i \mapsto \mathbf{e}_{i+1}$ of the basis vectors. Hence N is a binary cyclic $[23, 12]$ code, which is probably the easiest definition of $\mathcal{G}_{23}$. □

Generalizing the idea of Lemma 2.1, we give another way of describing linear codes and extended codes and we associate a linear code to an arbitrary polynomial over a finite field.

DESCRIPTION 1. We regard a pair (V, B) of m-dimensional vector space V over $\mathbb{F}_q$ and a set of vectors $B = \{b_1, \dots, b_n\}$ as linear q-ary code as follows: Consider the linear transform $\phi : \mathbb{F}_q^n \to V$ defined by mapping the standard basis vectors $\mathbf{e}_i$ to b_i. Then the nullspace N of ϕ is a q-ary linear code. A homomorphism of a code (V, B) into a code (V', B') is a linear transformation of V into V' which maps $U(B)$ into $U(B')$, where $U(B)$ is the union of 1-dimensional subspaces of V spanned by b_i's.

Two codes (V, B) and (V', B') are said to be *equivalent* if there is an isomorphism between them, i.e., if there is an invertible linear transformation of V onto V', which is a homomorphism of codes. If $V = V'$ and B' is obtained by shuffling the order of vectors in B, then the two codes (V, B) and (V', B') are obviously equivalent.

In most cases below, B generates V. In fact, if we replace V by the subspace V' generated by B, then the two codes (V, B) and (V', B) are the same code in the usual sense, i.e., the nullspace N of ϕ as in the 1 does not change upon the change of V as long as B is contained in V.

If B generates V, then the code is an $[n, n-m]$ code. A codeword is a linear combination of vectors in B which becomes 0, and its weight is the number of vectors in the combination whose coefficient is not zero. Thus the minimum weight is the least number of vectors involved in a linear relation among the vectors in B. For example, the minimum weight is 1 if and only if $0 \in B$; 2 if and only if $0 \notin B$ and B has two distinct vectors which are scalar multiples of each other. Thus if no vector in B is a scalar multiple of another vector in B, then the minimum weight is at least 3.

DESCRIPTION 2. Suppose a code (V, B) is given as in Description 1. Suppose $\overline{V}$ is a vector space containing V as a proper subspace. Fix a vector $\bar{b}_{n+1} \in \overline{V} \setminus V$ and consider the set $\overline{B} = \{\bar{b}_{n+1}\} \cup \{\bar{b}_i = b_i + \bar{b}_{n+1} \mid 1 \le i \le n\}$. Then the code $(\overline{V}, \overline{B})$ is the extended code of (V, B).

To see this, consider the linear mappings $\phi : \mathbb{F}_q^n \to V$ as in Description 1 and $\bar{\phi} : \mathbb{F}_q^{n+1} \to \overline{V}$ defined by $\bar{\phi}(\bar{\mathbf{e}}_i) = \bar{b}_i$, where $\{\bar{\mathbf{e}}_i \mid 1 \le i \le n+1\}$ is the standard basis for $\mathbb{F}_q^{n+1}$. Let $\mathcal{C}$ and $\overline{\mathcal{C}}$ be the nullspaces of ϕ and $\bar{\phi}$, respectively. Then clearly we have that $\sum_{i=1}^{n+1} x_i \bar{\mathbf{e}}_i \in \overline{\mathcal{C}}$ if and only if $\sum_{i=1}^{n} x_i \mathbf{e}_i \in \mathcal{C}$ and $\sum_{i=1}^{n+1} x_i = 0$.

DESCRIPTION 3. Let k be a field of characteristic p and q be a power of p. Suppose a polynomial $f \in k[Y]$ is given. Fix a splitting field $\widetilde{V}$ of f over k and let B be the set of roots of f in $\widetilde{V}$. If $k = \mathbb{F}_q$, then we take $V = \widetilde{V}$. Otherwise, we take V to be the $\mathbb{F}_q$-subspace of $\widetilde{V}$ generated by B. The q-ary linear code (V, B) is uniquely determined by f. We will denote this code by $\mathcal{C}_f$.

THEOREM 2.2. *Let L be the splitting field of Λ_F over $\mathbb{F}_2(X)$. Let V be the set of roots of Λ_F in L and B the set of roots of F in L. Let $\overline{L}$ be the splitting field of $\Lambda_{\overline{F}}$ over $\mathbb{F}_2(X, T)$. Let $\overline{V}$ be the vector subspace generated by the roots of $\Lambda_{\overline{F}}$ in $\overline{L}$ and $\overline{B}$ be the set of roots of $\overline{F}$ in $\overline{L}$. Then $\mathcal{C}_F = (V, B)$ is the Golay code $\mathcal{G}_{23}$ and $\mathcal{C}_{\overline{F}} = (\overline{V}, \overline{B})$ is the Golay code $\mathcal{G}_{24}$.*

Proof. Let R be the local ring obtained by localizing the integral closure of $A = \mathbb{F}_2[X]$ in L at a prime ideal lying above XA. Then clearly we have $B \subset V \subset R$. Let γ be the residue class map of R. Since $\gamma(F) = Y^{23} + 1$ and $\gamma(\Lambda_F) = Y^{2^{11}} + Y$, the vector space V is mapped onto $\mathbb{F}_{2^{11}}$ and B is mapped onto $\{\zeta^i\}$. By Lemma 2.1, (V, B) is isomorphic to the Golay code $\mathcal{G}_{23}$.

Now let $\overline{R}$ be the local ring obtained by localizing the integral closure of $\overline{A} = \mathbb{F}_2(X)[T]$ in $\overline{L}$ at a prime ideal lying above $T\overline{A}$. Then clearly we have $\overline{B} \subset \overline{V} \subset \overline{R}$. Let $\bar{\gamma}$ be the residue class map of $\overline{R}$. Since $\bar{\gamma}(\overline{F}) = YF$ and $\bar{\gamma}(\Lambda_{\overline{F}}) = \Lambda_F$, the vector space $\overline{V}$ is mapped onto V and $\overline{B}$ is mapped onto $B \cup \{0\}$.

There is exactly one root $\bar{b}_0 \in \overline{B}$ such that $\bar{\gamma}(\bar{b}_0) = 0$. Let us denote the roots of $\overline{F}$ by $\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{23}$ and consider the set $B_0 = \{b_i = \bar{b}_i - \bar{b}_0 \mid 1 \leq i \leq 23\}$. Then B_0 is bijectively mapped onto B by $\bar{\gamma}$ and generates a subspace V_0 of $\overline{V}$ which is again bijectively mapped onto V. Namely, (V_0, B_0) is isomorphic to (V, B) as codes. It is clear from Description 2 that $(\overline{V}, \overline{B})$ is the extended code of (V_0, B_0). Therefore $(\overline{V}, \overline{B})$ is isomorphic to the Golay code $\mathcal{G}_{24}$. $\square$

3. An ad hoc construction of Golay code.

In this section we prove that the code $\mathcal{C}_{f_{24}+X}$ is again $\mathcal{G}_{24}$. First, we need the following.

LEMMA 3.1. *Let p be an odd prime such that $\Phi_p(Y) = \frac{Y^p-1}{Y-1}$ is irreducible in $\mathbb{F}_2[Y]$. Let f be an irreducible polynomial of degree p in $\mathbb{F}_2[Y]$. Then the only possible $\mathbb{F}_2$-linear relation among the roots $1, r_1, r_2, \ldots, r_p$ of $(Y+1)f$ is*

$$r_1 + r_2 + \cdots + r_p = 0 \quad \text{or} \quad 1,$$

according as the term Y^{p-1} is absent or present in f.

Proof. Let V be the power set of $P = \{1, 2, \ldots, p\}$. Then we can think of V as a vector space over $\mathbb{F}_2$ with the symmetric difference as its vector addition. Note that the singleton subsets form a basis for V.

Consider the obvious action of p-cycle $\sigma = (1, 2, \ldots, p)$ on V. Since $\Phi(Y)$ is irreducible over $\mathbb{F}_2$, there are exactly two nontrivial invariant subspaces under this action, i.e., $W_1 = \mathrm{Span}(P)$ and $W_{p-1} = \mathrm{Span}(\{i, i+1\},\ i = 1, \ldots, p-1)$.

Now number the roots so that $r_i^2 = r_{i+1}$ for $i = 1, 2, \ldots, p-1$ and $r_p^2 = r_1$. Note that then we have $r_{\sigma(i)} = \varphi(r_i)$, where φ is the Frobenious automorphism of $\mathbb{F}_{2^p}$. Clearly, the subspace $W = \{S \in V \mid \sum_{i \in S} r_i = 0\}$ of V is invariant under the action of σ. If W contained W_{p-1}, then f would have had a single p-fold root. Thus W is either W_1, in which case we have $r_1 + r_2 + \cdots + r_p = 0$, or the trivial subspace, in which case we have $r_1 + r_2 + \cdots + r_p = 1$.

Suppose $\sum_{i\in S} r_i = 1$ for some nonempty subset S of P. Let $S' = \sigma S$. Then $\sum_{i\in S+S'} r_i = \sum_{i\in S} r_i + \sum_{i\in S'} r_i = \sum_{i\in S} r_i + \varphi(\sum_{i\in S} r_i) = 0$. Hence $S + S' \in W$. Since $S + S'$ has an even number of elements, this can happen only if $S + S'$ is empty. Thus $S = S' = P$. $\square$

The irreducibility condition on $\Phi_p(Y)$ is necessary. For example, take $p = 7$ and $f = Y^7 + Y + 1$. If α is a root of f, then α^2 and $\alpha^2 + \alpha$ are also roots of f. Note that $\Phi_7(Y) = (Y^3 + Y + 1)(Y^3 + Y^2 + 1)$.

LEMMA 3.2. *Let $p > 5$ be a prime such that $\Phi_p(Y)$ is irreducible in $\mathbb{F}_2[Y]$. Let $g_1(Y)$ and $g_2(Y)$ be distinct irreducible polynomials of degree p in $\mathbb{F}_2[Y]$ and let $f = (Y + 1)g_1(Y)g_2(Y)$. Then one of the followings holds.*

1. *The minimum weight of the code $\mathcal{C}_f$ is bigger than* 3.
2. *The code $\mathcal{C}_f$ has precisely p codewords of weight* 3.

Proof. Since f does not have any multiple roots, the minimum weight d of $\mathcal{C}_f$ is at least 3. And $d = 3$ if and only if f has three roots whose sum is zero. In order to prove the theorem, it is enough to show that if $d = 3$, then $\mathcal{C}_f$ has precisely p codewords of weight 3. Thus for the rest of the proof, we assume $d = 3$.

Suppose that $r_0 + r_1 + r_2$ is a codeword, i.e., r_0, r_1, r_2 are roots of f such that $r_0 + r_1 + r_2 = 0$. There are two cases. Namely, either (1) 1 is among r_0, r_1, r_2, or (2) 1 is not among r_0, r_1, r_2. In view of Lemma 3.1, we may assume that, in case (1), $r_0 = 1$ and $g_1(r_1) = g_2(r_2) = 0$, or in case (2) we may assume that $g_1(r_0) = 0$ and $g_2(r_1) = g_2(r_2) = 0$.

We first show that the two cases cannot occur simultaneously. Suppose that $r_0 + r_1 + r_2$ and $s_0 + s_1 + s_2$ are two codewords such that $r_0 = 1$, $g_1(r_1) = g_2(r_2) = 0$, $g_1(s_0) = 0$, and $g_2(s_1) = g_2(s_2) = 0$. Then we have $s_0 = r_1^{2^i}$ for some $i = 0, 1, \ldots, p-1$. Since $r_0^{2^i} + r_1^{2^i} + r_2^{2^i} = (r_0 + r_1 + r_2)^{2^i} = 0$, we have $1 + r_2^{2^i} + s_1 + s_2 = 0$. Since $p > 5$, there can't be such a relation between roots of $g_2(Y)$ by Lemma 3.1.

Now we fix a codeword $r_0 + r_1 + r_2$ of weight 3 and let $s_0 + s_1 + s_2$ be any codeword of weight 3. Suppose we are in case (1). That is, $r_0 = s_0 = 1$, $g_1(r_1) = g_1(s_1) = 0$, and $g_2(r_2) = g_2(s_2) = 0$. We have $s_1 = r_1^{2^i}$ and $s_2 = r_2^{2^j}$ for some $i, j = 0, 1, \ldots, p-1$. Then we have $0 = (1 + r_1 + r_2)^{2^i} + (s_0 + s_1 + s_2) = r_2^{2^i} + r_2^{2^j}$. Thus we have $i = j$. It follows that, in Case (1), any codeword of weight 3 is of the form $\phi^i(r_0 + r_1 + r_2)$, $i = 0, 1, \ldots, p-1$, where ϕ is the Frobenius automorphism of $\mathbb{F}_{2^p}$. Hence there are precisely p codewords of weight 3.

Suppose now we are in case (2). That is, $g_1(r_0) = g_1(s_0) = 0$, and $g_2(r_1) = g_2(r_2) = g_2(s_1) = g_2(s_2) = 0$. We have $s_0 = r_0^{2^i}$ for some $i = 0, 1, \ldots, p-1$. Similarly as above, we have $s_1 + s_2 + r_1^{2^i} + r_2^{2^i} = 0$. Since $p > 5$, in view of Lemma 3.1, we must have $s_1 = r_1^{2^i}$ and $s_2 = r_2^{2^i}$ (or $s_1 = r_2^{2^i}$ and $s_2 = r_1^{2^i}$). Hence there are precisely p codewords of weight 3.

Therefore, if $d = 3$, there are precisely p codewords of weight 3 in either cases. □

Note that Φ_{11} is irreducible over $\mathbb{F}_2$. Applying these two lemmas to the case when $p = 11$, we have the following.

THEOREM 3.3. *Let $g_1(Y)$ and $g_2(Y)$ be distinct irreducible polynomials of degree 11 in $\mathbb{F}_2[Y]$ and let $f = (Y+1)g_1(Y)g_2(Y)$. Suppose that $\mathcal{C}_f$ has doubly even dual code. Then one of the followings holds.*

1. $\mathcal{C}_f = \mathcal{G}_{23}$.
2. *The weight enumerator for the dual code $\mathcal{C}^\perp$ of $\mathcal{C}_f$ is given by*

$$\begin{aligned} W_{\mathcal{C}^\perp}(x, y) = x^{23} &+ 55x^{19}y^4 + 330x^{15}y^8 \\ &+ 1486x^{11}y^{12} + 165x^7y^{16} + 11x^3y^{20}. \end{aligned}$$

Proof. By Lemma 3.1, we know that the roots of f generates $\mathbb{F}_{2^{11}}$ as a vector space over $\mathbb{F}_2$. Hence $\mathcal{C}_f$ is a binary $[23, 12]$-code, the minimum weight $d \geq 3$, and the dual code $\mathcal{C}^\perp$ is a binary $[23, 11]$-code. The doubly even code $\mathcal{C}^\perp$ has weight enumerator of the form

$$\begin{aligned} W_{\mathcal{C}^\perp}(x, y) = x^{23} &+ A_4x^{19}y^4 + A_8x^{15}y^8 \\ &+ A_{12}x^{11}y^{12} + A_{16}x^7y^{16} + A_{20}x^3y^{20}. \end{aligned}$$

Moreover, in the 'usual sense' of linear codes, $\mathcal{C}^\perp$ is a subspace of $\mathcal{C}_f$ and the unique vector of weight 23 belongs to $\mathcal{C}_f$ but not to $\mathcal{C}^\perp$. Therefore we have

$$W_{\mathcal{C}}(x, y) = W_{\mathcal{C}^\perp}(x, y) + W_{\mathcal{C}^\perp}(y, x).$$

Especially, A_{20} is the number of weight 3 codewords in $\mathcal{C}_f$. On the other hand, we have MacWilliam's identity

$$W_{\mathcal{C}^\perp}(x, y) = \frac{1}{2^{12}} W_{\mathcal{C}}(x + y, x - y).$$

Solving these equations, we obtain

$$A_4 = 5t,\ A_8 = 506 - 16t,\ A_{12} = 1288 + 18t,\ A_{16} = 253 - 8t,\ A_{20} = t.$$

Thus if $d = 3$, then by Lemma 3.2 we have $A_{20} = 11$ and

$$W_{\mathcal{C}^\perp}(x,y) = x^{23} + 55x^{19}y^4 + 330x^{15}y^8 + 1486x^{11}y^{12} + 165x^7y^{16} + 11x^3y^{20}.$$

If $d > 3$, then we have $A_{20} = 0$ and

$$W_{\mathcal{C}_f}(x,y) = x^{23} + 253x^{16}y^7 + 506x^{15}y^8 + 1288x^{12}y^{11} + 1288x^{11}y^{12} + 506x^8y^{15} + 253x^7y^{16} + y^{23}.$$

But this weight enumerator is that of the Golay code $\mathcal{G}_{23}$ and by Theorem 5 of [6] we conclude that $\mathcal{C}_f$ is $\mathcal{G}_{23}$. □

REMARK 1. Suppose $\mathcal{C}_f$ is as in the above theorem and the minimum weight $d = 3$. A codeword of weight 3 is always of case (1) as in the proof of Lemma 3.2. That is, case (2) never occurs if the dual code $\mathcal{C}^\perp$ is doubly even.

REMARK 2. Suppose $\mathcal{C}_f$ is as in the above theorem and the minimum weight $d = 3$. The extended code $\overline{\mathcal{C}}$ of $\mathcal{C}_f$ is a doubly even binary $[24, 12]$-code. $\overline{\mathcal{C}}$ appears as E_{24} in [7]. Following their notation, we denote by E_{24} a binary $[24, 12, 4]$-code isomorphic to $\overline{\mathcal{C}}$. Similarly, by E_{23} we denote a binary $[23, 12, 3]$-code isomorphic to $\mathcal{C}_f$.

REMARK 3. The group of automorphisms of the codes E_{23} and E_{24} are the semidirect products $T_{10} \rtimes S_{11}$ and $T_{11} \rtimes S_{12}$, respectively, where T_n is the elementary abelian group of order 2^n and S_n is the symmetric group of degree n.

THEOREM 3.4. *Let L be the splitting field of $\Lambda_{f_{24}+X}$ over $\mathbb{F}_2(X)$. Let $\overline{V}$ be the vector subspace generated by the roots of $\Lambda_{f_{24}+X}$ in L and $\overline{B}$ the set of roots of $f_{24} + X$ in L. Then $\mathcal{C}_{f_{24}+X} = (\overline{V}, \overline{B})$ is the Golay code $\mathcal{G}_{24}$.*

Proof. Let R be the local ring obtained by localizing the integral closure of $A = \mathbb{F}_2[X]$ in L at a prime ideal lying above XA. Then clearly we have $\overline{B} \subset \overline{V} \subset R$. Let γ be the residue class map of R. Since $\gamma(\Lambda_{f_{24}+X}) = Y^{2^{11}} + Y$, the vector space $\overline{V}$ is mapped onto $\mathbb{F}_{2^{11}}$.

There is exactly one root $\bar{b}_0 \in \overline{B}$ such that $\gamma(\bar{b}_0) = 0$. Let us denote the roots of $f_{24} + X$ by $\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{23}$ and consider the sets $V = \{v - \bar{b}_0 \mid v \in R,\ \Lambda_{f_{24}+X}(v) = 0\}$ and $B = \{b_i = \bar{b}_i - \bar{b}_0 \mid 1 \le i \le 23\}$. The set V is in fact a subspace of $\overline{V}$ and $B \subset V$. Clearly, the code $(\overline{V}, \overline{B})$ is the extended code of (V, B).

Since $\gamma(f_{24}+X) = f_{24}$, the set B is bijectively mapped onto the set of roots of $f_{23} = \frac{f_{24}}{Y}$ by γ. Note that $f_{23} = (Y+1)f_{11}f'_{11}$, where $f_{11} = \frac{f_{12}}{Y}$ and $f'_{11} = \frac{f_{12}+1}{Y+1}$ are irreducible polynomials of degree 11. Thus by Lemma 3.1, we see that the roots of f_{23} generate $\mathbb{F}_{2^{11}}$ as an $\mathbb{F}_2$-vector space. Hence the 11-dimensional $\mathbb{F}_2$-vector space V is generated by B and is bijectively mapped onto $\mathbb{F}_{2^{11}}$ by γ. Thus the code (V, B) is a $[23,12]$-code, which is isomorphic to $\mathcal{C}_{f_{23}}$. Therefore it is enough to show that $\mathcal{C}_{f_{23}}$ is isomorphic to the Golay code $\mathcal{G}_{23}$.

We will show that the minimum weight of $\mathcal{C}_{f_{23}}$ is bigger than 3 and the dual code $\mathcal{C}^{\perp}$ of $\mathcal{C}_{f_{23}}$ is doubly even. Then the theorem will follow from Theorem 3.3. We need to find all the roots of f_{23} to do this.

We will use the fixed basis $\{1, \eta, \eta^2, \dots, \eta^{10}\}$, where $\eta \in \mathbb{F}_{2^{11}}$ is a root of the irreducible polynomial $Y^{11}+Y^2+1 \in \mathbb{F}_2[Y]$. Then $f_{11}(\eta^8+\eta^6) = 0$ and $f'_{11}(\eta^5+\eta^3+\eta^2+\eta+1) = 0$. Thus, we order the roots $b_1, b_2, \dots, b_{23}$ of f_{23} so that $b_1 = 1$, $b_{2+i} = (\eta^8+\eta^6)^{2^i}$, and $b_{13+i} = (\eta^5+\eta^3+\eta^2+\eta+1)^{2^i}$ for $i = 0, 1, \dots, 10$. The matrix A of the linear transform $\phi : \mathbb{F}_2^{23} \to \mathbb{F}_{2^{11}}$ defined as in Description 1 is given in Figure 1.

$$\begin{pmatrix}
1&0&0&0&1&0&0&0&0&0&1&0&1&1&0&0&0&1&0&1&0&1&0\\
0&0&1&0&1&1&1&1&0&0&1&0&1&0&1&0&1&0&1&0&1&1&0\\
0&0&0&1&1&0&1&1&1&0&1&0&1&1&1&1&0&0&1&0&1&0&0\\
0&0&1&1&1&1&0&0&1&0&0&1&1&0&1&0&1&0&1&1&0&1&0\\
0&0&0&0&1&1&0&1&1&1&0&1&0&1&1&1&1&0&0&1&0&1&0\\
0&0&1&1&0&0&0&0&1&1&0&0&1&0&0&1&1&1&1&1&1&1&0\\
0&1&0&1&1&1&1&0&0&1&0&0&0&1&0&1&0&1&0&1&1&0&1\\
0&0&1&0&0&1&1&1&0&1&1&0&0&0&0&0&0&0&1&1&0&0&0\\
0&1&0&0&0&1&1&0&1&1&1&0&0&0&1&1&1&1&0&0&1&0&1\\
0&0&0&0&1&0&0&0&0&0&1&0&0&0&1&1&1&0&1&0&1&0&1\\
0&0&0&1&1&0&0&0&0&1&1&0&0&1&0&0&1&1&1&1&1&1&1
\end{pmatrix}$$

FIGURE 1. The matrix A of ϕ

Note that each column of A represents a root of f_{23}. Apparently, there is no $0 \le i \le 10$ such that $b_2 + b_{13+i} = 1$. Note also that A is a parity check matrix of $\mathcal{C}_{f_{23}}$ as well as a generating matrix of the dual code $\mathcal{C}^{\perp}$ of $\mathcal{C}_{f_{23}}$. From the matrix product $A\,A^T$ given in Figure 2, we see that each generating codeword of $\mathcal{C}^{\perp}$ is of weight divisible by 4 and any pair of generating codewords has even number of 1's at common coordinates. It follows that $\mathcal{C}^{\perp}$ is doubly even. In view of Remark 1, we have that the minimum weight of $\mathcal{C}_{f_{23}}$ is bigger than 3. $\square$

$$\begin{pmatrix} 8 & 4 & 4 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 6 \\ 4 & 12 & 8 & 8 & 6 & 6 & 4 & 6 & 6 & 6 & 6 \\ 4 & 8 & 12 & 6 & 6 & 6 & 6 & 4 & 6 & 6 & 6 \\ 4 & 8 & 6 & 12 & 8 & 8 & 4 & 4 & 4 & 4 & 6 \\ 4 & 6 & 6 & 8 & 12 & 6 & 6 & 4 & 6 & 4 & 6 \\ 4 & 6 & 6 & 8 & 6 & 12 & 6 & 4 & 6 & 4 & 8 \\ 4 & 4 & 6 & 4 & 6 & 6 & 12 & 4 & 8 & 4 & 8 \\ 2 & 6 & 4 & 4 & 4 & 4 & 4 & 8 & 4 & 2 & 4 \\ 2 & 6 & 6 & 4 & 6 & 6 & 8 & 4 & 12 & 6 & 6 \\ 2 & 6 & 6 & 4 & 4 & 4 & 4 & 2 & 6 & 8 & 6 \\ 6 & 6 & 6 & 6 & 6 & 8 & 8 & 4 & 6 & 6 & 12 \end{pmatrix}$$

FIGURE 2. $A\,A^T$

4. Mathieu group coverings associated with E_{24}

Let us, at this moment, briefly review the proof of (1.1) in Section 6 of [4] and see how Galois groups act as the groups of automorphisms of associated codes. If we let X^* be an element of the algebraic closure of $k(X)$ such that $X^{*2} + X^* = X$, then we have $(f_{12}(Y) + X^*)(f_{12}(Y) + X^* + 1)$. This factorization gives a partition of B into two subsets $B_1 =$ the roots of $f_{12}(Y) + X^*$ and $B_2 =$ the roots of $f_{12}(Y) + X^* + 1$. The Galois group $\mathrm{Gal}(f_{24} + X, k(X))$ preserves this partition (any automorphism which sends X^* to X^*+1 interchanges B_1 and B_2). Since the splitting fields of $f_{24} + X$, $f_{12} + X^*$, and $f_{12} + X^* + 1$ over $k(X)$ coincide, $\mathrm{Gal}(f_{12}(Y) + X^*, k(X^*))$ and $\mathrm{Gal}(f_{12}(Y) + X^* + 1, k(X^*))$ are subgroups of $\mathrm{Gal}(f_{24}+X, k(X))$ and they stabilize B_1, B_2. The subgroup $\mathrm{Gal}(f_{12}(Y) + X^* + 1, k(X^*, x))$, where x is a root of $f_{12}(Y) + X^*$, fixes one vector in B_1 i.e., x, but in view of (3.2) of [4], $\mathrm{Gal}(f_{12}(Y) + X^* + 1, k(X^*, x))$ permutes B_2 transitively. By Theorem 15 in Chapter 10 of [5], $\mathrm{Gal}(f_{24} + X, k(X))$ and $\mathrm{Gal}(f_{12} + X^*, k(X))$ are subgroups of $\mathrm{Aut}(\mathcal{G}_{24}) \approx M_{24}$ which are respectively isomorphic to $\mathrm{Aut}(M_{12})$ and M_{12}.

The key features of the polynomial $f_{12} + X$ that make the above observations work are (1) $f'_{11} = \frac{f_{12}+1}{Y+1}$ is irreducible in $\mathbb{F}_2[Y]$, (2) $f_{23} = (Y+1)f_{11}f'_{11}$ and $\mathcal{C}_{f_{23}}$ is isomorphic to $\mathcal{G}_{23}$, (3) $f_{24} + X = Yf_{23} + X$ can be linearized at 11. (By a *linearized polynomial* over a field K of characteristic p we mean a polynomial $\Lambda \in K[Y]$ each term of which has Y-degree 0 or a power of p. A polynomial $H \in K[Y]$ *can be linearized at* n if H divides a linearized polynomial Λ_H of degree p^n. And in such a case, we call Λ_H a *linearization* of H. Since the linearization process is very much algorithmic in nature and there are enough examples in the literature, when we apply the process to a polynomial H we will present the linearization Λ_H only without giving lengthy computation in detail.)

Thus, hoping to find a new M_{12} covering of the affine line which is strong genus zero (i.e., a covering given by a polynomial of the form $F + X$ with $F \in k[Y]$), we searched, using Maple V, for irreducible polynomials $h = h(Y) \in \mathbb{F}_2[Y]$ of degree 11 such that $h' = h'(Y) = \frac{Yh(Y)+1}{Y+1}$ remain irreducible. Out of 186 irreducible polynomials of degree 11, only 27 have this property. Among these 27 polynomials, 9 were found to be in the situation that the dual code of $C_{\widetilde{h}}$ is doubly even and H can be linearized at 11, where we have let $\widetilde{h} = (Y+1)hh'$ and $H = Y\widetilde{h}+X$. We list these 9 polynomials below in three distinct groups.

THEOREM 4.1. *Let k be a field of characteristic 2. Let $f_{11} = f_{11}(Y) = y^{11}+y^5+y^3+y+1$ and $f_{11}^* = f_{11}^*(Y) = y^{11}+y^7+y^5+y^3+1$. Then the splitting fields of $Yf_{11}+X$ and Yf_{11}^*+X over $k(X)$ coincide. We have Gal$(Yf_{11}+X, k(X)) = M_{12}$ and $\mathcal{C}_{\widetilde{f}}$ and $\mathcal{C}_{\widetilde{f}^*}$ are isomorphic to $\mathcal{G}_{23}$, where $\widetilde{f} = f_{11}(Yf_{11}+1)$ and $\widetilde{f}^* = f_{11}^*(Yf_{11}^*+1)$.*

Proof. Note that $Yf_{11}^*+X+1 = (Y+1)f_{11}(Y+1)+X$. The theorem immediately follows from Section 6 of [4] and Theorem 3.4. □

THEOREM 4.2. *Let k be a field of characteristic 2. Let $g = g(Y) = y^{11}+y^9+y^5+y^3+1$ and $g^* = g^*(Y) = y^{11}+y^9+y^7+y^5+y^3+y+1$. Then the splitting fields of $Yg+X$ and Yg^*+X over $k(X)$ coincide. We have Gal$(Yg+X, k(X)) = M_{12}$ and $C_{\widetilde{g}}$ and $C_{\widetilde{g}^*}$ are isomorphic to E_{23}, where $\widetilde{g} = g(Yg+1)$ and $\widetilde{g}^* = g^*(Yg^*+1)$.*

Proof. Since g and g^* are irreducible over $\mathbb{F}_2$, Galois groups of $Yg+X$ and Yg^*+X over $\mathbb{F}_2(X)$ have cycles of length 11, and hence they are 2-transitive subgroups of the symmetric group S_{12} of degree 12. Upto isomorphism, there are six 2-transitive groups, S_{12}, the alternating group A_{12}, M_{12}, and $M_{11}(12)$ – the Mathieu group of degree 11 acting 3-transitively on 12 letters, PGL$(2,11)$ and PSL$(2,11)$. If we let $\overline{k}$ be an algebraic closure of k, then Gal$(Yg+X, \overline{k}(X))$ is a normal subgroup of Gal$(Yg+X, \mathbb{F}_2(X))$. Since each group in the list of possible Gal$(Yg+X, \mathbb{F}_2(X))$ is simple or almost simple, Gal$(Yg+X, \overline{k}(X))$, and hence Gal$(Yg+X, k(X))$ also, must stay in the same list.

If we let $\Gamma = (Yg+X)(Yg^*+X)$, then Γ can be linearized at 11 with the linearization

$$\begin{aligned}\Lambda_\Gamma = {}& Y^{2048} + (X^{64}+X^{16}+X^8+X^2+1)Y^{1024} \\ & + (X^{128}+X^{64}+X^{32}+X^{16}+X^8+X^2+1)Y^{512}\end{aligned}$$

$$\begin{aligned}
&+(X^{72}+X^{66}+X^{24}+X^{18}+X^8+X^2+1)Y^{256}\\
&+(X^{128}+X^{96}+X^{66}+X^{64}+X^{48}+X^{40}+X^{34}\\
&\quad+X^{32}+X^{18}+X^{10}+X^8+X^2+1)Y^{128}\\
&+(X^8+X^2+1)Y^{64}\\
&+(X^{96}+X^{72}+X^{64}+X^{48}+X^{40}+X^{34}\\
&\quad+X^{24}+X^{10}+X^8+X^2+1)Y^{32}\\
&+(X^{64}+X^{32}+X^4+X^2+1)Y^{16}\\
&+(X^{80}+X^{72}+X^{68}+X^{66}+X^{64}+X^{32}+X^{20}+X^{16}\\
&\quad+X^{12}+X^8+X^6+X^2+1)Y^8\\
&+(X^{72}+X^{66}+X^{64}+X^{24}+X^{18}+X^8+X^4+X^2+1)Y^4\\
&+(X^{64}+X^{16}+X^8+X^2+1)Y^2+(X^{64}+X^{16}+X^8+X^2)Y\\
&+X^{88}+X^{76}+X^{65}+X^{48}+X^{32}+X^{28}+X^{26}\\
&+X^{17}+X^{14}+X^{10}+X^3+X^2.
\end{aligned}$$

Hence we can apply the arguments in Section 6 of [4]. That is, 2-transitivity of the Galois groups and the linearization together affirm that the polynomials $Yg+X$, Yg^*+X, and Γ have common splitting field L.

Among the listed 2-transitive groups of degree 12, M_{12} is the only one that has a subgroup which is not a point stabilizer but is isomorphic to a point stabilizer as an abstract group, namely, $M_{11}(12)$. (One point stabilizer of $M_{11}(12)$ is isomorphic to $\mathrm{PSL}(2,11)$. But $\mathrm{PSL}(2,11)$, in the usual degree 12 action, has an involution which is the product of 6 disjoint transpositions. Since $M_{11}(12)$ does not have an element with this property, $\mathrm{PSL}(2,11)$ is a subgroup of $M_{11}(12)$ only as a point stabilizer.)

Now let x be a root of $Yg+X$ in the common splitting field. Then $k(X)(x)=k(x)$ is a rational function field over k. And $\mathrm{Gal}(L,k(x))$, thought of as a permutation group on the roots of $Yg+X$, is a point-stabilizer. Thus $\mathrm{Gal}(L,k(x))$, thought of as a permutation group on the roots of Yg^*+X, must be a point-stabilizer or, in case $\mathrm{Gal}(L,k(X))\approx M_{12}$, possibly the 3-transitive subgroup $M_{11}(12)$. In terms of polynomials, this means that $Yg^*+xg(x)$ is factored into a linear factor and a factor of degree 11, or is irreducible in which case $\mathrm{Gal}(L,k(X))$ must be M_{12}. But by the 'Factor Theorem', it is easy to see that $Yg^*+xg(x)$ does not have a linear factor over the rational function field $k(x)$. Therefore, we conclude $\mathrm{Gal}(Yg + X, k(X)) = M_{12}$.

The reason why we cannot apply the arguments in Section 6 of [4] directly to compute $\mathrm{Gal}(Yg+X, k(X))$ or $\mathrm{Gal}(Yg^*+X, k(X))$ separately is that $(Y+1)g(Y+1) = Yg+1$ (ditto for g^*) and thus if x is a root of $Yg + X$ then $x+1$ is a root of $Yg + xg(x) + 1$. But, because of the same reason, we would get $\mathcal{C}_{\widetilde{g}} = E_{23}$, once we prove that the dual code of $\mathcal{C}_{\widetilde{g}}$ is doubly even. This follows from the complete factorization of $\widetilde{g}$ as in the proof of Theorem 3.4. □

THEOREM 4.3. *Let k be a field of characteristic 2 and let h be one of the following polynomials: $Y^{11}+Y^9+Y^7+Y^4+Y^3+Y^2+1$, $Y^{11}+Y^9+Y^4+Y^2+1$, $Y^{11}+Y^8+Y^5+Y^2+1$, $Y^{11}+Y^8+Y^7+Y^5+Y^3+Y^2+1$, $Y^{11}+Y^8+Y^4+Y+1$. We have $\mathrm{Gal}(Yh + X, k(X)) = S_{12}$ and $\mathcal{C}_{\widetilde{h}}$ is isomorphic to E_{23}, where $\widetilde{h} = h(Yh+1)$.*

Proof. We will sketch the proof. By substituting various values for X in $Yh + X$ and then factoring, we can obtain enough cycle types for the Galois groups and conclude the Galois group $\mathrm{Gal}(Yh + X, k(X))$ is either S_{12} or A_{12}. Then using the Resultant Criterion in [2], we conclude the Galois group is indeed S_{12} for each of the polynomials listed above. From the complete factorization of $\widetilde{h}$, we see that the dual code of $\mathcal{C}_{\widetilde{h}}$ is doubly even. Finally, by noting that $(Y+1)h(Y+1) = Yh+1$ for each of the polynomial, we conclude $\mathcal{C}_{\widetilde{h}} = E_{23}$. □

5. Large Mathieu group coverings

In this section we will consider the polynomials

$$\begin{aligned}\mathcal{F} &= \mathcal{F}(U,V,X,Y)\\ &= Y^{23} + U^2Y^{15} + (U^4+V^6)Y^7 + VY^5 + XY^3 + UVY + 1\end{aligned}$$

and

$$\begin{aligned}\overline{\mathcal{F}} &= \overline{\mathcal{F}}(U,V,X,T,Y)\\ &= Y^{24} + U^2Y^{16} + (U^4+V^6)Y^8 + VY^6 + XY^4 + UVY^2 + Y + T.\end{aligned}$$

These polynomials include all the previously known polynomials to which the linearization process was applied so far as special cases. That is, $F = \mathcal{F}(0,0,X,Y)$, $\overline{F} = \overline{\mathcal{F}}(0,0,X,T,Y)$, $f_{24} + X = \overline{\mathcal{F}}(0,1,0,X,Y)$, and $Y\widetilde{f}^* + X = \overline{\mathcal{F}}(1,1,0,X,Y)$. We obtained a linearization of $\overline{\mathcal{F}}$ of degree 2^{11} and present it in Appendix A.

Concerning these polynomials we prove the following.

THEOREM 5.1. *Let U, V be arbitrary elements in k and let X be an indeterminate over k. Then the equation $\mathcal{F} = 0$ gives an unramified covering of the affine X-line L_k over k, with* $\mathrm{Gal}(\mathcal{F}, k(X)) = M_{23}$.

THEOREM 5.2. *Let U, V be arbitrary elements in k and let X, T be indeterminates over k. Then the equation $\overline{\mathcal{F}} = 0$ gives an unramified covering of the affine T-line $L_{k(X)}$ over $k(X)$, with* $\mathrm{Gal}(\overline{\mathcal{F}}, k(X,T)) = M_{24}$.

If $V = 0$, then we can drop the condition that X is an indeterminate over k. That is, if we let

$$\mathcal{F}^* = \mathcal{F}^*(U, X, Y) = Y^{23} + UY^{15} + U^2Y^7 + XY^3 + 1$$

and

$$\overline{\mathcal{F}}^* = \overline{\mathcal{F}}^*(U, X, T, Y) = Y^{24} + UY^{16} + U^2Y^8 + XY^4 + Y + T$$

then we have the following.

THEOREM 5.3. *Let X be an arbitrary element in k and let U be an indeterminate over k. The equation $\mathcal{F}^* = 0$ gives an unramified covering of the affine U-line L_k over k, with* $\mathrm{Gal}(\mathcal{F}^*, k(U)) = M_{23}$.

THEOREM 5.4. *Let U, X be arbitrary elements in k and let T be an indeterminate over k. Then the equation $\overline{\mathcal{F}}^* = 0$ gives an unramified covering of the affine T-line L_k over k, with* $\mathrm{Gal}(\overline{\mathcal{F}}^*, k(T)) = M_{24}$.

Proof. To prove these Theorems, observe, first of all, that the polynomials in Theorems 5.1, 5.2, 5.4 are irreducible since they are linear in an indeterminate. Obviously, the Y-discriminants of the polynomials equal to 1 and hence the polynomials give unramified coverings of appropriate affine lines (that is, of course if we prove that $\mathcal{F}^*$ is irreducible).

Now the twisted derivatives of $\overline{\mathcal{F}}'$ and $\overline{\mathcal{F}}^{*\prime}$ are given by

$$\begin{aligned}\overline{\mathcal{F}}' &= Y^{-1}[\overline{\mathcal{F}}(U, V, X, T, Y + \xi) - \overline{\mathcal{F}}(U, V, X, T, \xi)] \\ &= Y^{23} + (U^2 + \xi^8)Y^{15} + (U^4 + V^6 + \xi^{16})Y^7 \\ &\quad + VY^5 + (X + \xi^2 V)Y^3 + (UV + \xi^4 V)Y + 1 \\ &= \mathcal{F}(U + \xi^4, V, X + \xi^2 V, Y)\end{aligned}$$

and

$$\begin{aligned}\overline{\mathcal{F}}^{*\prime} &= Y^{-1}[\overline{\mathcal{F}}^*(U, X, T, Y + \xi) - \overline{\mathcal{F}}^*(U, X, T, \xi)] \\ &= Y^{23} + (U + \xi^8)Y^{15} + (U^2 + \xi^{16})Y^7 + (X + \xi^2)Y^3 + 1 \\ &= \mathcal{F}^*(U + \xi^8, X, Y).\end{aligned}$$

We put ξ to be a root of $\overline{\mathcal{F}}$ (or of $\overline{\mathcal{F}}^*$). Then T is a polynomial in U, V, X, ξ. Moreover, $\mathrm{Gal}(\mathcal{F}(U+\xi^4, V, X+\xi^2 V, Y), k(U,V,X,\xi))$ and $\mathrm{Gal}(\mathcal{F}^*(U+\xi^8, X, Y), k(U,X,\xi))$ are respective point stabilizers of $\mathrm{Gal}(\overline{\mathcal{F}}, k(U,V,X,T))$ and $\mathrm{Gal}(\overline{\mathcal{F}}^*, k(U,X,T))$. If X is an indeterminate over k, then $\mathrm{Gal}(\mathcal{F}(U+\xi^4, V, X+\xi^2 V, Y), k(U,V,X,\xi))$ is isomorphic to $\mathrm{Gal}(\mathcal{F}(U+\xi^4, V, X, Y), k(U,V,X,\xi))$ and hence Theorem 5.1 implies Theorem 5.2. If U is an indeterminate over k, then $\mathrm{Gal}(\mathcal{F}^*(U+\xi^8, X, Y), k(U,X,\xi))$ is isomorphic to $\mathrm{Gal}(\mathcal{F}^*(U,X,Y), k(U,X,\xi))$ and hence Theorem 5.3 and Theorem 5.4 are equivalent.

Assume for a moment that U, V are arbitrary elements of k and X is an indeterminate over k. By solving the equation $\mathcal{F} = 0$ for X we see that the valuation $X = \infty$ of $k(X)$ over k splits into two distinct valuations in $k(X,\xi)$ with respective ramification indices 3 and 20. Thus $\mathcal{F}$ factors into two irreducible factors of degrees 3 and 20 in $k((\frac{1}{X}))[Y]$ and hence we see that the order of the Galois group $G = \mathrm{Gal}(\mathcal{F}, k(X))$ is divisible by 3 as well as 20. By Burnside's theorem, G is either a subgroup of $\mathrm{AGL}(1,23)$ (i.e., the group of transformation of type $x \mapsto ax+b$ of $\mathbb{F}_{23}$) or 2-transitive. Since $|\mathrm{AGL}(1,23)| = 23\cdot 22$ is not divisible by 3, G is not a subgroup of $\mathrm{AGL}(1,23)$. On the other hand, we see that G is not A_{23} or S_{23} from the linearization of $\overline{\mathcal{F}}$. In view of the classification of 2-transitive groups, G must be M_{23}. Thus Theorems 5.1 and 5.2 are proved.

Now assume that U, X are arbitrary elements in k and let T be an indeterminate over k. Again from the linearization of $\overline{\mathcal{F}}$ we see that $G^* = \mathrm{Gal}(\overline{\mathcal{F}}^*, k(T))$ is not A_{24} or S_{24}. On the other hand, we can apply the Transitivity Lemma of [3] without any change to get G^* is a 2-transitive permutation group of degree 24 whose order is divisible by 7. In view of the classification of 2-transitive groups, G^* must be M_{24}. Thus Theorems 5.3 and 5.4 are proved. $\square$

If $V = 1$ and $U = X^2$, then we have different Galois group. To see this, let

$$\begin{aligned}\widetilde{F}_{24} &= \widetilde{F}_{24}(X,T,Y)\\ &= Y^{24} + X^4Y^{16} + (X^8+1)Y^8 + Y^6 + XY^4 + X^2Y^2 + Y + T\end{aligned}$$

and

$$\begin{aligned}\widetilde{F}_{12} &= \widetilde{F}_{12}(X,T,Y)\\ &= Y^{12} + X^2Y^8 + (X^4+X+1)Y^4 + (X^2+1)Y^2 + Y + T.\end{aligned}$$

Then we have the following.

THEOREM 5.5. *Let X be an arbitrary element in k. Then the equations $\widetilde{F}_{24} = 0$ and $\widetilde{F}_{12} = 0$ give unramified coverings of the affine T-line L_k over k, with* $\mathrm{Gal}(\widetilde{F}_{24}, k(T)) = \mathrm{Aut}(M_{12})$ *and* $\mathrm{Gal}(\widetilde{F}_{12}, k(T)) = M_{12}$.

Proof. Obviously we have

$$\begin{aligned}
\widetilde{F}_{24}(X, T^2 + T, Y) &= \widetilde{F}_{12}(X, T, Y)(\widetilde{F}_{12}(X, T, Y) + 1), \\
\widetilde{F}_{24}(X, T, Y) &= f_{24}(Y + \sqrt{X}) + T + X^{12} + X^4 + X^3 + \sqrt{X}, \\
\widetilde{F}_{12}(X, T, Y) &= f_{12}(Y + \sqrt{X}) + T + X^6 + X^3 + X^2 + X + \sqrt{X}.
\end{aligned}$$

Thus, for $n = 12, 24$, the Galois group $\mathrm{Gal}(\widetilde{F}_n, k(T, \sqrt{X}))$ (which is obviously isomorphic to $\mathrm{Gal}(\widetilde{F}_n, k(T))$) of $\widetilde{F}_n$ over the inseparable extension $k(T, \sqrt{X})$ is isomorphic to the previously known Galois group $\mathrm{Gal}(f_n + T, k(T))$. $\square$

Appendix A. The linearization of $\overline{\mathcal{F}}$

In this section, we present the linearization Λ of $\overline{\mathcal{F}}$:

$$\begin{aligned}
\Lambda =& Y^{2^{11}} + C_{10}Y^{2^{10}} + C_9Y^{2^9} + C_8Y^{2^8} + C_7Y^{2^7} + C_6Y^{2^6} + C_5Y^{2^5} \\
&+ C_4Y^{2^4} + C_3Y^{2^3} + C_2Y^{2^2} + C_1Y^2 + C_0Y + C_{-1}.
\end{aligned}$$

The coefficients $C_{10}, C_9, \ldots, C_0, C_{-1}$ are given below:

$$\begin{aligned}
C_{10} =& V^{384} + V^{108} + X^{16}V^{80} + (U^{24} + X^4U^4 + T^4)V^{72} + X^2V^{70} + U^4V^{56} + X^8V^{48} + V^{39} \\
&+ U^4X^{16}V^{28} + (U^{28} + X^4U^8 + T^4U^4 + X^{24})V^{20} + U^4X^2V^{18} + V^{16} \\
&+ (X^8U^{24} + X^{12}U^4 + T^4X^8)V^{12} + X^{16}V^{11} + X^{10}V^{10} + (U^{24} + X^4U^4 + T^4)V^3 + X^2V
\end{aligned}$$

$$\begin{aligned}
C_9 =& U^{128}V^{384} + V^{300} + X^{16}V^{272} + (U^{24} + X^4U^4 + T^4)V^{264} + X^2V^{262} + U^4V^{248} \\
&+ X^8V^{240} + V^{231} + U^4X^{16}V^{220} + (U^{28} + X^4U^8 + T^4U^4 + X^{24})V^{212} + U^4X^2V^{210} \\
&+ (X^8U^{24} + X^{12}U^4 + T^4X^8)V^{204} + X^{16}V^{203} + X^{10}V^{202} \\
&+ (U^{24} + X^4U^4 + T^4)V^{195} + X^2V^{193} + U^4V^{156} + X^8V^{148} + V^{139} + U^8V^{104} \\
&+ X^{16}V^{88} + V^{70} + U^4V^{64} + X^{32}V^{60} + X^8V^{56} + U^{12}V^{52} + V^{17} + (U^{48} + T^8)V^{44} \\
&+ X^4V^{40} + U^8V^{35} + X^{48}V^{32} + (U^{28} + X^4U^8 + T^4U^4)V^{28} + U^4X^2V^{26} \\
&+ (X^{32}U^{24} + X^{36}U^4 + T^4X^{32})V^{24} + X^{34}V^{22} + (X^8U^{24} + X^{12}U^4 + T^4X^8)V^{20} \\
&+ (U^4 + X^{10})V^{18} + (X^{16}U^{48} + X^{24}U^8 + T^8X^{16})V^{16} + X^{20}V^{12} \\
&+ (U^{24} + X^4U^4 + T^4)V^{11} + X^8V^{10} + X^2V^9 \\
&+ (U^{72} + X^4U^{52} + T^4U^{48} + X^8U^{32} + T^8U^{24} + X^{12}U^{12} + T^4X^8U^8 + T^8X^4U^4 + T^{12})V^8 \\
&+ (X^2U^{48} + X^{10}U^8 + T^8X^2)V^6 + (X^4U^{24} + X^8U^4 + T^4X^4)V^4 + X^6V^2 + V \\
&+ T^{64} + U^{16} + X^{64}U^{64} + U^{384}
\end{aligned}$$

$$
\begin{aligned}
C_8 = & V^{304} + U^{64}V^{300} + U^{64}X^{16}V^{272} + (U^{88} + X^4U^{68} + T^4U^{64})V^{264} + U^{64}X^2V^{262} \\
& + U^{32}V^{256} + U^4V^{252} + U^{68}V^{248} + X^8V^{244} + U^{64}X^8V^{240} + V^{235} + U^{64}V^{231} \\
& + U^{68}X^{16}V^{220} + (U^{92} + X^4U^{72} + T^4U^{68} + X^{24}U^{64} + 1)V^{212} + U^{68}X^2V^{210} + U^{64}V^{208} \\
& + (X^8U^{88} + X^{12}U^{68} + T^4X^8U^{64})V^{204} + U^{64}X^{16}V^{203} + U^{64}X^{10}V^{202} + U^8V^{200} \\
& + (U^{88} + X^4U^{68} + T^4U^{64})V^{195} + U^{64}X^2V^{193} + X^{64}V^{192} + X^{16}V^{184} + V^{166} \\
& + X^{32}V^{156} + U^{12}V^{148} + (U^{48} + T^8)V^{140} + X^4V^{136} + U^8V^{131} + X^{48}V^{128} \\
& + (U^{28} + X^4U^8 + T^4U^4)V^{124} + U^4X^2V^{122} \\
& + (X^{32}U^{24} + X^{36}U^4 + T^4X^{32} + 1)V^{120} + X^{34}V^{118} + (X^8U^{24} + X^{12}U^4 + T^4X^8)V^{116} \\
& + (U^4 + X^{10})V^{114} + (X^{16}U^{48} + X^{24}U^8 + T^8X^{16})V^{112} \\
& + (U^{192} + X^{32}U^{32} + X^{20} + T^{32})V^{108} + (U^{24} + X^4U^4 + T^4)V^{107} + X^8V^{106} + X^2V^{105} \\
& + (U^{72} + X^4U^{52} + T^4U^{48} + X^8U^{32} + T^8U^{24} + X^{12}U^{12} + T^4X^8U^8 + T^8X^4U^4 + T^{12})V^{104} \\
& + (X^2U^{48} + X^{10}U^8 + T^8X^2)V^{102} + (X^4U^{24} + X^8U^4 + T^4X^4)V^{100} + X^6V^{98} + V^{97} \\
& + U^{16}V^{96} + (U^{24} + X^4U^4 + T^4)V^{84} + X^2V^{82} \\
& + (X^{16}U^{192} + X^{48}U^{32} + X^{16}U^8 + T^{32}X^{16})V^{80} + V^{74} \\
& + (U^{216} + X^4U^{196} + T^4U^{192} + X^{32}U^{56} + X^{36}U^{36} + T^4X^{32}U^{32} + U^{32} \\
& \quad + T^{32}U^{24} + X^4U^{12} + T^4U^8 + T^{32}X^4U^4 + T^{36})V^{72} \\
& + (X^2U^{192} + X^{34}U^{32} + X^2U^8 + T^{32}X^2)V^{70} + (U^{196} + X^{32}U^{36} + T^{32}U^4)V^{56} \\
& + (X^8U^{192} + U^{48} + X^{40}U^{32} + X^8U^8 + T^8 + T^{32}X^8)V^{48} + X^4V^{44} + U^4X^{16}V^{40} \\
& + (U^{192} + X^{32}U^{32} + T^{32})V^{39} + X^{48}V^{36} + X^{24}V^{32} \\
& + (X^{16}U^{196} + X^{48}U^{36} + X^{32}U^{24} + X^{16}U^{12} + T^{32}X^{16}U^4 + X^{36}U^4 + T^4X^{32})V^{28} \\
& + X^{34}V^{26} + X^{16}V^{23} + U^4V^{22} \\
& + (U^{220} + X^4U^{200} + T^4U^{196} + X^{24}U^{192} + X^{32}U^{60} + X^{16}U^{48} + X^{36}U^{40} \\
& \quad + U^{36} + T^4X^{32}U^{36} + X^{56}U^{32} + T^{32}U^{28} + X^4U^{16} + T^4U^{12} + T^{32}X^4U^8 + T^{36}U^4 \\
& \quad + T^8X^{16} + T^{32}X^{24})V^{20} \\
& + (X^2U^{196} + X^{34}U^{36} + X^2U^{12} + T^{32}X^2U^4)V^{18} + (U^{192} + X^{32}U^{32} + X^{20} + T^{32})V^{16} \\
& + X^8V^{14} \\
& + (X^8U^{216} + X^{12}U^{196} + T^4X^8U^{192} + U^{72} + X^{40}U^{56} + X^4U^{52} + T^4U^{48} + X^{44}U^{36} \\
& \quad + T^4X^{40}U^{32} + T^8U^{24} + T^{32}X^8U^{24} + T^8X^4U^4 + T^{32}X^{12}U^4 + T^{12} + T^{36}X^8)V^{12} \\
& + (X^{16}U^{192} + X^{48}U^{32} + X^{16}U^8 + T^{32}X^{16})V^{11} \\
& + (X^{10}U^{192} + X^2U^{48} + X^{42}U^{32} + T^8X^2 + T^{32}X^{10})V^{10} \\
& + (X^4U^{24} + X^8U^4 + T^4X^4)V^8 + X^6V^6 + V^5 \\
& + (U^{216} + X^4U^{196} + T^4U^{192} + X^{32}U^{56} + X^{36}U^{36} + T^4X^{32}U^{32} + U^{32} + T^{32}U^{24} \\
& \quad + X^4U^{12} + T^4U^8 + T^{32}X^4U^4 + T^{36})V^3 \\
& + (X^2U^{192} + X^{34}U^{32} + X^2U^8 + T^{32}X^2)V + X^{16}
\end{aligned}
$$

$$
C_7 = V^{352} + U^{32}V^{304} + V^{260} + U^{64}V^{256} + U^{36}V^{252} + U^{32}X^8V^{244} + U^{16}V^{236} + U^{32}V^{235}
$$

$$
\begin{aligned}
&+ (U^{96} + T^{16})V^{208} + X^{32}V^{204} + (X^4U^{20} + T^4U^{16} + X^8)V^{200} + U^{16}X^2V^{198} + V^{191}\\
&+ (X^{16}U^{32} + U^{20})V^{184} + (X^8U^{16} + X^{48})V^{176} + (X^{32}U^{24} + X^{36}U^4 + T^4X^{32} + 1)V^{168}\\
&+ U^{16}V^{167} + (U^{32} + X^{34})V^{166} + U^{36}V^{160} + (U^{100} + X^{32}U^{32} + U^8 + T^{16}U^4)V^{156}\\
&+ (X^8U^{32} + X^{32}U^4)V^{152} + (X^8U^{96} + X^4U^{24} + T^4U^{20} + X^8U^4 + T^{16}X^8)V^{148}\\
&+ U^{20}X^2V^{146} + (U^{16} + X^{40})V^{144} + U^{32}V^{143}\\
&+ (U^{80} + X^8U^{40} + T^8U^{32} + X^{12}U^{20} + T^4X^8U^{16} + X^{16})V^{140} + (U^{96} + U^4 + T^{16})V^{139}\\
&+ U^{16}X^{10}V^{138} + U^{32}X^4V^{136} + X^{32}V^{135} + (U^{24} + X^4U^4 + T^4)V^{132}\\
&+ (X^4U^{20} + T^4U^{16})V^{131} + X^2V^{130} + U^{16}X^2V^{129} + (X^{48}U^{32} + X^{16}U^8)V^{128}\\
&+ (U^{60} + X^4U^{40} + T^4U^{36} + X^{48}U^4)V^{124} + (X^2U^{36} + 1)V^{122}\\
&+ (X^{32}U^{56} + X^{36}U^{36} + U^{32} + T^4X^{32}U^{32} + X^4U^{12} + T^4U^8)V^{120}\\
&+ (X^{34}U^{32} + X^2U^8)V^{118}\\
&+ (X^8U^{56} + X^{12}U^{36} + T^4X^8U^{32} + X^{32}U^{28} + X^{36}U^8 + T^4X^{32}U^4 + U^4 + X^{56})V^{116}\\
&+ (U^{36} + X^{10}U^{32} + X^{34}U^4)V^{114} + (X^{16}U^{80} + X^{24}U^{40} + T^8X^{16}U^{32} + X^{32})V^{112}\\
&+ (X^{20}U^{32} + X^{40}U^{24} + X^{44}U^4 + X^8 + T^4X^{40})V^{108} + (U^{56} + X^4U^{36} + T^4U^{32} + X^{48})V^{107}\\
&+ (X^8U^{32} + X^{42})V^{106} + U^{32}X^2V^{105}\\
&+ (X^4U^{84} + T^4U^{80} + X^8U^{64} + T^8U^{56} + X^{12}U^{44} + T^4X^8U^{40} + T^8X^4U^{36} + T^{12}U^{32}\\
&\quad + U^{12} + T^{16}U^8 + X^{20}U^4 + T^4X^{16})V^{104}\\
&+ (X^2U^{80} + X^{10}U^{40} + T^8X^2U^{32} + X^{18})V^{102} + (X^4U^{56} + X^8U^{36} + T^4X^4U^{32})V^{100}\\
&+ (X^{32}U^{24} + X^{36}U^4 + T^4X^{32})V^{99} + U^{32}X^6V^{98} + (U^{32} + X^{34})V^{97} + (X^8U^8 + T^8)V^{96}\\
&+ X^4V^{92} + (X^{16}U^{96} + X^{32}U^{16} + T^{16}X^{16})V^{88} + X^{48}V^{84} + (U^{28} + X^4U^8 + T^4U^4 + X^{24})V^{80}\\
&+ U^4X^2V^{78} + (X^{32}U^{24} + X^{16}U^{12} + X^{36}U^4 + T^4X^{32})V^{76} + X^{34}V^{74}\\
&+ (X^8U^{24} + X^{12}U^4 + T^4X^8)V^{72} + (U^{96} + X^{16}U^{16} + X^{10} + T^{16})V^{70}\\
&+ (X^{16}U^{48} + U^{36} + X^4U^{16} + T^4U^{12} + T^8X^{16})V^{68} + U^{12}X^2V^{66}\\
&+ (U^{192} + U^{100} + X^{16}U^{20} + T^{16}U^4 + X^{20} + T^{32})V^{64} + X^8V^{62}\\
&+ (X^{32}U^{96} + U^{72} + X^4U^{52} + T^4U^{48} + T^8U^{24} + X^{48}U^{16} + T^8X^4U^4 + T^{16}X^{32} + T^{12})V^{60}\\
&+ U^8X^{16}V^{59} + (X^2U^{48} + T^8X^2)V^{58} + (X^8U^{96} + X^4U^{24} + X^{24}U^{16} + T^{16}X^8 + T^4X^4)V^{56}\\
&+ X^6V^{54} + V^{53} + (U^{108} + U^{16} + T^{16}U^{12} + X^{20}U^8 + T^4X^{16}U^4 + X^{40})V^{52}\\
&+ (U^{32} + X^4U^{12} + T^4U^8)V^{51} + U^4X^{18}V^{50} + U^8X^2V^{49} + (U^{96} + X^{16}U^{16} + T^{16})V^{47}\\
&+ (U^{144} + T^8U^{96} + X^{16}U^{64} + U^{52} + T^{16}U^{48} + X^{24}U^{24} + T^8X^{16}U^{16} + X^{28}U^4\\
&\quad + T^8U^4 + T^{24} + T^4X^{24})V^{44}\\
&+ X^{32}V^{43} + X^{26}V^{42} + (X^4U^{96} + X^{20}U^{16} + X^4U^4 + T^{16}X^4)V^{40} + X^8V^{39} + U^8X^{16}V^{36}\\
&+ (U^{104} + U^{12} + T^{16}U^8 + X^{20}U^4 + T^4X^{16})V^{35} + X^{18}V^{33}\\
&+ (X^{48}U^{96} + X^{48}U^4 + T^{16}X^{48})V^{32}\\
&+ (U^{124} + X^4U^{104} + T^4U^{100} + X^{16}U^{44} + T^{16}U^{28} + X^{20}U^{24}\\
&\quad + T^4X^{16}U^{20} + T^{16}X^4U^8 + T^{20}U^4)V^{28}\\
&+ (X^2U^{100} + X^{18}U^{20} + T^{16}X^2U^4)V^{26}\\
&+ (X^{32}U^{120} + X^{36}U^{100} + T^4X^{32}U^{96} + X^{48}U^{40} + X^{32}U^{28} + T^{16}X^{32}U^{24} + X^{52}U^{20}\\
&\quad + T^4X^{48}U^{16} + X^{36}U^8 + T^4X^{32}U^4 + T^{16}X^{36}U^4 + T^{20}X^{32})V^{24}
\end{aligned}
$$

$$
\begin{aligned}
&+ (X^{34}U^{96} + X^{50}U^{16} + X^{34}U^4 + T^{16}X^{34})V^{22} \\
&+ (X^8U^{120} + X^{12}U^{100} + T^4X^8U^{96} + X^{24}U^{40} + X^8U^{28} + T^{16}X^8U^{24} + X^{28}U^{20} + T^4X^{24}U^{16} \\
&\quad + X^{12}U^8 + T^4X^8U^4 + T^{16}X^{12}U^4 + T^{20}X^8 + X^{32})V^{20} \\
&+ U^4X^{16}V^{19} + (U^{100} + X^{10}U^{96} + X^{16}U^{20} + X^{26}U^{16} + U^8 + X^{10}U^4 + T^{16}U^4 + T^{16}X^{10})V^{18} \\
&+ (X^{16}U^{144} + X^{24}U^{104} + T^8X^{16}U^{96} + X^{32}U^{64} + X^{16}U^{52} + T^{16}X^{16}U^{48} + X^{40}U^{24} \\
&\quad + T^8X^{32}U^{16} + X^{24}U^{12} + T^{16}X^{24}U^8 + T^8X^{16}U^4 + T^{24}X^{16})V^{16} \\
&+ (X^{20}U^{96} + X^{16}U^{24} + X^{36}U^{16} + T^{16}X^{20} + T^4X^{16})V^{12} \\
&+ (U^{120} + X^4U^{100} + T^4U^{96} + X^{16}U^{40} + T^{16}U^{24} + X^{20}U^{20} + T^4X^{16}U^{16} \\
&\quad + T^{16}X^4U^4 + X^{24} + T^{20})V^{11} \\
&+ (X^8U^{96} + X^{24}U^{16} + X^8U^4 + X^{18} + T^{16}X^8)V^{10} + (X^2U^{96} + X^{18}U^{16} + T^{16}X^2)V^9 \\
&+ (U^{168} + X^4U^{148} + T^4U^{144} + X^8U^{128} + T^8U^{120} + X^{12}U^{108} + T^4X^8U^{104} + T^8X^4U^{100} \\
&\quad + T^{12}U^{96} + X^{16}U^{88} + U^{76} + T^{16}U^{72} + X^{20}U^{68} + T^4X^{16}U^{64} + X^4U^{56} \\
&\quad + T^{16}X^4U^{52} + T^4U^{52} + X^{24}U^{48} + T^{20}U^{48} + T^8X^{16}U^{40} + X^8U^{36} + T^{16}X^8U^{32} \\
&\quad + T^8U^{28} + X^{28}U^{28} + T^{24}U^{24} + T^4X^{24}U^{24} + T^8X^{20}U^{20} + X^{12}U^{16} + T^{12}X^{16}U^{16} \\
&\quad + T^{16}X^{12}U^{12} + T^4X^8U^{12} + T^{20}X^8U^8 + T^8X^4U^8 + T^{24}X^4U^4 + T^{12}U^4 + T^{28})V^8 \\
&+ (X^2U^{144} + X^{10}U^{104} + T^8X^2U^{96} + X^{18}U^{64} + X^2U^{52} + T^{16}X^2U^{48} + X^{26}U^{24} \\
&\quad + T^8X^{18}U^{16} + X^{10}U^{12} + T^{16}X^{10}U^8 + T^8X^2U^4 + T^{24}X^2)V^6 \\
&+ (X^4U^{120} + X^8U^{100} + T^4X^4U^{96} + X^{20}U^{40} + X^4U^{28} + T^{16}X^4U^{24} + X^{24}U^{20} \\
&\quad + T^4X^{20}U^{16} + X^8U^8 + T^4X^4U^4 + T^{16}X^8U^4 + T^{20}X^4)V^4 \\
&+ (X^8U^{24} + X^{12}U^4 + T^4X^8)V^3 + (X^6U^{96} + X^{22}U^{16} + X^6U^4 + T^{16}X^6)V^2 \\
&+ (U^{96} + X^{16}U^{16} + U^4 + T^{16} + X^{10})V + X^{96}
\end{aligned}
$$

$$
\begin{aligned}
C_6 =& V^{284} + U^{16}V^{260} + X^{16}V^{256} + (U^{24} + X^4U^4 + T^4)V^{248} + X^2V^{246} + U^{32}V^{236} + U^{12}V^{220} \\
&+ V^{215} + (U^{48} + T^8)V^{212} + (X^{16}U^{32} + U^{20} + X^4)V^{208} + U^8V^{203} \\
&+ (U^{56} + X^4U^{36} + T^4U^{32} + X^8U^{16})V^{200} + U^{32}X^2V^{198} + (U^{28} + X^4U^8 + T^4U^4)V^{196} \\
&+ U^4X^2V^{194} + V^{192} + U^{16}V^{191} + (X^8U^{24} + X^{12}U^4 + T^4X^8)V^{188} + X^{10}V^{186} + U^{36}V^{184} \\
&+ U^8V^{180} + (U^{24} + X^4U^4 + T^4)V^{179} + X^2V^{177} + U^{32}X^8V^{176} + U^{32}V^{167} + X^{16}V^{164} \\
&+ U^4V^{163} + (U^{52} + X^8U^{12} + T^8U^4)V^{160} + (X^{16}U^{36} + X^4U^4)V^{156} + X^8V^{155} \\
&+ (X^8U^{48} + X^{16}U^8 + T^8X^8)V^{152} + (U^{60} + X^4U^{40} + T^4U^{36} + X^{24}U^{32} + X^{12})V^{148} \\
&+ U^{36}X^2V^{146} + U^{32}V^{144} + (U^{48} + X^8U^8 + T^8)V^{143} \\
&+ (U^{96} + X^8U^{56} + X^{12}U^{36} + T^4X^8U^{32} + U^4 + T^{16})V^{140} + (X^{16}U^{32} + X^4)V^{139} \\
&+ U^{32}X^{10}V^{138} + X^{32}V^{136} + U^8V^{134} + (U^{40} + X^4U^{20} + T^4U^{16})V^{132} \\
&+ (U^{56} + X^4U^{36} + T^4U^{32})V^{131} + U^{16}X^2V^{130} + U^{32}X^2V^{129} + U^{24}X^{16}V^{128} + U^{16}V^{122} \\
&+ (X^4U^{28} + T^4U^{24} + X^8U^8 + T^8)V^{120} + (X^2U^{24} + X^{16})V^{118} + (U^{20} + X^4)V^{116} \\
&+ (X^{16}U^{96} + X^{32}U^{16} + X^{16}U^4 + T^{16}X^{16})V^{112} + U^8V^{111} + U^{16}X^8V^{108} \\
&+ (U^{120} + X^4U^{100} + T^4U^{96} + U^{28} + T^{16}U^{24} + X^4U^8 + T^{16}X^4U^4 + T^4U^4 + T^{20})V^{104} \\
&+ (X^2U^{96} + X^2U^4 + T^{16}X^2)V^{102} + U^{12}X^{16}V^{100} + U^{16}V^{99} + (U^{64} + X^8U^{24} + T^8U^{16})V^{96}
\end{aligned}
$$

$$
\begin{aligned}
&+ X^{16}V^{95} + U^4V^{94} + (X^{16}U^{48} + U^{36} + T^4U^{12} + T^8X^{16})V^{92} + U^{12}X^2V^{90} \\
&+ (U^{100} + X^{16}U^{20} + T^{16}U^4 + X^{20})V^{88} + X^8V^{86} \\
&+ (U^{72} + X^4U^{52} + T^4U^{48} + T^8U^{24} + X^{48}U^{16} + X^{32}U^4 + T^8X^4U^4 + T^{12})V^{84} + U^8X^{16}V^{83} \\
&+ (X^2U^{48} + U^{12} + T^8X^2)V^{82} \\
&+ (X^8U^{96} + X^{16}U^{56} + X^4U^{24} + T^8X^{16}U^8 + T^{16}X^8 + T^4X^4 + X^{64})V^{80} + X^6V^{78} \\
&+ (X^{32}U^{40} + X^{36}U^{20} + T^4X^{32}U^{16} + U^{16} + T^4X^{16}U^4 + X^{40})V^{76} \\
&+ (U^{32} + X^4U^{12} + T^4U^8)V^{75} + (U^{48} + X^{34}U^{16} + X^{18}U^4 + T^8)V^{74} + U^8X^2V^{73} \\
&+ (U^{80} + X^4U^{60} + T^4U^{56} + X^8U^{40} + T^8U^{32} + X^{12}U^{20} + T^4X^8U^{16} + X^{32}U^{12} + T^8X^4U^{12} \\
&\quad + T^{12}U^8 + X^{16})V^{72} \\
&+ (U^{96} + X^{16}U^{16} + T^{16})V^{71} + (X^2U^{56} + U^{20} + X^{10}U^{16} + T^8X^2U^8 + X^4)V^{70} \\
&+ (X^{16}U^{64} + T^4U^{28} + X^{24}U^{24} + T^8X^{16}U^{16} + T^4X^4U^8 + T^8U^4 + X^{28}U^4 + T^4X^{24})V^{68} \\
&+ (X^2U^{28} + X^6U^8 + X^{16}U^4 + X^{26})V^{66} + U^8V^{65} + (U^{24} + X^{20}U^{16} + X^{40}U^8 + X^4U^4)V^{64} \\
&+ X^8V^{63} + U^{16}X^8V^{62} \\
&+ (X^{16}U^{100} + U^{88} + X^4U^{68} + T^4U^{64} + X^8U^{48} + T^8U^{40} + T^8X^4U^{20} + T^{12}U^{16} \\
&+ X^{16}U^8 + T^{16}X^{16}U^4 + T^8X^8)V^{60} \\
&+ (U^{12} + X^{20}U^4 + T^4X^{16})V^{59} + (X^2U^{64} + T^8X^2U^{16} + X^{24})V^{58} + X^{18}V^{57} \\
&+ (X^{16}U^{72} + U^{60} + X^{20}U^{52} + T^4X^{16}U^{48} + X^4U^{40} + X^{24}U^{32} + T^8X^{16}U^{24} + T^4X^4U^{16} \\
&\quad + T^8U^{12} + X^{28}U^{12} + T^4X^{24}U^8 + T^8X^{20}U^4 + X^{12} + T^{12}X^{16})V^{56} \\
&+ U^8X^{32}V^{55} + (X^{18}U^{48} + X^6U^{16} + X^{26}U^8 + T^8X^{18})V^{54} + U^{16}V^{53} \\
&+ (U^{124} + X^4U^{104} + T^4U^{100} + X^{24}U^{96} + T^{16}U^{28} + X^{20}U^{24} + X^4U^{12} + T^{16}X^4U^8 \\
&\quad + T^{20}U^4 + T^4X^{20} + T^{16}X^{24})V^{52} \\
&+ (U^{48} + X^4U^{28} + T^4U^{24} + X^8U^8)V^{51} + (X^2U^{100} + T^{16}X^2U^4 + X^{22})V^{50} \\
&+ (X^2U^{24} + X^{16})V^{49} \\
&+ (U^{96} + X^8U^{56} + X^{32}U^{28} + X^{16}U^{16} + T^8X^8U^8 + X^{36}U^8 + T^4X^{32}U^4 + T^{16})V^{48} \\
&+ U^4X^{34}V^{46} \\
&+ (X^8U^{120} + X^{12}U^{100} + T^4X^8U^{96} + X^8U^{28} + T^{16}X^8U^{24} + X^{48}U^{12} + T^{16}X^{12}U^4 + T^4X^8U^4 \\
&\quad + T^{20}X^8)V^{44} \\
&+ (X^{16}U^{96} + X^{16}U^4 + T^{16}X^{16})V^{43} + (X^{10}U^{96} + U^8 + X^{10}U^4 + T^{16}X^{10})V^{42} \\
&+ (X^{40}U^{24} + X^{44}U^4 + X^8 + T^4X^{40})V^{40} + (U^{56} + X^8U^{16} + T^8U^8 + X^{48})V^{39} + X^{42}V^{38} \\
&+ (X^{48}U^{48} + X^{32}U^{36} + X^{16}U^{24} + X^{36}U^{16} + T^4X^{32}U^{12} + X^{20}U^4 + T^4X^{16} + T^8X^{48})V^{36} \\
&+ (U^{120} + X^4U^{100} + T^4U^{96} + T^{16}U^{24} + X^4U^8 + T^{16}X^4U^4 + T^{20})V^{35} + (X^{34}U^{12} + X^{18})V^{34} \\
&+ (X^2U^{96} + T^{16}X^2)V^{33} \\
&+ (U^{76} + X^4U^{56} + T^4U^{52} + X^8U^{36} + T^8U^{28} + X^{12}U^{16} + T^4X^8U^{12} + X^{32}U^8 + T^8X^4U^8 \\
&\quad + T^{12}U^4 + X^{52})V^{32} \\
&+ (X^2U^{52} + X^{10}U^{12} + T^8X^2U^4)V^{30} \\
&+ (X^{32}U^{72} + X^{36}U^{52} + T^4X^{32}U^{48} + X^4U^{28} + T^8X^{32}U^{24} + X^8U^8 \\
&\quad + T^8X^{36}U^4 + T^4X^4U^4 + X^{64}U^4 + T^{12}X^{32})V^{28} \\
&+ (X^8U^{24} + X^{48}U^8 + X^{12}U^4 + T^4X^8)V^{27} + (X^{34}U^{48} + X^6U^4 + X^{16} + T^8X^{34})V^{26}
\end{aligned}
$$

$$
\begin{aligned}
&+ (U^4 + X^{10})V^{25} \\
&+ (X^8U^{72} + X^{12}U^{52} + T^4X^8U^{48} + X^{16}U^{32} + T^8X^8U^{24} + X^{36}U^{24} + X^{20}U^{12} + T^4X^{16}U^8 \\
&\quad + T^8X^{12}U^4 + X^{40}U^4 + T^4X^{36} + T^{12}X^8)V^{24} \\
&+ (X^{16}U^{48} + X^{24}U^8 + T^8X^{16})V^{23} \\
&+ (U^{52} + X^{10}U^{48} + X^8U^{12} + X^{18}U^8 + T^8U^4 + X^{38} + T^8X^{10})V^{22} \\
&+ (X^{16}U^{96} + X^{48}U^{28} + X^{12}U^{24} + X^{32}U^{16} + X^{52}U^8 + X^{16}U^4 + T^4X^{48}U^4 \\
&\quad + T^4X^{12} + X^{72} + T^{16}X^{16})V^{20} \\
&+ (X^{32}U^{32} + X^{36}U^{12} + T^4X^{32}U^8 + X^{20})V^{19} + (X^4U^4 + X^{50}U^4 + X^{14})V^{18} \\
&+ (X^{34}U^8 + X^8)V^{17} + X^{48}V^{16} + (X^8U^{48} + X^{16}U^8 + T^8X^8)V^{14} \\
&+ (U^{120} + X^4U^{100} + T^4U^{96} + X^{16}U^{40} + X^{56}U^{24} + T^{16}U^{24} + X^{20}U^{20} + T^4X^{16}U^{16} \\
&\quad + X^{60}U^4 + T^{16}X^4U^4 + T^{20} + X^{24} + T^4X^{56})V^{12} \\
&+ X^{64}V^{11} + (X^2U^{96} + X^{18}U^{16} + X^{12} + T^{16}X^2 + X^{58})V^{10} + V^8 + (U^{48} + X^8U^8 + T^8)V^5 \\
&+ (X^8U^{24} + X^{12}U^4 + T^4X^8)V^4 + (X^{48}U^{24} + X^{52}U^4 + T^4X^{48})V^3 + X^{10}V^2 + (X^{50} + X^4)V
\end{aligned}
$$

$$
\begin{aligned}
C_5 =& U^{16}V^{272} + U^4V^{244} + X^8V^{236} + U^{12}V^{232} + V^{227} + (U^{48} + T^8)V^{224} + (U^{20} + X^4)V^{220} \\
&+ U^{16}X^8V^{212} + (U^{28} + X^4U^8 + T^4U^4)V^{208} + U^4X^2V^{206} + U^{16}V^{203} \\
&+ (X^8U^{24} + X^{12}U^4 + T^4X^8)V^{200} + X^{10}V^{198} + U^8V^{192} + U^{16}V^{180} + X^{16}V^{176} \\
&+ (U^{52} + X^8U^{12} + T^8U^4)V^{172} + (U^{24} + X^4U^4)V^{168} + (X^8U^{48} + T^8X^8)V^{164} + U^{12}V^{163} \\
&+ X^{12}V^{160} + V^{158} + (U^{32} + X^4U^{12} + T^4U^8)V^{156} + (U^{48} + T^8)V^{155} + U^8X^2V^{154} \\
&+ (X^{16}U^{16} + U^4)V^{152} + X^4V^{151} + X^8V^{144} + (X^{16}U^{24} + X^{20}U^4 + T^4X^{16})V^{140} \\
&+ (U^{28} + X^4U^8 + T^4U^4)V^{139} + X^{18}V^{138} + U^4X^2V^{137} + V^{135} + U^{16}V^{134} \\
&+ (X^8U^{24} + X^{12}U^4 + T^4X^8)V^{131} + X^{10}V^{129} + U^{16}X^{32}V^{124} + U^8V^{123} \\
&+ (U^{56} + X^8U^{16} + T^8U^8)V^{120} + (U^{28} + X^4U^8)V^{116} + V^{112} + U^{16}V^{111} + (U^{64} + T^8U^{16})V^{108} \\
&+ (X^{16}U^{48} + X^4U^{16} + X^{24}U^8 + T^8X^{16})V^{104} + X^{20}V^{100} + (X^4U^4 + T^4)V^{99} + X^2V^{97} \\
&+ (X^{48}U^{16} + X^{32}U^4)V^{96} + U^{12}V^{94} + (U^{44} + X^4U^{24} + T^4U^{20})V^{92} + U^{20}X^2V^{90} + V^{89} \\
&+ (X^{32}U^{40} + X^{36}U^{20} + U^{16} + T^4X^{32}U^{16} + X^{40})V^{88} + (U^{48} + X^{34}U^{16} + T^8)V^{86} \\
&+ (X^8U^{40} + X^{12}U^{20} + T^4X^8U^{16} + X^{32}U^{12} + X^{16})V^{84} + (U^{20} + X^{10}U^{16} + X^4)V^{82} \\
&+ (X^{16}U^{64} + U^{52} + X^{24}U^{24} + T^8X^{16}U^{16} + X^8U^{12} + T^8U^4)V^{80} + X^{32}V^{79} + U^4X^{16}V^{78} \\
&+ (X^{32}U^{48} + X^{20}U^{16} + T^4 + T^8X^{32})V^{76} + (U^{40} + X^4U^{20} + T^4U^{16})V^{75} + (X^8U^{16} + X^2)V^{74} \\
&+ U^{16}X^2V^{73} \\
&+ (U^{88} + X^4U^{68} + T^4U^{64} + T^8U^{40} + X^{12}U^{28} + T^4X^8U^{24} + T^8X^4U^{20} + T^{12}U^{16} + X^{16}U^8 \\
&\quad + T^8X^8 + X^{36})V^{72} \\
&+ (X^2U^{64} + X^{10}U^{24} + T^8X^2U^{16} + X^{24})V^{70} \\
&+ (X^4U^{40} + X^8U^{20} + T^4X^4U^{16} + X^{48}U^4 + X^{12})V^{68} + U^{16}X^6V^{66} + U^{16}V^{65} + U^{32}V^{64} \\
&+ (U^{96} + X^{16}U^{16} + U^4 + X^{56} + T^{16})V^{60} + U^{20}V^{59} + (X^{48}U^{12} + X^{32})V^{56} + U^8V^{54} \\
&+ (X^{16}U^{52} + U^{40} + X^4U^{20} + T^4U^{16} + X^{24}U^{12} + T^8X^{16}U^4)V^{52} + (U^{56} + T^8U^8 + X^{48})V^{51} \\
&+ U^{16}X^2V^{50} + (X^{48}U^{48} + X^{32}U^{36} + X^{36}U^{16} + T^4X^{32}U^{12} + X^{20}U^4 + T^8X^{48})V^{48}
\end{aligned}
$$

$$
\begin{aligned}
&+ U^8X^4V^{47} + U^{12}X^{34}V^{46}\\
&+ (U^{76} + X^4U^{56} + T^4U^{52} + X^{24}U^{48} + X^8U^{36} + T^8U^{28} + X^{12}U^{16} + T^4X^8U^{12} + T^8X^4U^8\\
&\quad + T^{12}U^4 + T^8X^{24} + X^{52})V^{44}\\
&+ (X^{32}U^{24} + X^{16}U^{12} + X^{36}U^4 + 1 + T^4X^{32})V^{43} + (X^2U^{52} + U^{16} + X^{10}U^{12} + T^8X^2U^4)V^{42}\\
&+ X^{34}V^{41}\\
&+ (X^{32}U^{72} + X^{16}U^{60} + X^{36}U^{52} + T^4X^{32}U^{48} + X^4U^{28} + T^8X^{32}U^{24} + X^{24}U^{20} + T^8X^{16}U^{12}\\
&\quad + X^8U^8 + T^8X^{36}U^4 + T^4X^4U^4 + X^{28} + T^{12}X^{32})V^{40}\\
&+ (X^{34}U^{48} + X^6U^4 + X^{16} + T^8X^{34})V^{38} + U^4V^{37}\\
&+ (X^8U^{72} + X^{12}U^{52} + T^4X^8U^{48} + X^{16}U^{32} + T^8X^8U^{24} + X^{36}U^{24} + T^4X^{16}U^8 + X^{40}U^4\\
&\quad + T^8X^{12}U^4 + T^{12}X^8 + T^4X^{36})V^{36}\\
&+ (X^{16}U^{48} + U^{36} + X^4U^{16} + T^4U^{12} + T^8X^{16})V^{35}\\
&+ (U^{52} + X^{10}U^{48} + X^8U^{12} + X^{18}U^8 + T^8U^4 + X^{38} + T^8X^{10})V^{34} + U^{12}X^2V^{33}\\
&+ (X^{16}U^{96} + U^{84} + X^4U^{64} + T^4U^{60} + X^{24}U^{56} + X^8U^{44} + T^8U^{36} + X^{48}U^{28} + T^4X^8U^{20}\\
&\quad + T^8X^4U^{16} + T^{12}U^{12} + T^8X^{24}U^8 + X^{52}U^8 + T^4X^{48}U^4 + T^{16}X^{16} + T^4X^{12})V^{32}\\
&+ X^{20}V^{31} + (X^2U^{60} + X^{10}U^{20} + T^8X^2U^{12} + X^4U^4 + X^{50}U^4 + X^{14})V^{30} + X^8V^{29}\\
&+ (X^4U^{36} + X^8U^{16} + T^4X^4U^{12} + X^{28}U^8)V^{28} + (X^8U^{32} + X^{12}U^{12} + T^4X^8U^8)V^{27}\\
&+ (X^8U^{48} + X^6U^{12} + X^{16}U^8 + T^8X^8)V^{26} + (U^{12} + X^{10}U^8)V^{25}\\
&+ (U^{120} + X^4U^{100} + T^4U^{96} + X^8U^{80} + X^{12}U^{60} + T^4X^8U^{56} + X^{32}U^{52} + T^8X^8U^{32} + U^{28}\\
&\quad + T^{16}U^{24} + X^{56}U^{24} + X^{40}U^{12} + T^8X^{12}U^{12} + T^{12}X^8U^8 + X^4U^8\\
&\quad + T^4U^4 + X^{60}U^4 + T^8X^{32}U^4 + T^{16}X^4U^4 + T^4X^{56} + T^{20})V^{24}\\
&+ (X^2U^{96} + X^{10}U^{56} + T^8X^{10}U^8 + X^2U^4 + X^{58} + X^{12} + T^{16}X^2)V^{22}\\
&+ (X^{12}U^{32} + X^{32}U^{24} + X^{16}U^{12} + T^4X^{12}U^8 + T^4X^{32})V^{20}\\
&+ (X^{16}U^{28} + X^{20}U^8 + T^4X^{16}U^4)V^{19} + (X^{14}U^8 + X^{34})V^{18} + (U^{48} + X^{18}U^4 + T^8)V^{17}\\
&+ (X^{16}U^{76} + X^{20}U^{56} + T^4X^{16}U^{52} + X^{40}U^{48} + X^{24}U^{36} + T^8X^{16}U^{28} + X^{28}U^{16} + T^4X^{24}U^{12}\\
&\quad + X^{48}U^8 + T^8X^{20}U^8 + T^{12}X^{16}U^4 + T^8X^{40})V^{16}\\
&+ X^{16}V^{15} + (X^{18}U^{52} + X^{26}U^{12} + T^8X^{18}U^4)V^{14} + X^4V^{13}\\
&+ (X^{20}U^{28} + X^{24}U^8 + T^4X^{20}U^4 + X^{44})V^{12} + (X^{24}U^{24} + X^{28}U^4 + T^4X^{24})V^{11}\\
&+ (X^{22}U^4 + X^{32})V^{10} + (X^{16}U^4 + X^{26})V^9\\
&+ (U^{100} + X^{24}U^{72} + X^{28}U^{52} + T^4X^{24}U^{48} + X^{32}U^{32} + T^8X^{24}U^{24} + X^{16}U^{20} + X^{36}U^{12}\\
&\quad + T^4X^{32}U^8 + U^8 + T^8X^{28}U^4 + T^{16}U^4 + T^{12}X^{24})V^8\\
&+ (U^{24} + X^4U^4 + T^4)V^7 + (X^{26}U^{48} + X^{34}U^8 + T^8X^{26})V^6 + X^2V^5\\
&+ (X^{28}U^{24} + X^{32}U^4 + T^4X^{28})V^4\\
&+ X^{30}V^2 + X^{24}V + T^{16}X^8 + X^{24}U^{16} + X^8U^4 + X^{64} + X^8U^{96}
\end{aligned}
$$

$$
\begin{aligned}
C_4 =& V^{256} + U^8V^{244} + (U^{24} + X^4U^4 + T^4)V^{220} + X^2V^{218} + V^{210} + U^2V^{207} + X^4V^{203}\\
&+ U^{12}V^{192} + U^8X^8V^{184} + U^8V^{175} + (U^{28} + X^4U^8 + T^4U^4)V^{168} + U^4X^2V^{166}\\
&+ (X^8U^{24} + X^{12}U^4 + T^4X^8)V^{160} + (U^4 + X^{10})V^{158} + U^6V^{155} + U^8V^{152} + (U^{24} + T^4)V^{151}
\end{aligned}
$$

$$
\begin{aligned}
&+ X^8V^{150} + X^2V^{149} + U^2X^8V^{147} + X^{12}V^{143} + V^{141} + U^2V^{138} + X^4V^{134} \\
&+ (U^{24} + X^4U^4 + T^4)V^{128} + X^2V^{126} + U^2V^{115} + (X^{16}U^{16} + U^4)V^{112} + X^4V^{111} + X^{32}V^{108} \\
&+ (U^{40} + X^4U^{20} + T^4U^{16} + X^8)V^{104} + U^{10}V^{103} + U^{16}X^2V^{102} + U^8X^4V^{99} + U^8X^{32}V^{96} \\
&+ (U^{48} + X^8U^8 + T^8)V^{92} + X^{16}V^{90} + X^4V^{88} + U^4X^{16}V^{84} + (U^{24} + X^4U^4 + T^4)V^{82} \\
&+ (U^{56} + X^8U^{16} + T^8U^8 + X^{48} + X^2)V^{80} + (U^{26} + X^4U^6 + T^4U^2)V^{79} + U^2X^2V^{77} \\
&+ (U^{28} + T^4U^4 + X^{24})V^{76} + (X^4U^{24} + X^8U^4 + T^4X^4)V^{75} + U^4X^2V^{74} + X^6V^{73} \\
&+ (X^{16}U^{12} + 1)V^{72} + U^2V^{69} + (X^8U^{24} + X^{48}U^8 + X^{12}U^4 + T^4X^8)V^{68} + (U^4 + X^{10})V^{66} \\
&+ X^4V^{65} + (U^{36} + X^4U^{16} + T^4U^{12} + X^{24}U^8)V^{64} + (X^2U^{12} + X^{32})V^{62} \\
&+ (X^{32}U^{32} + X^{16}U^{20} + X^{36}U^{12} + T^4X^{32}U^8 + U^8)V^{60} + U^2X^{32}V^{59} + (X^{34}U^8 + X^8)V^{58} \\
&+ (U^{72} + X^4U^{52} + T^4U^{48} + T^8U^{24} + X^{32}U^4 + T^8X^4U^4 + T^{12})V^{56} + (X^{16}U^8 + X^{36})V^{55} \\
&+ (X^2U^{48} + T^8X^2)V^{54} + (X^{16}U^{56} + U^{44} + T^4U^{20} + T^8X^{16}U^8 + X^8U^4 + T^4X^4)V^{52} \\
&+ U^{14}V^{51} + (X^2U^{20} + X^6)V^{50} + V^{49} + (X^{16}U^{28} + T^4X^{16}U^4 + X^{40})V^{48} + (U^{32} + T^4U^8)V^{47} \\
&+ (U^{48} + X^8U^8 + X^{18}U^4 + T^8)V^{46} + U^8X^2V^{45} \\
&+ (U^{80} + X^4U^{60} + T^4U^{56} + T^8U^{32} + X^{48}U^{24} + T^8X^4U^{12} + T^{12}U^8 + X^{52}U^4 \\
&\quad + X^{16} + T^4X^{48})V^{44} \\
&+ (U^{50} + X^{16}U^{16} + U^4 + T^8U^2)V^{43} + (X^2U^{56} + T^8X^2U^8 + X^4 + X^{50})V^{42} \\
&+ (X^4U^{32} + X^{24}U^{24} + X^8U^{12} + T^4X^4U^8 + X^{28}U^4 + T^4X^{24})V^{40} \\
&+ (X^4U^{48} + X^4U^2 + X^{32} + T^8X^4)V^{39} + (X^6U^8 + X^{26})V^{38} \\
&+ (X^{32}U^{48} + X^{40}U^8 + T^8X^{32})V^{36} + (U^{40} + X^4U^{20} + T^4U^{16} + X^{16}U^6)V^{35} \\
&+ (U^{10} + X^{48})V^{34} + U^{16}X^2V^{33} + (X^{16}U^8 + X^{36})V^{32} + (X^{16}U^{24} + X^{48}U^2 + T^4X^{16})V^{31} \\
&+ U^8X^4V^{30} + X^{18}V^{29} \\
&+ (X^{16}U^{72} + X^{20}U^{52} + T^4X^{16}U^{48} + X^{24}U^{32} + T^8X^{16}U^{24} + X^{28}U^{12} + T^4X^{24}U^8 + X^{48}U^4 \\
&\quad + T^8X^{20}U^4 + T^{12}X^{16})V^{28} \\
&+ (X^{24}U^2 + X^{52})V^{27} + (X^{18}U^{48} + X^{32}U^{24} + X^{26}U^8 + X^{36}U^4 + T^4X^{32} + T^8X^{18})V^{26} \\
&+ (U^{32} + X^{20}U^{24} + X^4U^{12} + T^4U^8 + X^{24}U^4 + X^{34} + T^4X^{20})V^{24} \\
&+ (X^{32}U^{26} + X^{36}U^6 + T^4X^{32}U^2 + X^{28})V^{23} + (X^2U^8 + X^{22})V^{22} + U^2X^{34}V^{21} \\
&+ (U^{96} + X^{32}U^{28} + X^{16}U^{16} + X^{36}U^8 + T^4X^{32}U^4 + U^4 + X^{56} + T^{16})V^{20} \\
&+ (X^{36}U^{24} + X^{40}U^4 + T^4X^{36})V^{19} + (X^{16}U^{48} + X^{24}U^8 + X^{34}U^4 + X^{16}U^2 + T^8X^{16})V^{18} \\
&+ (U^6 + X^{38})V^{17} + (X^{16}U^{50} + X^{24}U^{10} + X^{16}U^4 + T^8X^{16}U^2)V^{15} + U^4X^4V^{13} \\
&+ (X^{40}U^{24} + X^{44}U^4 + T^4X^{40})V^{12} + (X^{20}U^{48} + X^{28}U^8 + X^{20}U^2 + T^8X^{20} + X^{48})V^{11} \\
&+ (U^{72} + X^4U^{52} + T^4U^{48} + X^8U^{32} + T^8U^{24} + X^{12}U^{12} + T^4X^8U^8 + T^8X^4U^4 \\
&\quad + X^{42} + T^{12})V^{10} \\
&+ U^2X^8V^9 + (X^2U^{48} + X^{16}U^{24} + X^{10}U^8 + X^{20}U^4 + T^4X^{16} + T^8X^2)V^8 \\
&+ (U^{74} + X^4U^{54} + T^4U^{50} + X^8U^{34} + U^{28} + T^8U^{26} + X^{12}U^{14} + T^4X^8U^{10} + X^4U^8 \\
&\quad + T^8X^4U^6 + T^4U^4 + T^{12}U^2)V^7 \\
&+ (X^4U^{24} + X^8U^4 + X^{18} + T^4X^4)V^6 + (X^2U^{50} + X^{10}U^{10} + X^2U^4 + T^8X^2U^2 + X^{12})V^5 \\
&+ X^6V^4 \\
&+ (X^4U^{72} + X^8U^{52} + T^4X^4U^{48} + X^{12}U^{32} + X^4U^{26} + T^8X^4U^{24} + X^{32}U^{24} + X^{16}U^{12}
\end{aligned}
$$

$$
\begin{aligned}
&+T^4X^{12}U^8+X^8U^6+X^{36}U^4+T^8X^8U^4+T^4X^4U^2+T^4X^{32}+1+T^{12}X^4)V^3\\
&+(X^6U^{48}+X^{14}U^8+X^6U^2+T^8X^6+X^{34})V+U^{48}+X^8U^8+U^2+T^8
\end{aligned}
$$

$$
\begin{aligned}
C_3 =\,&V^{236}+U^8V^{224}+V^{213}+U^2V^{210}+X^4V^{206}+(U^{24}+X^4U^4+T^4)V^{200}+X^2V^{198}\\
&+U^4V^{184}+X^8V^{176}+U^{12}V^{172}+V^{167}+U^8X^8V^{164}+U^4V^{161}+U^6V^{158}+U^8V^{155}\\
&+U^4X^4V^{154}+X^8V^{153}+U^2X^8V^{150}+(U^{28}+X^4U^8+T^4U^4)V^{148}+(X^2U^4+X^{12})V^{146}\\
&+U^2V^{141}+(X^8U^{24}+X^{12}U^4+T^4X^8)V^{140}+X^{10}V^{138}+X^4V^{137}+U^8V^{132}\\
&+(U^{24}+X^4U^4+T^4)V^{131}+X^2V^{129}+V^{121}+U^{16}V^{120}+U^2V^{118}+X^4V^{114}+U^8V^{109}\\
&+(U^{24}+X^4U^4+T^4)V^{108}+(U^{10}+X^2)V^{106}+U^8X^{16}V^{104}+U^8X^4V^{102}\\
&+(U^{32}+X^4U^{12}+T^4U^8)V^{96}+U^8X^2V^{94}+X^{32}V^{88}+U^8V^{86}+(U^{24}+X^4U^4+T^4)V^{85}\\
&+X^2V^{83}+(U^{26}+X^4U^6+T^4U^2)V^{82}+(X^{16}U^{24}+X^{20}U^4+X^2U^2+T^4X^{16})V^{80}\\
&+(X^4U^{24}+X^8U^4+X^{18}+T^4X^4)V^{78}+(X^{32}U^8+X^6)V^{76}+V^{75}\\
&+(U^{48}+X^8U^8+U^2+T^8)V^{72}+X^{16}V^{70}+U^{20}V^{68}+X^{32}V^{65}+U^4X^{16}V^{64}+U^2X^{32}V^{62}\\
&+(U^{56}+T^8U^8+X^{48})V^{60}+X^{36}V^{58}+U^{12}V^{57}+(U^{28}+T^4U^4+X^{24})V^{56}\\
&+(U^{14}+X^2U^4)V^{54}+U^{16}V^{51}+U^{12}X^4V^{50}+(U^{48}+T^8)V^{49}\\
&+(X^8U^{24}+X^{48}U^8+X^{12}U^4+T^4X^8)V^{48}+X^{16}V^{47}+(U^{50}+T^8U^2+X^{10})V^{46}+X^4V^{45}\\
&+(X^{16}U^{48}+X^{24}U^8+T^8X^{16})V^{44}+(X^4U^{48}+X^4U^2+T^8X^4)V^{42}+U^4X^{16}V^{41}\\
&+(X^{32}U^{32}+X^{36}U^{12}+T^4X^{32}U^8+U^8+X^{20})V^{40}+(U^{24}+X^4U^4+T^4)V^{39}\\
&+(X^{34}U^8+X^{16}U^6+X^8)V^{38}+(U^{10}+X^{48}+X^2)V^{37}+(U^{12}+X^{20}U^4+X^{48}U^2)V^{34}\\
&+(X^4U^8+X^{24})V^{33}\\
&+(X^{16}U^{56}+X^{24}U^{16}+T^8X^{16}U^8)V^{32}+(X^{24}U^2+X^{52})V^{30}\\
&+(X^{32}U^{24}+X^{36}U^4+T^4X^{32})V^{29}+U^8X^{20}V^{28}+X^{34}V^{27}\\
&+(X^{32}U^{26}+X^8U^8+X^{36}U^6+T^4X^{32}U^2+X^{28})V^{26}\\
&+(U^{80}+X^4U^{60}+T^4U^{56}+X^8U^{40}+T^8U^{32}+X^{48}U^{24}+X^{12}U^{20}+T^4X^8U^{16}+T^8X^4U^{12}\\
&\quad+T^{12}U^8+X^{52}U^4+X^{34}U^2+T^4X^{48}+X^{16})V^{24}\\
&+U^4V^{23}+(X^2U^{56}+X^{36}U^{24}+X^{10}U^{16}+T^8X^2U^8+X^{40}U^4+T^4X^{36}+X^{50})V^{22}\\
&+(X^{16}U^{48}+X^{24}U^8+X^{16}U^2+T^8X^{16})V^{21}\\
&+(U^{52}+X^4U^{32}+T^4X^4U^8+U^6+T^8U^4+X^{38})V^{20}\\
&+(X^{16}U^{50}+X^{24}U^{10}+X^6U^8+X^{16}U^4+T^8X^{16}U^2)V^{18}+U^8V^{17}\\
&+(X^{32}U^{48}+X^{40}U^8+T^8X^{32})V^{16}+X^8V^{15}+(X^{20}U^{48}+X^{28}U^8+X^{20}U^2+T^8X^{20})V^{14}\\
&+(U^{72}+X^4U^{52}+T^4U^{48}+X^8U^{32}+T^8U^{24}+X^{12}U^{12}+T^4X^8U^8+T^8X^4U^4+T^{12})V^{13}\\
&+(X^8U^{48}+X^{16}U^8+X^8U^2+T^8X^8+X^{36})V^{12}+(X^2U^{48}+X^{10}U^8+T^8X^2)V^{11}\\
&+(U^{74}+X^4U^{54}+T^4U^{50}+X^8U^{34}+T^8U^{26}+X^{12}U^{14}+T^4X^8U^{10}+T^8X^4U^6+T^{12}U^2)V^{10}\\
&+(X^4U^{24}+X^8U^4+T^4X^4)V^9\\
&+(X^{16}U^{72}+X^{20}U^{52}+X^2U^{50}+T^4X^{16}U^{48}+X^{24}U^{32}+T^8X^{16}U^{24}+X^{28}U^{12}+X^{10}U^{10}\\
&\quad+T^4X^{24}U^8+T^8X^{20}U^4+T^8X^2U^2+T^{12}X^{16})V^8\\
&+X^6V^7
\end{aligned}
$$

$$
\begin{aligned}
&+ (X^4U^{72} + X^8U^{52} + X^{18}U^{48} + T^4X^4U^{48} + X^{12}U^{32} + X^4U^{26} + T^8X^4U^{24} + X^{16}U^{12} \\
&\quad + X^{26}U^8 + T^4X^{12}U^8 + X^8U^6 + T^8X^8U^4 + T^4X^4U^2 + T^8X^{18} + 1 + T^{12}X^4)V^6 \\
&+ (X^6U^{48} + X^{20}U^{24} + X^{14}U^8 + X^{24}U^4 + X^6U^2 + T^8X^6 + T^4X^{20})V^4 \\
&+ (U^{48} + X^8U^8 + U^2 + T^8)V^3 + (X^8U^{24} + X^{12}U^4 + T^4X^8 + X^{22})V^2 + X^{16}V \\
&+ U^4 + X^{16}U^{16} + T^{16} + U^{96} + X^{10}
\end{aligned}
$$

$$
\begin{aligned}
C_2 =&U^4V^{220} + (U^{12} + X^2U^2 + T^2)V^{208} + XV^{207} + V^{203} + U^8V^{168} + U^4X^8V^{160} \\
&+ (U^{16} + X^2U^6 + T^2U^4)V^{156} + U^4XV^{155} + (X^8U^{12} + X^{10}U^2 + T^2X^8)V^{148} + X^9V^{147} \\
&+ X^8V^{143} + (U^{12} + X^2U^2 + T^2)V^{139} + XV^{138} + V^{134} + U^4V^{128} + (X^2U^2 + T^2)V^{116} \\
&+ XV^{115} + V^{111} + U^2V^{108} + (U^{20} + X^2U^{10} + T^2U^8 + X^4)V^{104} + U^8XV^{103} + U^8V^{99} \\
&+ (U^{28} + X^4U^8 + T^4U^4)V^{92} + U^4X^2V^{90} + U^4V^{82} \\
&+ (U^{36} + X^2U^{26} + T^2U^{24} + X^4U^{16} + T^4U^{12} + X^6U^6 + T^2X^4U^4 \\
&\quad + T^4X^2U^2 + X^{16}U^2 + T^6)V^{80} \\
&+ (XU^{24} + X^5U^4 + T^4X)V^{79} + (X^2U^{12} + X^4U^2 + T^2X^2)V^{78} + X^3V^{77} + X^{20}V^{76} \\
&+ (U^{24} + X^4U^4 + T^4)V^{75} + X^2V^{73} + (U^{26} + X^4U^6 + X^{32}U^4 + T^4U^2)V^{72} + (U^{12} + T^2)V^{70} \\
&+ XV^{69} + (X^4U^{24} + X^8U^4 + T^4X^4)V^{68} + X^6V^{66} + V^{65} + U^{16}V^{64} \\
&+ (X^{32}U^{12} + X^{34}U^2 + T^2X^{32})V^{60} + X^{33}V^{59} + (U^{52} + U^6 + T^8U^4)V^{56} \\
&+ X^{32}V^{55} + (U^{24} + X^2U^{14} + T^2U^{12})V^{52} + U^{12}XV^{51} + (X^{16}U^8 + X^8U^2)V^{48} \\
&+ (U^{60} + X^2U^{50} + T^2U^{48} + T^8U^{12} + X^{48}U^4 + T^8X^2U^2 + X^{12} + T^{10})V^{44} \\
&+ (XU^{48} + T^8X)V^{43} + (X^4U^{12} + X^{24}U^4 + X^6U^2 + T^2X^4)V^{40} + (U^{48} + U^2 + T^8 + X^5)V^{39} \\
&+ (X^{32}U^{28} + X^{16}U^{16} + X^{36}U^8 + X^{18}U^6 + T^4X^{32}U^4 + T^2X^{16}U^4)V^{36} \\
&+ (U^{20} + X^2U^{10} + T^2U^8 + X^{17}U^4)V^{35} + (XU^8 + X^{34}U^4)V^{34} \\
&+ (X^{48}U^{12} + X^{50}U^2 + T^2X^{48})V^{32} + X^{49}V^{31} \\
&+ (X^{16}U^{52} + X^{16}U^6 + T^8X^{16}U^4 + X^{26}U^2 + T^2X^{24})V^{28} + (X^{48} + X^{25})V^{27} \\
&+ (X^{32}U^{36} + X^{34}U^{26} + T^2X^{32}U^{24} + X^{36}U^{16} + T^4X^{32}U^{12} + X^{38}U^6 + T^2X^{36}U^4 \\
&\quad + T^4X^{34}U^2 + T^6X^{32})V^{24} \\
&+ (X^{33}U^{24} + X^{37}U^4 + T^4X^{33} + X^{24})V^{23} + (X^{34}U^{12} + X^8U^4 + X^{36}U^2 + T^2X^{34})V^{22} \\
&+ X^{35}V^{21} \\
&+ (U^{76} + X^4U^{56} + T^4U^{52} + X^8U^{36} + U^{30} + T^8U^{28} + X^{12}U^{16} + T^4X^8U^{12} + X^4U^{10} \\
&\quad + T^8X^4U^8 + T^4U^6 + T^{12}U^4 + X^{21}U^2)V^{20} \\
&+ (X^{32}U^{24} + X^{16}U^{12} + X^{36}U^4 + X^{18}U^2 + T^2X^{16} + T^4X^{32})V^{19} \\
&+ (X^2U^{52} + U^{16} + X^{10}U^{12} + T^2U^4 + T^8X^2U^4 + X^{17})V^{18} + (XU^4 + X^{34})V^{17} \\
&+ (X^{16}U^{60} + X^{18}U^{50} + T^2X^{16}U^{48} + X^{24}U^{20} + T^8X^{16}U^{12} + X^{26}U^{10} + T^2X^{24}U^8 \\
&\quad + T^8X^{18}U^2 + X^{28} + T^{10}X^{16})V^{16} \\
&+ (X^{17}U^{48} + X^{25}U^8 + T^8X^{17})V^{15} + X^{16}V^{14} \\
&+ (X^8U^{26} + X^{20}U^{12} + X^{12}U^6 + T^4X^8U^2 + X^{22}U^2 + T^2X^{20})V^{12} \\
&+ (X^{16}U^{48} + X^{24}U^8 + X^{16}U^2 + T^8X^{16} + X^{21})V^{11} + (X^8U^{12} + T^2X^8)V^{10} + X^9V^9
\end{aligned}
$$

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\begin{aligned}
&+ (U^{84} + X^2U^{74} + T^2U^{72} + X^4U^{64} + T^4U^{60} + X^6U^{54} + T^2X^4U^{52} + T^4X^2U^{50} + T^6U^{48}\\
&\quad + X^8U^{44} + T^8U^{36} + X^{10}U^{34} + T^2X^8U^{32} + T^8X^2U^{26} + T^{10}U^{24} + T^4X^8U^{20} + T^8X^4U^{16}\\
&\quad + X^{14}U^{14} + T^2X^{12}U^{12} + T^{12}U^{12} + T^4X^{10}U^{10} + T^6X^8U^8 + T^8X^6U^6\\
&\quad + X^{16}U^4 + T^{10}X^4U^4 + T^{12}X^2U^2 + T^4X^{12} + T^{14})V^8\\
&+ (XU^{72} + X^5U^{52} + T^4XU^{48} + X^9U^{32} + T^8XU^{24} + X^{13}U^{12}\\
&\quad + T^4X^9U^8 + T^8X^5U^4 + T^{12}X)V^7\\
&+ (X^2U^{60} + X^4U^{50} + T^2X^2U^{48} + X^{10}U^{20} + T^8X^2U^{12} + X^{12}U^{10}\\
&\quad + T^2X^{10}U^8 + T^8X^4U^2 + T^{10}X^2 + X^{14})V^6\\
&+ (X^3U^{48} + X^{11}U^8 + T^8X^3 + X^8)V^5\\
&+ (X^4U^{36} + X^6U^{26} + T^2X^4U^{24} + X^8U^{16} + T^4X^4U^{12}\\
&\quad + X^{10}U^6 + T^2X^8U^4 + T^4X^6U^2 + T^6X^4)V^4\\
&+ (U^{72} + X^4U^{52} + T^4U^{48} + X^8U^{32} + U^{26} + T^8U^{24} + X^5U^{24} + X^{12}U^{12} + T^4X^8U^8 + X^4U^6\\
&\quad + X^9U^4 + T^8X^4U^4 + T^4U^2 + T^{12} + T^4X^5)V^3\\
&+ (X^6U^{12} + X^8U^2 + T^2X^6)V^2 + (X^2U^{48} + U^{12} + X^{10}U^8 + T^8X^2 + T^2 + X^7)V + X
\end{aligned}
$$

$$
\begin{aligned}
C_1 =\,& UV^{208} + X^2V^{206} + U^5V^{156} + U^4X^2V^{154} + UX^8V^{148} + X^{10}V^{146} + UV^{139} + X^2V^{137}\\
&+ UV^{116} + X^2V^{114} + V^{106} + U^9V^{104} + U^8X^2V^{102} + (U^{25} + X^4U^5 + T^4U)V^{80}\\
&+ (X^2U^{24} + X^6U^4 + X^2U + X^{16} + T^4X^2)V^{78} + X^4V^{76} + (U^{24} + X^4U^4 + U + T^4)V^{70}\\
&+ UX^{32}V^{60} + X^{34}V^{58} + U^4V^{54} + U^{13}V^{52} + U^{12}X^2V^{50} + X^8V^{46} + (U^{49} + T^8U)V^{44}\\
&+ (X^2U^{48} + T^8X^2)V^{42} + UX^4V^{40} + X^6V^{38} + V^{37} + U^5X^{16}V^{36} + U^9V^{35}\\
&+ U^4X^{18}V^{34} + U^8X^2V^{33} + UX^{48}V^{32} + X^{50}V^{30} + UX^{24}V^{28} + (X^{16}U^4 + X^{26})V^{26}\\
&+ (X^{32}U^{25} + X^{36}U^5 + T^4X^{32}U)V^{24} + (X^{34}U^{24} + X^{38}U^4 + X^{34}U + T^4X^{34})V^{22} + X^{36}V^{20}\\
&+ UX^{16}V^{19} + (U^{28} + X^4U^8 + U^5 + T^4U^4 + X^{24})V^{18} + X^{18}V^{17}\\
&+ (X^{16}U^{49} + X^{24}U^9 + T^8X^{16}U)V^{16} + (X^{18}U^{48} + X^{26}U^8 + T^8X^{18})V^{14} + UX^{20}V^{12}\\
&+ (X^8U^{24} + X^{12}U^4 + X^8U + T^4X^8 + X^{22})V^{10} + X^{16}V^9\\
&+ (U^{73} + X^4U^{53} + T^4U^{49} + X^8U^{33} + T^8U^{25} + X^{12}U^{13} + T^4X^8U^9 + T^8X^4U^5 + T^{12}U)V^8\\
&+ (X^2U^{72} + X^6U^{52} + X^2U^{49} + T^4X^2U^{48} + X^{10}U^{32} + T^8X^2U^{24} + X^{14}U^{12} + X^{10}U^9\\
&\quad + T^4X^{10}U^8 + T^8X^6U^4 + T^8X^2U + T^{12}X^2)V^6\\
&+ (X^4U^{48} + X^4U^{25} + X^{12}U^8 + X^8U^5 + T^4X^4U + T^8X^4)V^4\\
&+ (X^6U^{24} + X^{10}U^4 + X^6U + T^4X^6)V^2 + (U^{24} + X^4U^4 + U + T^4)V + X^8
\end{aligned}
$$

$$
\begin{aligned}
C_0 =\,& V^{207} + U^4V^{155} + X^8V^{147} + V^{138} + V^{115} + U^8V^{103} + (U^{24} + X^4U^4 + T^4)V^{79} + X^2V^{77}\\
&+ V^{69} + X^{32}V^{59} + U^{12}V^{51} + (U^{48} + T^8)V^{43} + X^4V^{39} + U^4X^{16}V^{35} + U^8V^{34} + X^{48}V^{31}\\
&+ X^{24}V^{27} + (X^{32}U^{24} + X^{36}U^4 + T^4X^{32})V^{23} + X^{34}V^{21} + X^{16}V^{18} + U^4V^{17}\\
&+ (X^{16}U^{48} + X^{24}U^8 + T^8X^{16})V^{15} + X^{20}V^{11} + X^8V^9\\
&+ (U^{72} + X^4U^{52} + T^4U^{48} + X^8U^{32} + T^8U^{24} + X^{12}U^{12} + T^4X^8U^8 + T^8X^4U^4 + T^{12})V^7\\
&+ (X^2U^{48} + X^{10}U^8 + T^8X^2)V^5 + (X^4U^{24} + X^8U^4 + T^4X^4)V^3 + X^6V + 1
\end{aligned}
$$

$$
\begin{aligned}
C_{-1} ={}& T^{16}V^{256} + T^{8}V^{236} + U^{4}T^{4}V^{220} + U^{16}T^{8}V^{212} + (T^{16}U^{32} + T^{4}U^{12})V^{208} + TV^{207} \\
&+ T^{2}X^{2}V^{206} + T^{4}V^{203} + (T^{8}U^{24} + T^{8}X^{4}U^{4} + T^{12})V^{200} + T^{8}X^{2}V^{198} + U^{4}T^{8}V^{184} \\
&+ T^{8}X^{8}V^{176} + U^{8}T^{4}V^{168} + T^{8}V^{167} + (T^{8}U^{20} + T^{4}X^{8}U^{4})V^{160} + (T^{16}U^{36} + T^{4}U^{16})V^{156} \\
&+ U^{4}TV^{155} + U^{4}T^{2}X^{2}V^{154} + U^{16}T^{8}X^{8}V^{152} \\
&+ (T^{16}X^{8}U^{32} + T^{8}U^{28} + T^{4}X^{8}U^{12} + T^{8}X^{4}U^{8} + T^{12}U^{4})V^{148} + TX^{8}V^{147} \\
&+ (T^{8}X^{2}U^{4} + T^{2}X^{10})V^{146} + T^{8}V^{144} + (T^{8}U^{16} + T^{4}X^{8})V^{143} \\
&+ (T^{8}X^{8}U^{24} + T^{8}X^{12}U^{4} + T^{12}X^{8})V^{140} + (T^{16}U^{32} + T^{4}U^{12})V^{139} + (T + T^{8}X^{10})V^{138} \\
&+ T^{2}X^{2}V^{137} + T^{4}V^{134} + (T^{8}U^{24} + T^{8}X^{4}U^{4} + T^{12})V^{131} + T^{8}X^{2}V^{129} + U^{4}T^{4}V^{128} \\
&+ U^{16}T^{8}V^{120} + TV^{115} + T^{2}X^{2}V^{114} + T^{4}V^{111} + (T^{32}U^{64} + T^{16}X^{32})V^{108} + T^{2}V^{106} \\
&+ (T^{16}U^{40} + T^{4}U^{20} + T^{8}X^{16}U^{8} + T^{4}X^{4})V^{104} + U^{8}TV^{103} + U^{8}T^{2}X^{2}V^{102} + U^{8}T^{4}V^{99} \\
&+ (T^{4}U^{28} + T^{4}X^{4}U^{8} + T^{8}U^{4})V^{92} + U^{4}T^{4}X^{2}V^{90} + (T^{16}X^{16}U^{32} + T^{8}X^{32})V^{88} \\
&+ (T^{8}U^{40} + T^{8}X^{4}U^{20} + T^{12}U^{16})V^{84} + (T^{8}X^{2}U^{16} + T^{4}U^{4})V^{82} \\
&+ (T^{32}X^{16}U^{64} + T^{4}U^{36} + T^{8}X^{16}U^{24} + T^{4}X^{4}U^{16} + T^{8}U^{12} + T^{16}X^{48})V^{80} \\
&+ (TU^{24} + TX^{4}U^{4} + T^{5})V^{79} + (T^{2}X^{2}U^{24} + T^{4}X^{2}U^{12} + T^{2}X^{6}U^{4} + T^{6}X^{2} + T^{2}X^{16})V^{78} \\
&+ TX^{2}V^{77} + (T^{4}X^{20} + T^{2}X^{4})V^{76} + (T^{4}U^{24} + T^{4}X^{4}U^{4} + T^{8})V^{75} + U^{16}T^{8}V^{74} + T^{4}X^{2}V^{73} \\
&+ (T^{32}U^{88} + T^{32}X^{4}U^{68} + T^{36}U^{64} + T^{8}U^{48} + T^{8}X^{4}U^{28} + T^{12}U^{24} + T^{16}X^{32}U^{24} \\
&\quad + T^{4}X^{32}U^{4} + T^{16}X^{36}U^{4} + T^{20}X^{32})V^{72} \\
&+ (T^{32}X^{2}U^{64} + T^{16}U^{32} + T^{8}X^{2}U^{24} + T^{2}U^{24} + T^{4}U^{12} + T^{2}X^{4}U^{4} \\
&\quad + T^{16}X^{34} + T^{8}X^{16} + T^{6})V^{70} \\
&+ TV^{69} + (T^{4}X^{4}U^{24} + T^{4}X^{8}U^{4} + T^{8}X^{4})V^{68} + T^{4}X^{6}V^{66} + T^{4}V^{65} \\
&+ (T^{16}U^{36} + T^{8}X^{32}U^{16} + T^{4}U^{16} + T^{8}X^{16}U^{4})V^{64} + (T^{16}X^{32}U^{32} + T^{4}X^{32}U^{12} + T^{8}X^{48})V^{60} \\
&+ TX^{32}V^{59} + T^{2}X^{34}V^{58} \\
&+ (T^{32}U^{68} + T^{4}U^{52} + T^{16}X^{8}U^{32} + T^{12}U^{4} + T^{16}X^{32}U^{4} + T^{8}X^{24})V^{56} \\
&+ T^{4}X^{32}V^{55} + U^{4}T^{2}V^{54} + (T^{16}U^{44} + T^{4}U^{24} + T^{8}X^{16}U^{12})V^{52} + U^{12}TV^{51} + U^{12}T^{2}X^{2}V^{50} \\
&+ (T^{8}U^{64} + T^{32}X^{8}U^{64} + T^{8}X^{8}U^{24} + T^{16}U^{16} + T^{4}X^{16}U^{8} + T^{16}X^{40})V^{48} \\
&+ (T^{16}U^{32} + T^{8}X^{16})V^{47} + T^{2}X^{8}V^{46} \\
&+ (T^{16}U^{80} + T^{4}U^{60} + T^{8}X^{16}U^{48} + T^{24}U^{32} + T^{8}X^{4}U^{16} + T^{12}U^{12} + T^{4}X^{48}U^{4} \\
&\quad + T^{16}X^{16} + T^{4}X^{12})V^{44} \\
&+ (TU^{48} + T^{9})V^{43} + (T^{2}X^{2}U^{48} + T^{10}X^{2})V^{42} \\
&+ (T^{16}X^{4}U^{32} + T^{8}X^{16}U^{20} + T^{4}X^{4}U^{12} + T^{4}X^{24}U^{4} + T^{8}X^{20})V^{40} \\
&+ (T^{32}U^{64} + T^{4}U^{48} + T^{16}X^{32} + TX^{4} + T^{12})V^{39} + T^{2}X^{6}V^{38} + T^{2}V^{37} \\
&+ (T^{4}X^{32}U^{28} + T^{8}X^{48}U^{16} + T^{4}X^{16}U^{16} + T^{4}X^{36}U^{8} + T^{8}X^{32}U^{4})V^{36} \\
&+ (T^{16}U^{40} + T^{4}U^{20} + T^{8}X^{16}U^{8} + TX^{16}U^{4})V^{35} + (TU^{8} + T^{4}X^{34}U^{4} + T^{2}X^{18}U^{4})V^{34} \\
&+ U^{8}T^{2}X^{2}V^{33} + (T^{16}X^{48}U^{32} + T^{8}X^{24}U^{16} + T^{4}X^{48}U^{12})V^{32} + TX^{48}V^{31} + T^{2}X^{50}V^{30} \\
&+ (T^{32}X^{16}U^{68} + T^{16}U^{60} + T^{4}X^{16}U^{52} + T^{16}X^{4}U^{40} + T^{8}X^{32}U^{40} + T^{20}U^{36} + T^{8}X^{36}U^{20} \\
&\quad + T^{12}X^{32}U^{16} + T^{8}X^{20}U^{8} + T^{16}X^{48}U^{4})V^{28}
\end{aligned}
$$

$$+ (T^4X^{48} + TX^{24})V^{27} + (T^{16}X^2U^{36} + T^8X^{34}U^{16} + T^2X^{16}U^4 + T^8X^{18}U^4 + T^2X^{26})V^{26}$$
$$+ (T^{16}X^{32}U^{56} + T^{16}X^{36}U^{36} + T^4X^{32}U^{36} + T^{20}X^{32}U^{32} + T^8X^{48}U^{24} + T^4X^{36}U^{16}$$
$$+ T^8X^{32}U^{12} + T^8X^{52}U^4 + T^{12}X^{48})V^{24}$$
$$+ (TX^{32}U^{24} + T^8X^{16}U^{16} + TX^{36}U^4 + T^4X^{24} + T^5X^{32})V^{23}$$
$$+ (T^{16}X^{34}U^{32} + T^2X^{34}U^{24} + T^8U^{20} + T^4X^{34}U^{12} + T^4X^8U^4 + T^2X^{38}U^4$$
$$+ T^6X^{34} + T^8X^{50})V^{22}$$
$$+ TX^{34}V^{21}$$
$$+ (T^{32}U^{92} + T^4U^{76} + T^{32}X^4U^{72} + T^{36}U^{68} + T^8X^{16}U^{64} + T^{32}X^{24}U^{64}$$
$$+ T^{16}X^8U^{56} + T^4X^4U^{56} + T^4X^8U^{36} + T^{16}X^{12}U^{36} + T^{20}X^8U^{32} + T^8X^4U^{32}$$
$$+ T^{16}X^{32}U^{28} + T^8X^{24}U^{24} + T^4X^{12}U^{16} + T^{16}X^{16}U^{16} + T^8X^8U^{12}$$
$$+ T^{16}X^{30}U^8 + T^{12}X^4U^8 + T^8X^{28}U^4 + T^{20}X^{32}U^4 + T^{16}U^4$$
$$+ T^{12}X^{24} + T^2X^{36} + T^{16}X^{56})V^{20}$$
$$+ (T^4X^{32}U^{24} + T^4X^{16}U^{12} + T^4X^{36}U^4 + T^8X^{32})V^{19}$$
$$+ (T^{32}X^2U^{68} + T^4X^2U^{52} + T^{16}U^{36} + T^{16}X^{10}U^{32} + T^2U^{28} + T^8X^2U^{28} + T^4U^{16}$$
$$+ T^4X^{10}U^{12} + T^2X^4U^8 + T^{16}X^{34}U^4 + T^{12}X^2U^4 + T^8X^{16}U^4 + T^6U^4$$
$$+ T^2X^{24} + TX^{16} + T^8X^{26})V^{18}$$
$$+ (TU^4 + T^2X^{18} + T^4X^{34})V^{17}$$
$$+ (T^{16}X^{16}U^{80} + T^{32}U^{64} + T^4X^{16}U^{60} + T^8X^{32}U^{48} + T^{16}X^{24}U^{40} + T^{24}X^{16}U^{32}$$
$$+ T^4X^{24}U^{20} + T^8X^{20}U^{16} + T^{12}X^{16}U^{12} + T^8X^{40}U^8 + T^4X^{28})V^{16}$$
$$+ (TX^{16}U^{48} + TX^{24}U^8 + T^9X^{16})V^{15}$$
$$+ (T^2X18U^{48} + T^8X^8U^{16} + T^2X^{26}U^8 + T^{10}X^{18} + T^4X^{16})V^{14}$$
$$+ (T^8U^{88} + T^{32}X^8U^{88} + T^{32}X^{12}U^{68} + T^8X^4U^{68} + T^{12}U^{64} + T^{36}X^8U^{64} + T^{16}U^{40}$$
$$+ T^{16}X^{20}U^{32} + T^{16}X^{40}U^{24} + T^{16}X^4U^{20} + T^{20}U^{16} + T^4X^{20}U^{12} + T^{16}X^{44}U^4$$
$$+ T^{20}X^{40} + T^8X^{36})V^{12}$$
$$+ (T^{32}X^{16}U^{64} + T^{16}U^{56} + T^4X^{16}U^{48} + T^{16}X^4U^{36} + T^{20}U^{32} + T^4X^{24}U^8 + T^8X^{20}U^4$$
$$+ T^{16}X^{48} + TX^{20})V^{11}$$
$$+ (T^8X^2U^{64} + T^{32}X^{10}U^{64} + T^{16}X^8U^{32} + T^2X^8U^{24} + T^{16}X^2U^{16} + T^4X^8U^{12} + T^2X^{12}U^4$$
$$+ T^6X^8 + T^8X^{24} + T^{16}X^{42} + T^2X^{22})V^{10}$$
$$+ (T^{16}X^2U^{32} + T^8X^{18} + TX^8 + T^2X^{16})V^9$$
$$+ (T^{16}U^{104} + T^4U^{84} + T^{16}X^4U^{84} + T^{20}U^{80} + T^8X^{16}U^{72} + T^4X^4U^{64} + T^{16}X^8U^{64} + T^8U^{60}$$
$$+ T^{24}U^{56} + T^8X^{20}U^{52} + T^{12}X^{16}U^{48} + T^4X^8U^{44} + T^{16}X^{12}U^{44}$$
$$+ T^8X^4U^{40} + T^{20}X^8U^{40} + T^{12}U^{36} + T^{24}X^4U^{36} + T^8X^{24}U^{32} + T^{28}U^{32}$$
$$+ T^{16}X^{16}U^{24} + T^8X^{28}U^{12} + T^{16}U^{12} + T^{12}X^{24}U^8 + T^{16}X^{20}U^4 + T^4X^{16}U^4$$
$$+ T^8X^{12} + T^{20}X^{16})V^8$$
$$+ (TU^{72} + TX^4U^{52} + T^5U^{48} + TX^8U^{32} + T^9U^{24} + TX^{12}U^{12} + T^5X^8U^8$$
$$+ T^9X^4U^4 + T^{13})V^7$$
$$+ (T^{16}X^2U^{80} + T^2X^2U^{72} + T^4X^2U^{60} + T^2X^6U^{52} + T^6X^2U^{48} + T^8X^{18}U^{48} + T^{16}X^{10}U^{40}$$
$$+ T^{24}X^2U^{32} + T^2X^{10}U^{32} + T^{10}X^2U^{24} + T^4X^{10}U^{20} + T^8X^6U^{16} + T^2X^{14}U^{12} + T^{12}X^2U^{1}$$

$$
\begin{aligned}
&+T^8X^{26}U^8+T^6X^{10}U^8+T^{10}X^6U^4+T^4X^{14}+T^{14}X^2+T^{16}X^{18})V^6\\
&+(TX^2U^{48}+T^8U^{16}+TX^{10}U^8+T^4X^8+T^9X^2)V^5\\
&+(T^{16}X^4U^{56}+T^2X^4U^{48}+T^4X^4U^{36}+T^{16}X^8U^{36}+T^{20}X^4U^{32}+T^8X^{20}U^{24}+T^4X^8U^{16}\\
&\quad+T^8X^4U^{12}+T^2X^{12}U^8+T^8X^{24}U^4+T^{10}X^4+T^{12}X^{20})V^4\\
&+(T^{32}U^{88}+T^4U^{72}+T^{32}X^4U^{68}+T^{36}U^{64}+T^4X^4U^{52}+T^4X^8U^{32}+T^8X^4U^{28}\\
&\quad+T^{16}X^{32}U^{24}+TX^4U^{24}+T^4X^{12}U^{12}+T^8X^8U^8+T^{12}X^4U^4+TX^8U^4+T^{16}X^{36}U^4\\
&\quad+T^{16}+T^{20}X^{32}+T^5X^4)V^3\\
&+(T^{16}X^6U^{32}+T^2X^6U^{24}+T^4X^6U^{12}+T^2X^{10}U^4+T^8X^{22}+T^6X^6)V^2\\
&+(T^{32}X^2U^{64}+T^4X^2U^{48}+T^{16}U^{32}+T^2U^{24}+T^8X^2U^{24}+T^4U^{12}+T^4X^{10}U^8+T^2X^4U^4\\
&\quad+T^{16}X^{34}+T^6+T^{12}X^2+TX^6+T^8X^{16})V\\
&+T^8U^{96}+T^{64}U^{128}+T^8X^{16}U^{16}+T^{16}U^{48}+T^{32}X^{64}+T^{24}+T^2X^8+T
\end{aligned}
$$

References

[1] S. Abhyankar, *Mathieu group coverings in characteristic two*, C. R. Acad. Sci. Paris **316** (1993), 63–68.

[2] S. Abhyankar, J. Ou, and A. Sathaye, *Alternating group coverings of the affine line for characteristic two*, Discrete Math. **133** (1994), 25–46.

[3] S. Abhyankar and I. Yie, *Some more Mathieu group coverings of the affine line in characteristic two*, Proc. Amer. Math. Soc. **122** (1994), 1–23.

[4] ———, *Small Mathieu group coverings of the affine line in characteristic two*, Proc. Amer. Math. Soc. **123** (1995), 1319–1329.

[5] J. H. Conway and N. J. A. Sloan *Sphere packings, lattices and groups*, Springer Verlag, New York, 1993

[6] V. Pless, *On the uniqueness of the Golay codes*, J. Combin. Theory **5** (1968), 215–228.

[7] V. Pless and N. J. A. Sloane, *On the classification and enumeration of self-dual codes*, J. Combin. Theory Ser. A **18** (1975), 313–335.

Department of Mathematics
Inha University
Inchon 402-751, Korea
E-mail: ikyie@math.inha.ac.kr