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ON SOME COMPLETE COMPLEX SUBMANIFOLDS IN A LOCALLY SYMMETRIC KAEHLER MANIFOLD

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ABSTRACT. The purpose of this paper is to study complete complex submanifolds of a locally symmetric Kaehler manifold with non-positive totally real normal bisectional curvature and k-pinched totally real tangent bisectional curvature.

1. Introduction

The theory of complex submanifolds of a complex space form is one of the most interesting topics in differential geometry and it has been investigated by many geometers from the various different points of view ([1], [2] and [5]-[9]).

Let M be an n-dimensional complex submanifold of an $(n+p)$-dimensional Kaehler manifold M'. We denote by $H'(P', Q')$ the holomorphic bisectional curvature of M' for any holomorphic planes P' and Q'. It is totally real bisectional curvature if P' and Q' are orthogonal ([4]). It is also said to be *tangent* (resp. *normal*) if P' and Q' are both tangent to M (resp. either P' or Q' is normal to M). The normal holomorphic bisectional curvature is closely related to the normal curvature of M.

It seems to be interesting for us to give an information about the squared norm $|\alpha|^2 = h_2$ of the second fundamental form α of M in order to solve the Chern-type problem in Kaehler geometry which is given as below.

Problem. *For an n-dimensional complete complex submanifold M of an $(n+p)$-dimensional complex space form $M^{n+p}(c)$ of constant holo-*

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morphic sectional curvature c, does there exist a constant h in such a way that M is totally geodesic provided $h_2 < h$ (or $h_2 > h$)?

The purpose of this paper is to research the Chern-type problem in the case where the ambient space is a locally symmetric Kaehler manifold which has non-positive totally real normal bisectional curvature and k-pinched totally real tangent bisectional curvature. More precisely, we shall prove

THEOREM. *Let* M' *be an* $(n+p)$*-dimensional locally symmetric Kaehler manifold which has non-positive totally real normal bisectional curvature and* k*-pinched totally real tangent bisectional curvature* ($\frac{4}{5} < k \leqq \frac{10}{11}$). *Let* M *be an* n*-dimensional complete complex submanifold of which normal connection in* M' *is proper. Then there exists a constant* h *in such a way that* M *is totally geodesic provided* $h_2 < h$.

2. Kaehler manifolds

Let M be a complex $m(\geqq 2)$-dimensional connected Kaehler manifold equipped with Kaehler metric g and almost complex structure J. For the Kaehler structure $\{g, J\}$, it follows that J is integrable. We can choose a local field $\{E_\alpha\} = \{E_A, E_{A^*}\} = \{E_1, \dots, E_m, E_{1^*}, \dots, E_{m^*}\}$ of orthonormal frames on a neighborhood of M, where $E_{A^*} = JE_A$ and $A^* = m + A$. Here the indices $A, B, \dots$ run from 1 to m and the indices $\alpha, \beta, \dots$ run from 1 to $2m = m^*$. We set $U_A = \frac{1}{\sqrt{2}}(E_A - iE_{A^*})$ and $\overline{U}_A = \frac{1}{\sqrt{2}}(E_A + iE_{A^*})$, where i is the imaginary unit. Then $\{U_A\}$ constitutes a local field of unitary frames on the neighborhood of M. This is a complex linear frame which is orthonormal with respect to the Kaehler metric, that is, $g(U_A, \overline{U}_B) = \delta_{AB}$. Let $\{\theta_\alpha\}$, $\{\theta_{\alpha\beta}\}$ and $\{\Theta_{\alpha\beta}\}$ be the canonical form, the connection form and the curvature form, respectively on M with respect to the local field $\{E_\alpha\} = \{E_A, E_{A^*}\}$ of orthonormal frames. Then we have the following structure equations

$$
\begin{aligned}
&d\theta_\alpha + \sum_\beta \theta_{\alpha\beta} \wedge \theta_\beta = 0, \ \theta_{\alpha\beta} - \theta_{\alpha^*\beta^*} = 0, \\
(2.1) \qquad &\theta_{\alpha^*\beta} + \theta_{\alpha\beta^*} = 0, \ \theta_{\alpha\beta} + \theta_{\beta\alpha} = 0, \ \theta_{\alpha\beta^*} - \theta_{\beta\alpha^*} = 0, \\
&d\theta_{\alpha\beta} + \sum_\gamma \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta} = \Theta_{\alpha\beta}, \ \Theta_{\alpha\beta} = -\frac{1}{2}\sum_{\gamma,\delta} K_{\alpha\beta\gamma\delta}\theta_\gamma \wedge \theta_\delta,
\end{aligned}
$$

where $K_{\alpha\beta\gamma\delta}$ denotes the components of the Riemannian curvature tensor of M.

Now, let $\{\omega_A\}$ be the dual coframe field with respect to the local field $\{U_A\}$ of unitary frames on the neighborhood of M. Then $\{\omega_A\} = \{\omega_1, \dots, \omega_m\}$ consists of complex valued 1-forms of type $(1, 0)$ on M such that $\omega_A(U_B) = \delta_{AB}$ and $\omega_1, \dots, \omega_m, \overline{\omega}_1, \dots, \overline{\omega}_m$ are linearly independent. The Kaehler metric g of M can be expressed as $g = 2\sum_A \omega_A \otimes \overline{\omega}_A$. Associated with the frame field $\{U_A\}$, there exist complex valued forms ω_{AB}, which are usually called *connection forms* on M such that they satisfy the structure equations of M:

$$
\begin{aligned}
&d\omega_A + \sum_B \omega_{AB} \wedge \omega_B = 0, \\
&\omega_{AB} + \overline{\omega}_{BA} = 0, \\
&d\omega_{AB} + \sum_C \omega_{AC} \wedge \omega_{CB} = \Omega_{AB}, \\
&\Omega_{AB} = \sum_{C,D} R_{\bar{A}BC\bar{D}} \omega_C \wedge \overline{\omega}_D,
\end{aligned} \tag{2.2}
$$

where Ω_{AB} (resp. $R_{\bar{A}BC\bar{D}}$) denotes the curvature form (resp. the components of the Riemannian curvature tensor R) of M. So, by (2.1) and (2.2), we obtain

$$
R_{\bar{A}BC\bar{D}} = -\{(K_{ABCD} + K_{A^*BC^*D}) + i(K_{A^*BCD} - K_{ABC^*D})\}. \tag{2.3}
$$

(2.2) implies the skew-Hermitian symmetry of Ω_{AB}, which is equivalent to the symmetric condition

$$
R_{\bar{A}BC\bar{D}} = \overline{R}_{\bar{B}AD\bar{C}}. \tag{2.4}
$$

Moreover, the first Bianchi equation $\sum_B \Omega_{AB} \wedge \omega_B = 0$ is given by the exterior differential of the first equation and the third equation of (2.2), which implies the further symmetric relations

$$
R_{\bar{A}BC\bar{D}} = R_{\bar{A}CB\bar{D}} = R_{\bar{D}CB\bar{A}} = R_{\bar{D}BC\bar{A}}. \tag{2.5}
$$

Next, relative to the frame field chosen above, the Ricci tensor S of M can be expressed as follows:

$$
S = \sum_{A,B} (S_{A\bar{B}} \omega_A \otimes \overline{\omega}_B + S_{\bar{A}B} \overline{\omega}_A \otimes \omega_B),
$$

where $S_{A\bar{B}} = \sum_C R_{\bar{C}CA\bar{B}} = S_{\bar{B}A} = \overline{S}_{\bar{A}B}$. The scalar curvature r of M is also given by $r = 2\sum_A S_{A\bar{A}}$.

The components $R_{\bar{A}BC\bar{D}:E}$ and $R_{\bar{A}BC\bar{D}:\bar{E}}$ of the covariant derivative of the Riemannian curvature tensor R are given by

$$\begin{aligned}(2.6)\qquad &\sum_E (R_{\bar{A}BC\bar{D}:E}\omega_E + R_{\bar{A}BC\bar{D}:\bar{E}}\overline{\omega}_E) \\ &= dR_{\bar{A}BC\bar{D}} - \sum_E (R_{\bar{E}BC\bar{D}}\overline{\omega}_{EA} + R_{\bar{A}EC\bar{D}}\omega_{EB} \\ &\qquad + R_{\bar{A}BE\bar{D}}\omega_{EC} + R_{\bar{A}BC\bar{E}}\overline{\omega}_{ED}).\end{aligned}$$

The second Bianchi identity is given by

$$(2.7)\qquad R_{\bar{A}BC\bar{D}:E} = R_{\bar{A}BE\bar{D}:C}.$$

On the other hand, if a plane P is invariant by the complex structure J, then it said to be *holomorphic*. For the plane P spanned by X and Y in P, the sectional curvature $K(P)$ is usually defined by

$$K(P) = K(X,Y) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

and the sectional curvature $K(P)$ of the holomorphic plane P is called the *holomorphic sectional curvature*, which is denoted by $H(P)$. The Kaehler manifold M is said to be *of constant holomorphic sectional curvature* if its holomorphic sectional curvatures $H(P)$ are constant for all holomorphic planes at all points on M. Then M is called a *complex space form*, which is denoted by $M^m(c)$, provided that it is of constant holomorphic sectional curvature c, of complex dimension m. It is seen in Wolf [11] that the standard models of complex space forms are the following three kinds : the complex projective space CP^m, the complex Euclidean space C^m or the complex hyperbolic space CH^m, according as $c > 0$, $c = 0$ or $c < 0$. It is also shown in [11] that they are complete simply connected complex space forms of dimension m. The Riemannian curvature tensor $R_{\bar{A}BC\bar{D}}$ of $M^m(c)$ is given by

$$R_{\bar{A}BC\bar{D}} = \frac{c}{2}(\delta_{AB}\delta_{CD} + \delta_{AC}\delta_{BD}).$$

Given two holomorphic planes P and Q in T_xM at any point x in M, the holomorphic bisectional curvature $H(P,Q)$ determined by the two planes P and Q of M is defined by

$$(2.8)\qquad H(P,Q) = \frac{g(R(X,JX)JY,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2},$$

where X (resp. Y) is a non-zero vector in P (resp. Q). It is a simple matter to verify that the right hand side in (2.8) depends only on P and Q and so it is well defined. It may be also denoted by $H(P,Q) = H(X,Y)$. It is easily seen that $H(P,P) = H(P) = H(X,X) =: H(X)$ is the holomorphic sectional curvature determined by the holomorphic plane P. We denote by P_A the holomorphic plane $[E_A, JE_A]$ spanned by E_A and $JE_A = E_{A^*}$. We set

$$H(P_A, P_B) = H(E_A, E_B) =: H_{AB} \quad (A \neq B),$$
$$H(P_A, P_A) = H(P_A) = H_{AA} =: H_A.$$

The holomorphic bisectional curvature H_{AB} $(A \neq B)$ and the holomorphic sectional curvature H_A are given by

$$H_{AB} = \frac{g(R(E_A, JE_A)JE_B, E_B)}{g(E_A, E_A)g(E_B, E_B)} = -K_{AA^*BB^*}, \quad A \neq B,$$
$$H_A = \frac{g(R(E_A, JE_A)JE_A, E_A)}{g(E_A, E_A)g(E_A, E_A)} = -K_{AA^*AA^*}.$$

By (2.3), we have

$$H_{AB} = R_{\bar{A}AB\bar{B}} \ (A \neq B), \quad H_A = R_{\bar{A}AA\bar{A}}. \tag{2.9}$$

Using the first Bianchi identity the holomorphic bisectional curvature $H(X,Y)$ for any non-zero vectors X and Y can be reformed as

$$H(X,Y) = -\frac{g(R(JX,JY)X,Y) + g(R(JY,X)JX,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

Since the Kaehler connection ∇ on M is almost complex, we have $\nabla J = 0$, from which it follows that the Riemannian curvature tensor R of the Kaehler manifold possesses the properties

$$R(X,Y) \circ J = J \circ R(X,Y), \quad R(JX,JY) = R(X,Y).$$

So we see that

$$H(X,Y) = -\frac{g(R(X,Y)X,Y) + g(R(JY,X)JX,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

Suppose that X, Y, JX and JY are orthonormal. Then we get

$$H(X,Y) = K(X,Y) + K(X,JY). \tag{2.10}$$

For the orthonormal set $\{X, JX, Y, JY\}$, we set $X' = \frac{1}{\sqrt{2}}(X+Y)$ and $Y' = \frac{1}{\sqrt{2}}J(X-Y)$. Then it is easily seen that the set $\{X', JX', Y', JY'\}$ is also the orthonormal one. So (2.10) implies

$$H(X', Y') = K(X', Y') + K(X', JY'). \tag{2.11}$$

On the other hand, we have

$$\begin{aligned} H(X', Y') &= g(R(X', JX')JY', Y') \\ &= \frac{1}{4}\{H(X) + H(Y) + 2H(X, Y) - 4K(X, JY)\} \end{aligned}$$

and hence we have

$$4H(X', Y') = 2H(X, Y) + H(X) + H(Y) - 4K(X, JY). \tag{2.12}$$

Next we set $X'' = \frac{1}{\sqrt{2}}(X + JY)$ and $Y'' = \frac{1}{\sqrt{2}}(JX + Y)$. Then it is easily seen that the set $\{X'', JX'', Y'', JY''\}$ is also the orthonormal one. Similarly it follows from (2.12) that

$$4H(X'', Y'') = 2H(X, Y) + H(X) + H(Y) - 4K(X, Y). \tag{2.13}$$

Summing up (2.12) and (2.13) and using (2.10) we have

$$2H(X', Y') + 2H(X'', Y'') = H(X) + H(Y) \tag{2.14}$$

for any orthonormal vectors X, Y, JX, and JY.

Now, we introduce a fundamental property for the generalized maximum principal due to Omori [10] and Yau [13].

THEOREM O-Y. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below on M. If a C^2-function f is bounded from above on M, then, for any positive constant ε, there exists a point P such that*

$$|\nabla f(P)| < \varepsilon, \quad \Delta f(P) < \varepsilon, \quad \sup f - \varepsilon < f(P).$$

If a C^2-function f is bounded from below on M, then, for any positive constant ε, there exists a point P such that

$$|\nabla f(P)| < \varepsilon, \quad \Delta f(P) > -\varepsilon, \quad \inf f + \varepsilon > f(P),$$

where ∇f is the gradient of the function f and Δ denotes the Laplacian operator on M.

3. Complex submanifolds

Let M' be an $(n+p)$-dimensional connected Kaehler manifold with the Kaehler structure (g', J'). Let M be an n-dimensional connected complex submanifold of M' and let g be the induced Kaehler metric on M from g'. We can choose a local field $\{U_A\} = \{U_j, U_x\} = \{U_1, \dots, U_{n+p}\}$ of unitary frames on a neighborhood of M' in such a way that restricted to M, $U_1, \dots, U_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices is used throughout this paper, unless otherwise stated :

$$A,\ B,\ C,\ \cdots = 1,\ \dots,\ n,\ n+1,\ \dots,\ n+p\ ;$$
$$i,\ j,\ k,\ \cdots = 1,\ \dots,\ n; \qquad x,\ y,\ z,\ \cdots = n+1,\ \dots,\ n+p.$$

With respect to the frame field $\{U_A\}$, let $\{\omega_A\} = \{\omega_i, \omega_x\}$ be its dual frame field. Then the Kaehler metric tensor g' of M' is given by $g' = 2\sum_A \omega_A \otimes \overline{\omega}_A$. The canonical forms ω_A and the connection forms ω_{AB} of the ambient space M' satisfy the structure equations appeared in (2.2). Restricting these forms to the submanifold M, we have

$$\omega_x = 0, \tag{3.1}$$

and the induced Kaehler metric tensor g of M is given by $g = 2\sum_j \omega_j \otimes \overline{\omega}_j$. Moreover $\{U_j\}$ is a local unitary frame field with respect to the induced metric and $\{\omega_j\}$ is a local dual frame field due to $\{U_j\}$, which consists of complex valued 1-forms of type (1.0) on M. Of course, $\omega_1, \dots, \omega_n, \overline{\omega}_1, \dots, \overline{\omega}_n$ are linearly independent, and $\{\omega_j\}$ is the canonical forms on M. It follows from (3.1) and Cartan's lemma that the exterior derivative of (3.1) give rise to

$$\omega_{xi} = \sum_j h_{ij}{}^x \omega_j, \quad h_{ij}{}^x = h_{ji}{}^x. \tag{3.2}$$

The quadratic form $\alpha = \sum_{x,i,j} h_{ij}{}^x \omega_i \otimes \omega_j \otimes U_x$ with values in the normal bundle NM on M in M' is called the *second fundamental form* of the submanifold M. From the structure equations for M', the structure equations for M are similarly given by

$$\begin{aligned} &d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \overline{\omega}_{ji} = 0, \\ &d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \quad \Omega_{ij} = \sum_{k.l} R_{\bar{i}jk\bar{l}} \omega_k \wedge \overline{\omega}_l, \end{aligned} \tag{3.3}$$

where $\Omega = (\Omega_{ij})$ (resp. $R_{\bar{i}jk\bar{l}}$) denotes the curvature form (resp. the component of the Riemannian curvature tensor R) of M.

Moreover, the following relationships are obtained :

$$d\omega_{xy} + \sum_{z} \omega_{xz} \wedge \omega_{zy} = \Omega_{xy}, \quad \Omega_{xy} = \sum_{k,l} R_{\bar{x}yk\bar{l}} \omega_k \wedge \overline{\omega}_l, \tag{3.4}$$

where Ω_{xy} is called the *normal curvature form* of M and $R_{\bar{x}yk\bar{l}}$ are the components of the normal curvature tensor of M. For the Riemannian curvature tensors R and R' of M and M' respectively, it follows from (2.2) and (3.1)-(3.3) that we have the Gauss equation

$$R_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} - \sum_{x} h_{jk}{}^{x} \overline{h}_{il}{}^{x}. \tag{3.5}$$

And also by means of (2.2), (3.1), (3.2), and (3.4), we have

$$R_{\bar{x}yk\bar{l}} = R'_{\bar{x}yk\bar{l}} + \sum_{j} h_{kj}{}^{x} \overline{h}_{jl}{}^{y}. \tag{3.6}$$

The components $S_{i\bar{j}}$ of the Ricci tensor S and the scalar curvature r of M are given by

$$S_{i\bar{j}} = \sum_{k} R'_{\bar{j}ik\bar{k}} - h_{i\bar{j}}{}^{2}, \quad r = 2(\sum_{j,k} R'_{\bar{j}jk\bar{k}} - h_2), \tag{3.7}$$

where we put $h_{i\bar{j}}{}^2 = h_{\bar{j}i}{}^2 = \sum_{x,k} h_{ik}{}^x \overline{h}_{kj}{}^x$ and $h_2 = \sum_j h_{j\bar{j}}{}^2$.

On the other hand, the components $h_{ijk}{}^x$ and $h_{ij\bar{k}}{}^x$ of the covariant derivative of the second fundamental form on M are given by

$$\begin{aligned} &\sum_{k} (h_{ijk}{}^{x} \omega_k + h_{ij\bar{k}}{}^{x} \overline{\omega}_k) \\ &= dh_{ij}{}^{x} - \sum_{k} (h_{kj}{}^{x} \omega_{ki} + h_{ik}{}^{x} \omega_{kj}) + \sum_{y} h_{ij}{}^{y} \omega_{xy}. \end{aligned} \tag{3.8}$$

Substituting $dh_{ij}{}^x$ in this definition into the exterior derivative of (3.2) and using (2.2), (3.1)-(3.3), and (3.8), we have

$$h_{ijk}{}^{x} = h_{ikj}{}^{x}, \quad h_{ij\bar{k}}{}^{x} = -R'_{\bar{x}ij\bar{k}}. \tag{3.9}$$

Similarly, the components $h_{ijkl}{}^x$ and $h_{ijk\bar{l}}{}^x$ (resp. $h_{ij\bar{k}l}{}^x$ and $h_{ij\bar{k}\bar{l}}{}^x$) of the covariant derivative of $h_{ijk}{}^x$ (resp. $h_{ij\bar{k}}{}^x$) can be defined by

$$\begin{aligned}&\sum_l (h_{ijkl}{}^x \omega_l + h_{ijk\bar{l}}{}^x \overline{\omega}_l) \\ &= dh_{ijk}{}^x - \sum_l (h_{ljk}{}^x \omega_{li} + h_{ilk}{}^x \omega_{lj} + h_{ijl}{}^x \omega_{lk}) \\ &\quad + \sum_y h_{ijk}{}^y \omega_{xy},\end{aligned} \tag{3.10}$$

$$\begin{aligned}&\sum_l (h_{ij\bar{k}l}{}^x \omega_l + h_{ij\bar{k}\bar{l}}{}^x \overline{\omega}_l) \\ &= dh_{ij\bar{k}}{}^x - \sum_l (h_{lj\bar{k}}{}^x \omega_{li} + h_{il\bar{k}}{}^x \omega_{lj} + h_{ij\bar{l}}{}^x \overline{\omega}_{lk}) \\ &\quad + \sum_y h_{ij\bar{k}}{}^y \omega_{xy}.\end{aligned} \tag{3.11}$$

Differentiating (3.8) exteriorly and using the properties $d^2 = 0$, (3.3), (3.4) and (3.7)-(3.9), we have the following Ricci formula for the second fundamental form

$$h_{ijkl}{}^x = h_{ijlk}{}^x, \quad h_{ij\bar{k}\bar{l}}{}^x = h_{ij\bar{l}\bar{k}}{}^x, \tag{3.12}$$

$$h_{ijk\bar{l}}{}^x - h_{ij\bar{l}k}{}^x = \sum_r (R_{\bar{l}ki\bar{r}} h_{rj}{}^x + R_{\bar{l}kj\bar{r}} h_{ri}{}^x) - \sum_y R_{\bar{x}yk\bar{l}} h_{ij}{}^y. \tag{3.13}$$

4. The Laplacian operator

In this section, the Laplacian of the squared norm of the second fundamental form on a complex submanifold of a Kaehler manifold will be calculated. Let M' be an $(n+p)$-dimensional Kaehler manifold and let M be an n-dimensional complex submanifold of M'. Let f be any smooth C^2-function on M. The components f_i and $f_{\bar{i}}$ of the exterior derivative df of f are given by

$$df = \sum_i (f_i \omega_i + f_{\bar{i}} \overline{\omega}_i).$$

Moreover, the components f_{ij} and $f_{i\bar{j}}$ (resp. $f_{\bar{i}j}$ and $f_{\bar{i}\bar{j}}$) of the covariant derivative of f_i (resp. $f_{\bar{i}}$) can be defined by

$$\sum_j (f_{ij}\omega_j + f_{i\bar{j}}\overline{\omega}_j) = df_i - \sum_j f_j\omega_{ji},$$
$$\sum_j (f_{\bar{i}j}\omega_j + f_{\bar{i}\bar{j}}\overline{\omega}_j) = df_{\bar{i}} - \sum_j f_{\bar{j}}\overline{\omega}_{ji}.$$

The Laplacian of the function f is by definition given as

$$\Delta f = \sum_j (f_{j\bar{j}} + f_{\bar{j}j}) = 2\sum_j f_{j\bar{j}}. \tag{4.1}$$

Now, we calculate the Laplacian of the squared norm $h_2 = |\alpha|^2$ of the second fundamental form α on M. By (3.11) and the second equation of (3.9), we have

$$\begin{aligned}
&\sum_l (h_{ij\bar{k}l}{}^x\omega_l + h_{ij\bar{k}\bar{l}}{}^x\overline{\omega}_l) \\
&= -dR'_{\bar{x}ij\bar{k}} + \sum_l (R'_{\bar{x}lj\bar{k}}\omega_{li} + R'_{\bar{x}il\bar{k}}\omega_{lj} + R'_{\bar{x}ij\bar{l}}\overline{\omega}_{lk}) - \sum_y R'_{\bar{y}ij\bar{k}}\omega_{xy} \\
&= -dR'_{\bar{x}ij\bar{k}} + \sum_A (R'_{\bar{x}Aj\bar{k}}\omega_{Ai} + R'_{\bar{x}iA\bar{k}}\omega_{Aj} + R'_{\bar{x}ij\bar{A}}\overline{\omega}_{Ak}) - \sum_A R'_{\bar{A}ij\bar{k}}\omega_{xA} \\
&\qquad - \sum_y (R'_{\bar{x}yj\bar{k}}\omega_{yi} + R'_{\bar{x}iy\bar{k}}\omega_{yj} + R'_{\bar{x}ij\bar{y}}\overline{\omega}_{yk}) + \sum_l R'_{\bar{l}ij\bar{k}}\omega_{xl},
\end{aligned}$$

from which together with (2.2), (2.6) and (3.2), it follows that

$$\begin{aligned}
&\sum_l (h_{ij\bar{k}l}{}^x\omega_l + h_{ij\bar{k}\bar{l}}{}^x\overline{\omega}_l) \\
&= -\sum_A (R'_{\bar{x}ij\bar{k}:A}\omega_A + R'_{\bar{x}ij\bar{k}:\bar{A}}\overline{\omega}_A) \\
&\quad - \sum_{y,l} (R'_{\bar{x}yj\bar{k}}h_{il}{}^y\omega_l + R'_{\bar{x}iy\bar{k}}h_{jl}{}^y\omega_l + R'_{\bar{x}ij\bar{y}}\overline{h}_{kl}{}^y\overline{\omega}_l) \\
&\quad + \sum_{l,r} R'_{\bar{r}ij\bar{k}}h_{rl}{}^x\omega_l.
\end{aligned}$$

Comparing the coefficients of ω_l in the above equation, we have

$$h_{ij\bar{k}l}{}^x = -R'_{\bar{x}ij\bar{k}:l} - \sum_y (R'_{\bar{x}yj\bar{k}}h_{il}{}^y + R'_{\bar{x}iy\bar{k}}h_{jl}{}^y) + \sum_r R'_{\bar{r}ij\bar{k}}h_{rl}{}^x. \tag{4.2}$$

On the other hand, from (3.13), we get

$$
\begin{aligned}
(4.3)\qquad & h_{ijk\bar{l}}{}^{x} - h_{ij\bar{l}k}{}^{x} \\
&= \sum_{r}(R'_{\bar{r}ki\bar{l}}h_{rj}{}^{x} + R'_{\bar{r}kj\bar{l}}h_{ri}{}^{x}) - \sum_{y} R'_{\bar{l}ky\bar{x}}h_{ij}{}^{y} \\
&\quad - \sum_{y,r}(h_{ik}{}^{y}\overline{h}_{rl}{}^{y}h_{rj}{}^{x} + h_{jk}{}^{y}\overline{h}_{rl}{}^{y}h_{ri}{}^{x}) - \sum_{y,r} h_{kr}{}^{x}\overline{h}_{rl}{}^{y}h_{ij}{}^{y}
\end{aligned}
$$

with the help of (3.5) and (3.6). Combining (4.2) with (4.3), we obtain

$$
\begin{aligned}
h_{ijk\bar{l}}{}^{x} &= - R'_{\bar{x}ij\bar{l}:k} \\
&\quad - \sum_{y}(R'_{\bar{x}yj\bar{l}}h_{ik}{}^{y} + R'_{\bar{x}yi\bar{l}}h_{jk}{}^{y} + R'_{\bar{x}yk\bar{l}}h_{ij}{}^{y}) \\
&\quad + \sum_{r}(R'_{\bar{r}jk\bar{l}}h_{ri}{}^{x} + R'_{\bar{r}ik\bar{l}}h_{rj}{}^{x} + R'_{\bar{r}ij\bar{l}}h_{rk}{}^{x}) \qquad (4.4) \\
&\quad - \sum_{y,r}(h_{ik}{}^{y}\overline{h}_{rl}{}^{y}h_{rj}{}^{x} + h_{jk}{}^{y}\overline{h}_{rl}{}^{y}h_{ri}{}^{x}) \\
&\quad - \sum_{y,r} h^{x}_{kr}\overline{h}_{rl}{}^{y}h_{ij}{}^{y}.
\end{aligned}
$$

The matrix $A = (A_y{}^x)$ of order p defined by $A_y{}^x = \sum_{i,j} h_{ij}{}^{x}\overline{h}_{ij}{}^{y}$ is a positive semi-definite Hermitian one. Summing up $k = l$ in (4.4), we have

$$
\begin{aligned}
\sum_{k} h_{ijk\bar{k}}{}^{x} &= - \sum_{k} R'_{\bar{x}ij\bar{k}:k} \\
&\quad - \sum_{y,k}(R'_{\bar{x}yj\bar{k}}h_{ik}{}^{y} + R'_{\bar{x}yi\bar{k}}h_{jk}{}^{y} + R'_{\bar{x}yk\bar{k}}h_{ij}{}^{y}) \qquad (4.5) \\
&\quad + \sum_{k,l}(R'_{\bar{l}jk\bar{k}}h_{li}{}^{x} + R'_{\bar{l}ik\bar{k}}h_{lj}{}^{x} + R'_{\bar{l}ij\bar{k}}h_{lk}{}^{x}) \\
&\quad - \sum_{k}(h_{i\bar{k}}{}^{2}h_{kj}{}^{x} + h_{j\bar{k}}{}^{2}h_{ki}{}^{x}) - \sum_{y} A_y{}^{x}h_{ij}{}^{y}.
\end{aligned}
$$

Moreover, by (4.1), we have

$$
(4.6)\qquad \Delta h_2 = \sum_{k}\{(\sum_{x,i,j} h_{ij}{}^{x}\overline{h}_{ij}{}^{x})_{k\bar{k}} + (\sum_{x,i,j} h_{ij}{}^{x}\overline{h}_{ij}{}^{x})_{\bar{k}k}\}.
$$

The first term in the right hand side of (4.6) is equivalent to

$$(4.7)\qquad \sum_{x,i,j,k}(h_{ijk\bar{k}}{}^{x}\overline{h}_{ij}{}^{x}+h_{ijk}{}^{x}\overline{h}_{ijk}{}^{x}+h_{ij\bar{k}}{}^{x}\overline{h}_{ij\bar{k}}{}^{x}+h_{ij}{}^{x}\overline{h}_{ij\bar{k}k}{}^{x}).$$

And the second term is expressed as

$$(4.8)\qquad \sum_{x,i,j,k}(h_{ij\bar{k}k}{}^{x}\overline{h}_{ij}{}^{x}+h_{ij\bar{k}}{}^{x}\overline{h}_{ij\bar{k}}{}^{x}+h_{ijk}{}^{x}\overline{h}_{ijk}{}^{x}+h_{ij}{}^{x}\overline{h}_{ijk\bar{k}}{}^{x}).$$

On the other hand, (4.2) gives

$$(4.9)\qquad \begin{aligned}\sum_{k}h_{ij\bar{k}k}{}^{x} = &-\sum_{k}R'_{\bar{x}ij\bar{k}:k}\\ &-\sum_{y,k}(R'_{\bar{x}yj\bar{k}}h_{ik}{}^{y}+R'_{\bar{x}yi\bar{k}}h_{jk}{}^{y})+\sum_{k,r}R'_{\bar{r}ij\bar{k}}h_{rk}{}^{x}.\end{aligned}$$

Since A is the positive semi-definite Hermitian matrix of order p, its eigenvalues λ_x are all non-negative real valued functions on M and it is easily seen that

$$(4.10)\qquad \begin{aligned}&h_2{}^2 \geqq h_4 \geqq \frac{1}{n}h_2{}^2,\quad \sum_{x}\lambda_x = TrA = h_2,\\ &h_2{}^2 \geqq TrA^2 = \sum_{x}\lambda_x{}^2 \geqq \frac{1}{p}h_2{}^2,\end{aligned}$$

where we put $h_4 = \sum_{i,j} h_{i\bar{j}}{}^2 h_{j\bar{i}}{}^2$. Substituting (4.7)-(4.9) into (4.6), we obtain the formula for the Laplacian of the squared norm h_2 of the second fundamental form α on M. That is, we have

$$(4.11)\qquad \begin{aligned}\Delta h_2 =&2|\nabla\alpha|^2 - 2\sum_{x,i,j,k}R'_{\bar{x}ij\bar{k}:k}\overline{h}_{ij}{}^{x} - 2\sum_{x,i,j,k}R'_{\bar{i}xk\bar{j}:\bar{k}}h_{ij}{}^{x}\\ &-8\sum_{x,y,i,j,k}R'_{\bar{x}yj\bar{k}}h_{ki}{}^{y}\overline{h}_{ij}{}^{x} - 2\sum_{x,y,k}A_x{}^{y}R'_{\bar{x}yk\bar{k}}\\ &+4\sum_{x,i,j,k,l}R'_{\bar{k}ij\bar{l}}h_{kl}{}^{x}\overline{h}_{ij}{}^{x} + 4\sum_{i,j,k}R'_{\bar{i}jk\bar{k}}h_{i\bar{j}}{}^{2}\\ &-4h_4 - 2TrA^2,\end{aligned}$$

with the aid of (2.4), (2.5), (2.7), (4.5), and (4.6), where the squared norm $|\nabla\alpha|^2$ of the covariant derivative $\nabla\alpha$ of α is defined by

$$(4.12)\qquad |\nabla\alpha|^2 = \sum_{x,i,j,k}(h_{ijk}{}^{x}\overline{h}_{ijk}{}^{x}+h_{ij\bar{k}}{}^{x}\overline{h}_{ij\bar{k}}{}^{x}).$$

5. Normal curvature tensor

Let M' be an $(n+p)$-dimensional Kaehler manifold equipped with Kaehler structure $\{g', J'\}$ and let M be an n-dimensional complex submanifold of M' endowed with induced Kaehler structure $\{g, J\}$ from the Kaehler structure $\{g', J'\}$. Let us denote by $\nabla^{\perp}$ the normal connection on M, namely, it is the mapping of $TM \times NM$ into NM defined by

$$\nabla^{\perp}(X, V) = \nabla^{\perp}{}_X V = \text{the normal part of } \nabla'_X V$$

for any tangent vector field X in TM and any normal vector field V in NM, where ∇' is the Kaehler connection on M', and TM and NM are the tangent bundle and the normal bundle of M, respectively ([12]). The normal curvature tensor $R^{\perp}$ on M is defined by

$$R^{\perp}(X, Y)V = (\nabla^{\perp}{}_X \nabla^{\perp}{}_Y - \nabla^{\perp}{}_Y \nabla^{\perp}{}_X - \nabla^{\perp}{}_{[X,Y]})V,$$

where $X, Y \in TM$ and $V \in NM$. If it satisfies

$$R^{\perp}(X, Y)V = f\ g(X, JY)J'V$$

for a function f on M, then the normal connection $\nabla^{\perp}$ is said to be *proper*. In particular, if f is a non-zero constant or zero, then it is said to be *semi-flat* or *flat*, respectively (for details, see [3] and [12]).

Now, in order to consider the normal curvature transformation, we first investigate the local version of the normal curvature tensor. By means of (3.6) of the normal curvature tensor, we can define a linear transformation T_N on the np-dimensional complex vector space Ξ^{np} consisting of tensors (ξ_{xk}) at each point on M by

$$T_N(\xi_{xk}) = (\eta_{xk}), \quad \eta_{xk} = \sum_{y,l} R_{\bar{x}yk\bar{l}}\xi_{yl}.$$

We denote by $(R_{yl}{}^{xk})$ the matrix of the linear transformation T_N. Then T_N is the self-adjoint operator with respect to the definite metric canonically defined on Ξ^{np}. The linear operator defined by the $np \times np$ Hermitian matrix $(R_{yl}{}^{xk})$ is called the *normal curvature operator* on M. We assume that the matrix $(R_{yl}{}^{xk})$ is diagonalizable, namely, it satisfies

$$R_{\bar{x}yk\bar{l}} = f_{xk}\delta_{yl}{}^{xk} = f_{xk}\delta_{xy}\delta_{kl}, \tag{5.1}$$

where every eigenvalue f_{xk} of T_N is a real valued function on M. Then the normal connection is said to be *proper*. In this case, we can choose suitably an unitary frame field $\{U_A\} = \{U_j, U_x\}$ in such a way that the matrix $({R_{yl}}^{xk})$ is of the form (5.1). By (3.6) and (5.1), we have

$$R'_{\bar{x}yk\bar{l}} = f_{xk}\delta_{xy}\delta_{kl} - \sum_j {h_{kj}}^x {\overline{h}_{jl}}^y. \tag{5.2}$$

6. Locally symmetric spaces

In this section, let (M', g') be an $(n+p)$-dimensional Kaehler manifold and let M be an n-dimensional complex submanifold of M'. Assume that M' is locally symmetric, the normal connection of M is proper and it satisfies the following two conditions concerning with holomorphic bisectional curvatures:

($*1$) A totally real tangent bisectional curvature is bounded from above by a_0 and from below by a_1.

($*2$) A totally real normal bisectional curvature is bounded from above by a_2.

Then M' is said to satisfy the condition ($*$) if it satisfies the above conditions ($*1$) and ($*2$). In particular, it is said to be $k(=\frac{a_1}{a_0})$*-pinched* provided that $a_0 > 0$.

For the local field $\{E_A, E_{A*}\}$ of orthonormal frames associated with the submanifold chosen in Section 2, it follows from (2.9) that

$$H'(P'_j, P'_k) = H'(E_j, E_k) = H'_{jk} = R'_{\bar{j}jk\bar{k}} \quad (j \neq k),$$
$$H'(P'_x, P'_k) = H'(E_x, E_k) = H'_{xk} = R'_{\bar{x}xk\bar{k}}.$$

If M' satisfies the condition ($*$), then we have

$$a_1 \leqq H'_{jk} \leqq a_0 \quad (j \neq k), \quad H'_{xk} \leqq a_2.$$

REMARK 6.1. Let M' be an $(n+p)$-dimensional complex space form $M^{n+p}(c)$ and of constant holomorphic sectional curvature c. Then M' is locally symmetric and it satisfies the condition ($*$) with $a_0 = a_1 = a_2 = \frac{c}{2}$ and it is 1-pinched ($c \neq 0$).

In the rest of this section, *we denote by* $\mathcal{T}(i)$ *the* i_{th} *term in the right hand side of* (4.11). First of all, we estimate Δh_2 from the below on the complex submanifold M. In order to estimate $\mathcal{T}(4)$ and $\mathcal{T}(5)$, we prepare some basic formulas.

First of all we check the relation between the normal curvature and the totally real bisectional curvature $H'(P',Q')$ for a tangent holomorphic plane P' and a normal holomorphic plane Q' in M'. A totally real plane $[X,Y]$ is defined by a plane $\{X,Y\}$ of the orthogonal pair X and Y, and its image $\{J'X,J'Y\}$ by the almost complex structure J'. For two holomorphic planes $P' = [X,J'X]$ and $Q' = [Y,J'Y]$, where X and Y are orthogonal vectors, the *totally real bisectional curvature* $H'(P',Q') = H'(X,Y)$ on M' is defined by

$$H'(P',Q') = H'(X,Y) = \frac{g'(R'(X,J'X)J'Y,Y)}{g'(X,X)g'(Y,Y)}.$$

Accordingly, we have by (5.2)

$$H'_{xk} = H'(E_x,E_k) = R'_{\bar{x}xk\bar{k}} = f_{xk} - \sum_j h_{kj}{}^x \overline{h}_{kj}{}^x. \tag{6.1}$$

Between the totally real bisectional curvature and the normal curvature, we have the following relation. By (6.1) and the condition (*2), the normal curvature satisfies

$$f_{xk} = H'_{xk} + \sum_j h_{kj}{}^x \overline{h}_{kj}{}^x \leqq a_2 + \sum_j h_{kj}{}^x \overline{h}_{kj}{}^x, \tag{6.2}$$

and consequently $\mathcal{T}(4)$ and $\mathcal{T}(5)$ can be estimated as follows:

$$\begin{aligned}
\mathcal{T}(4) &= -8 \sum_{x,y,i,j,k} R'_{\bar{x}yj\bar{k}} h_{ki}{}^y \overline{h}_{ij}{}^x \\
&= -8 \sum_{x,i,j} f_{xj} h_{ij}{}^x \overline{h}_{ij}{}^x + 8 \sum_{j,k} h_{j\bar{k}}{}^2 h_{k\bar{j}}{}^2 \\
&\geqq -8 \sum_{x,i,j} (a_2 + \sum_k h_{jk}{}^x \overline{h}_{jk}{}^x) h_{ij}{}^x \overline{h}_{ij}{}^x + 8h_4 \\
&= -8a_2h_2 + 8h_4 - 8 \sum_{x,j} (\sum_k h_{jk}{}^x \overline{h}_{jk}{}^x)^2,
\end{aligned}$$

where the second equality follows from (5.2) and the third inequality is derived by (6.2). Since $\sum_{\alpha=1}^m x_\alpha{}^2 \leqq (\sum_{\alpha=1}^m x_\alpha)^2$ for non-negative numbers $x_1, \ldots, x_m$, it is clear that

$$-8 \sum_{x,j} (\sum_k h_{jk}{}^x \overline{h}_{jk}{}^x)^2 \geqq -8 \sum_j (\sum_{x,k} h_{jk}{}^x \overline{h}_{jk}{}^x)^2$$

and consequently

$$\mathcal{T}(4) \geqq -8a_2h_2 + 8h_4 - 8{h_2}^2. \tag{6.3}$$

On the other hand, let A be the positive semi-definite Hermitian matrix defined by $({A_y}^x) = (\sum_{j,k} {h_{jk}}^x \overline{h}_{jk}{}^y)$ and let λ_x be its eigenvalue. Then $\mathcal{T}(5)$ can be estimated as follows ;

$$\begin{aligned}\mathcal{T}(5) &= -2\sum_{x,y,k} \lambda_x \delta_{xy} R'_{\bar{x}yk\bar{k}} = -2\sum_{x,k} \lambda_x R'_{\bar{x}xk\bar{k}} \\ &\geqq -2a_2 \sum_{x,k} \lambda_x = -2na_2 \sum_x \lambda_x\end{aligned}$$

with the help of $(*2)$, from which and (4.10) we have

$$\mathcal{T}(5) \geqq -2na_2h_2. \tag{6.4}$$

Next, we estimate $\mathcal{T}(6)$ and $\mathcal{T}(7)$. For the sake of the estimation, we consider the curvature operator T' on M'. From the symmetric relation (2.5), on the n^2-dimensional complex vector space $\Xi_x^{n^2} = T_xM^C \times T_xM^C$ which consists of symmetric tensor (ξ_{ij}) at each point x on M, we can define a linear transformation T' by

$$T'(\xi_{ij}) = (\eta_{ij}), \quad \eta_{ij} = \sum_{l,k} R'_{\bar{k}ij\bar{l}}\xi_{kl}.$$

We denote by $({R'_{kl}}^{ij})$ the matrix of the linear transformation T'. The linear operator T' defined by the $n^2 \times n^2$ matrix $({R'_{kl}}^{ij})$ is called the *curvature operator* on the submanifold M (for details, see [9]). Since T' is the self-adjoint operator with respect to the metric canonically induced on $\Xi_x^{n^2}$, every eigenvalue R'_{ij} of T' is a real valued function. So we have

(6.5)
$$R'_{\bar{i}jk\bar{l}} = {R'_{il}}^{jk} = R'_{jk}\delta_{ij}\delta_{lk}, \quad R'_{ij} = R'_{\bar{i}ij\bar{j}} = H'(E_i, E_j) = H'_{ij} \quad (i \neq j).$$

For distinct indices i and j, vectors E_i, E_j, JE_i and JE_j are orthonormal. So, by (2.14) we get

$$H'(E_i) + H'(E_j) = 2H'(E'_i, E'_j) + 2H'(E''_i, E''_j),$$

where $\{E'_i, E'_j, E''_i, E''_j\}$ are the orthonormal set associated with E_i and E_j chosen in Section 2, from which together with the assumption $(*1)$ it follows that

$$4a_1 \leqq H'(E_i) + H'(E_j) \leqq 4a_0 \quad (i \neq j). \tag{6.6}$$

Summing up (6.6), we have $4\sum_{i<j} a_1 \leqq \sum_{i<j}(H'_i + H'_j) \leqq 4\sum_{i<j} a_0$, which implies that

$$2na_1 \leqq \sum_j H'_j \leqq 2na_0, \tag{6.7}$$

where the first or second equality holds if and only if $H'_j = 2a_1$ or $2a_0$ for any index j. From (3.5) and $(*1)$, the scalar curvature r is given by

$$\begin{aligned} r &= 2\sum_j S_{j\bar{j}} = 2\sum_{j,k} R_{\bar{j}jk\bar{k}} = 2(\sum_j R'_{\bar{j}jj\bar{j}} + \sum_{j\neq k} R'_{\bar{j}jk\bar{k}}) - 2h_2 \\ &= 2(\sum_j H'_j + \sum_{j\neq k} H'_{jk}) - 2h_2 \\ &\geqq 2\sum_j H'_j + 2n(n-1)a_1 - 2h_2. \end{aligned}$$

Similarly, by $(*1)$ we have

$$r \leqq 2\sum_j H'_j + 2n(n-1)a_0 - 2h_2.$$

Consequently we have

$$r/2 + h_2 - n(n-1)a_0 \leqq \sum_j H'_j \leqq r/2 + h_2 - n(n-1)a_1, \tag{6.8}$$

where the first or second equality holds if and only if $H'_{ij} = R'_{\bar{i}ij\bar{j}} = a_0$ or a_1 for any distinct indices i and j. By (6.7) and (6.8), we have

$$2n(n+1)a_1 \leqq r + 2h_2 \leqq 2n(n+1)a_0, \tag{6.9}$$

where the first or second equality holds if and only if $H'_{ij} = a_1$ or a_0 for any distinct indices i and j. On the other hand, from (6.6), we have

$$\sum_{j(\neq k)} (H'_k + H'_j) \geqq 4(n-1)a_1 \quad \text{for } k = 1, \ldots, n,$$

which and (6.8) imply

$$\begin{aligned}(n-2)H'_k &\geqq 4(n-1)a_1 - \sum_j H'_j \\ &\geqq 4(n-1)a_1 - \{r/2 + h_2 - n(n-1)a_1\} \\ &= (n-1)(n+4)a_1 - r/2 - h_2,\end{aligned}$$

and consequently

$$H'_j \geqq \frac{(n-1)(n+4)a_1 - r/2 - h_2}{n-2}, \quad n \geqq 3 \tag{6.10}$$

for any index j.

LEMMA 6.1. *Let M' be an $(n+p)$-dimensional Kaehler manifold satisfying the condition $(*)$ and let M be an n $(\geqq 3)$-dimensional complex submanifold of M'. If the scalar curvature of M is bounded from below, then the Ricci curvature of M is bounded from below.*

Proof. Since the Ricci curvature $S_{j\bar{j}}$ of M is given by

$$S_{j\bar{j}} = \sum_k R_{\bar{j}jk\bar{k}} = R'_{\bar{j}jj\bar{j}} + \sum_{k(\neq j)} R'_{\bar{j}jk\bar{k}} - \sum_{x,k} h_{jk}{}^x \overline{h}_{jk}{}^x,$$

by using the condition $n \geqq 3$, $(*1)$, (6.9) and (6.10), we have

$$\begin{aligned}S_{j\bar{j}} &= H'_j + \sum_{k(\neq j)} H'_{jk} - \sum_{x,k} h_{jk}{}^x \overline{h}_{jk}{}^x \\ &\geqq \frac{(n-1)(n+4)a_1 - r/2 - h_2}{n-2} + (n-1)a_1 - h_2 \\ &= \frac{2(n-1)(n+1)a_1 - r/2 - h_2}{n-2} - h_2 \\ &\geqq \frac{n+1}{n-2}\{2(n-1)a_1 - na_0\} - h_2.\end{aligned}$$

By (6.9) the function $r+2h_2$ is bounded and therefore the result together with the assumption about the scalar curvature of M yield the fact that h_2 is bounded. Hence the Ricci curvature of M is bounded from below, which completes the proof. □

Now, we estimate $\mathcal{T}(6)$. By (6.5), we have

$$\begin{aligned}\mathcal{T}(6) &= 4 \sum_{x,i,j,k,l} R'_{\bar{k}ij\bar{l}} h_{kl}{}^x \overline{h}_{ij}{}^x \\ &= 4 \sum_{x,i,j,k,l} R'_{ij} \delta_{ik}\delta_{jl} h_{kl}{}^x \overline{h}_{ij}{}^x \\ &= 4 \sum_{x,i,j} R'_{ij} h_{ij}{}^x \overline{h}_{ij}{}^x,\end{aligned}$$

from which together with (*1), (6.5) and (6.10), it follows that

$$\begin{aligned}\mathcal{T}(6) &= 4 \sum_{i,j} H'_{ij} \sum_x h_{ij}{}^x \overline{h}_{ij}{}^x \\ &= 4 \sum_i (H'_i \sum_x h_{ii}{}^x \overline{h}_{ii}{}^x + \sum_{j(\neq i)} H'_{ij} \sum_x h_{ij}{}^x \overline{h}_{ij}{}^x) \\ &\geqq 4 \sum_i \Big\{ \frac{(n-1)(n+4)a_1 - r/2 - h_2}{n-2} \sum_x h_{ii}{}^x \overline{h}_{ii}{}^x \\ &\quad + \sum_{j(\neq i)} a_1 \sum_x h_{ij}{}^x \overline{h}_{ij}{}^x \Big\} \\ &= 4 \Big\{ \frac{(n-1)(n+4)a_1 - r/2 - h_2}{n-2} \sum_{x,i} h_{ii}{}^x \overline{h}_{ii}{}^x \\ &\quad + a_1 (h_2 - \sum_{x,i} h_{ii}{}^x \overline{h}_{ii}{}^x) \Big\} \\ &= 4 \Big\{ \frac{(n^2+2n-2)a_1 - r/2 - h_2}{n-2} \sum_{x,i} h_{ii}{}^x \overline{h}_{ii}{}^x + a_1 h_2 \Big\}.\end{aligned}$$

Accordingly we have

$$\begin{aligned}\mathcal{T}(6) &\geqq 4a_1h_2 + \frac{2}{n-2}\{2(n^2+2n-2)a_1 - r - 2h_2\} \sum_{x,i} h_{ii}{}^x \overline{h}_{ii}{}^x \\ &\geqq 4a_1h_2 + \frac{4}{n-2}\{(n^2+2n-2)a_1 - n(n+1)a_0\} \sum_{x.i} h_{ii}{}^x \overline{h}_{ii}{}^x,\end{aligned}$$

where the second inequality is derived by (6.9). Suppose that the constant $(n^2+2n-2)a_1 - n(n+1)a_0 \geqq 0$. Then we have

$$\mathcal{T}(6) \geqq 4a_1h_2. \tag{6.11}$$

By the way if $(n^2+2n-2)a_1 - n(n+1)a_0 \leqq 0$, then we have

$$\mathcal{T}(6) \geqq \frac{4}{n-2}\{(n-1)(n+4)a_1 - n(n+1)a_0\}h_2. \tag{6.12}$$

Finally, we estimate $\mathcal{T}(7)$. The matrix $(h_{i\bar{j}}{}^2)$ is a positive semi-definite Hermitian one, whose eigenvalues λ_j's are non-negative real functions, i.e., $h_{i\bar{j}}{}^2 = \lambda_j \delta_{ij}$. So $\mathcal{T}(7)$ is estimated as follows ;

$$\begin{aligned}
\mathcal{T}(7) &= 4\sum_{i,j,k} R'_{\bar{i}jk\bar{k}} h_{i\bar{j}}{}^2 = 4\sum_{j,k} \lambda_j R'_{\bar{j}jk\bar{k}} \\
&= 4\sum_{j} \lambda_j (R'_{\bar{j}jj\bar{j}} + \sum_{k(\neq j)} R'_{\bar{j}jk\bar{k}}) \\
&\geqq 4\sum_{j} \lambda_j \left\{ \frac{(n-1)(n+4)a_1 - r/2 - h_2}{n-2} + \sum_{k(\neq j)} a_1 \right\}
\end{aligned}$$

by virtue of $(*1)$ and (6.10). Thus we have

$$\mathcal{T}(7) \geqq \frac{4(n+1)}{n-2}\{2(n-1)a_1 - na_0\}h_2. \tag{6.13}$$

By using the above inequalities, we can prove the following proposition.

PROPOSITION 6.2. *Let M' be an $(n+p)$-dimensional locally symmetric Kaehler manifold satisfying the condition $(*)$ and let M be an $n(\geqq 3)$-dimensional complete complex submanifold of M'. If the normal connection of M is proper, then the following statements hold good :*

(1) *If $(n^2+2n-2)a_1 - n(n+1)a_0 \geqq 0$ and $A_1 > 0$, then there exists a positive constant h in such a way that M is totally geodesic, provided $h_2 < h$.*
(2) *If $(n^2+2n-2)a_1 - n(n+1)a_0 \leqq 0$ and $A_2 > 0$, then there exists a positive constant h in such a way that M is totally geodesic, provided $h_2 < h$.*

Here,

$$\begin{aligned}
A_1 &= -2n(n+1)a_0 + 2(2n^2+n-4)a_1 - (n-2)(n+4)a_2, \\
A_2 &= -4n(n+1)a_0 + 6(n-1)(n+2)a_1 - (n-2)(n+4)a_2.
\end{aligned}$$

Proof. We first prove the assertion (2) holds good. Since the ambient space is locally symmetric, by using (4.12), (6.3), (6.4), (6.12), and (6.13), the equation (4.11) can be estimated as follows ;

$$\begin{aligned}\Delta h_2 \geqq & -8a_2h_2 + 8h_4 - 8{h_2}^2 - 2na_2h_2 \\ & + \frac{4}{n-2}\{(n-1)(n+4)a_1 - n(n+1)a_0\}h_2 \\ & + \frac{4(n+1)}{n-2}\{2(n-1)a_1 - na_0\}h_2 - 4h_4 - 2TrA^2\end{aligned}$$

and moreover

$$\Delta h_2 \geqq c_0{h_2}^2 + c_1h_2, \tag{6.14}$$

where we have put

$$c_0 = \frac{2}{n}(4-7n), \quad c_1 = \frac{2}{n-2}A_2.$$

Now, let f be the non-negative function h_2. Then, by (6.14), we have

$$\Delta f \geqq c_0f^2 + c_1f =: F(f), \tag{6.15}$$

where F is the polynomial of the variable f with the constant coefficients. Since $h_2 < h$, the scalar curvature r on M is also bounded from below and hence Lemma 6.1 implies that the Ricci curvature of M is bounded from below. From (6.9) we have $r + 2f \leqq 2n(n+1)a_0$, which implies that f is bounded from above. Thus we can apply Theorem O-Y to the function f and obtain that $F(\sup f) \leqq 0$, which gives $\sup f = 0$ or $\sup f \geqq -\frac{c_1}{c_0} > 0$. Now we take a positive constant h in such a way that $h < -\frac{c_1}{c_0}$ and $h_2 < h$. Then $\sup f \leqq h < -\frac{c_1}{c_0}$, and therefore $\sup f = 0$. Since the function f is non-negative, it vanishes identically on M, which means that M is totally geodesic. The assertion (1) can be also proved by the same argument as in the proof of the assertion (2). □

By means of Proposition 6.2, we can prove the main theorem stated in Section 1.

Proof of the main theorem. In order to verify that the assumptions of Proposition 6.2 are satisfied, we first investigate the sign of the constant $(n^2+2n-2)a_1 - n(n+1)a_0$. We notice that it has the same sign as that of

$k - f(n)$, where $f(n) = n(n+1)/(n^2+2n-2)$. But since $f(n) \geqq f(4) = f(5) = 10/11$, we have $k - f(n) \leqq k - f(4) = k - 10/11$. This means that the condition $k \leqq 10/11$ yields that the above constant is non-positive. On the other hand we can also see that the sign of the constant A_2 is the same as that of $k - g(n)$, where $g(n) = 2n(n+1)/3\{(n-1)(n+2)\}$. Since $g(n)$ is monotonically decreasing with respect to n and it satisfies $g(3) = 4/5$ and $\lim_{n\to\infty} g(n) = 2/3$, we have $2/3 \leqq g(n) \leqq 4/5$ and hence the constant A_2 is positive if $k > 4/5$. Thus we can apply Proposition 6.2(2) and complete the proof. □

In particular, we consider the case where the ambient space is a complex space form of constant holomorphic sectional curvature c. Then it is locally symmetric and it satisfies the condition $(*)$ such that $a_0 = a_1 = a_2 = c/2$. Furthermore, from (4.10), we have directly the following inequality:

(6.16)
$$\Delta f \geqq c_0 f^2 + c_1 f =: F(f),\ c_0 = \frac{2}{n}(4 - 7n),\ c_1 = (n+2)c,\ c < 0,$$

(6.17)
$$\Delta f \geqq c_0 f^2 + c_1' f =: F(f),\ c_0 = \frac{2}{n}(4 - 7n),\ c_1' = nc,\ c > 0.$$

By Lemma 6.1 the Ricci curvature of M is bounded from below, so is the scalar curvature r on M. On the other hand, since $r + 2f$ is bounded from above by (6.9), the function f is also bounded from above. Thus by means of (6.17) and Theorem O-Y, we obtain $F(\sup f) \leqq 0$. Thus we have

THEOREM 6.4. *Let M' be an $(n+p)$-dimensional complex space form $M^{n+p}(c)$, $c > 0$. Let M be an n-dimensional complete complex submanifold of M'. Then there exists a positive constant h in such a way that M is totally geodesic, provided $h_2 < h$.*

REMARK 6.2. When the ambient space is a complex space form of constant holomorphic sectional curvature $c < 0$,

$$(n^2 + 2n - 2)a_0 - n(n+1)a_1 = (n-2)c/2 < 0, \quad A_2 = n(n-2)c/2 < 0$$

because of $a_0 = a_1 = a_2 = c/2 < 0$. But unfortunately we have no information about the squared norm h_2 something like (6.16).

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