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COMPLETE PROLONGATION AND THE FROBENIUS INTEGRABILITY FOR OVERDETERMINED SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We study the compatibility conditions and the existence of solutions for overdetermined PDE systems that admit complete prolongation. For a complete system of order k there exists a submanifold of the $(k-1)$st jet space of unknown functions that is the largest possible set on which the initial conditions of $(k-1)$st order may take values. There exists a unique solution for any initial condition that belongs to this set if and only if the complete system satisfies the compatibility conditions on the initial data set. We prove by applying the Frobenius theorem to a Pfaffian differential system associated with the complete prolongation.

Introduction

In this paper we study the compatibility conditions and the existence of solutions for overdetermined systems of partial differential equations by means of complete prolongation and finding the Frobenius integrability conditions. Consider a system of partial differential equations of order q ($q \geq 1$) for unknown functions $u = (u^1, \cdots, u^m)$ of independent variables $x = (x^1, \cdots, x^n)$

$$\Delta_\lambda(x, u^{(q)}) = 0, \qquad \lambda = 1, \cdots, l, \tag{1}$$

which is overdetermined, that is, $m < l$. Here x varies in an open set $X \subset \mathbb{R}^n$, u in open set $U \subset \mathbb{R}^m$ and $u^{(q)}$ denotes all the partial derivatives of u up to order q and each $\Delta_\lambda(x, u^{(q)})$ is a polynomial in

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$u^{(q)}$ with coefficients that are smooth functions in x. We call such Δ_λ a differential polynomial of order q. For each positive integer p the set A_p of all differential polynomials of order $\leq p$ forms a commutative algebra over the ring of smooth functions in x. We call the set $J^q(X,U) := \{(x, u^{(q)}) : x \in X, u \in U\}$ the space of q-th jets of u. If $f : X \to U$ is smooth (C^∞), let $(j^k f)(x) = (x, f(x), \partial^\alpha f(x) : |\alpha| \leq k)$, then $j^k f$ is a smooth section of the vector bundle $J^k(X,U) \to X$ called the k-jet graph of f. The total derivative of a differential polynomial $H(x, u^{(k)})$ of order k with respect to x^λ is a differential polynomial of order $k+1$ defined by the chain rule:

$$(D_\lambda H)(x, u^{(k+1)}) := \frac{\partial H}{\partial x^\lambda} + \sum_{a=1}^{m} \sum_{J} u^a_{J,\lambda} \frac{\partial H}{\partial u^a_J}, \tag{2}$$

where $J = (j_1, \cdots, j_n)$ is a multi-index such that $|J| \leq k$ and $J, \lambda = (j_1, \cdots, j_\lambda + 1, \cdots, j_n)$.

By prolongation of (1) we simply mean any process of total differentiations and algebra operations on Δ to get new equations. Let $\mathcal{S}_\Delta$ be the subvariety of $J^q(X,U)$ defined by (1). A smooth function $u = f(x)$ is a solution of (1) if the q-jet graph

$$x \mapsto (x, f^{(q)}(x))$$

is a submanifold of $\mathcal{S}_\Delta$. Our viewpoint in this paper is purely local and we assume that the reference point $(x_0, u_0^{(q)}) \in \mathcal{S}_\Delta$ has a neighborhood Ω which is open in $J^q(X,U)$ such that

$$\begin{aligned} &\text{i)} \quad \mathcal{S}_\Delta \cap \Omega \quad \text{is a smooth manifold} \\ &\qquad \text{and} \\ &\text{ii)} \quad dx^1 \wedge \cdots \wedge dx^n \neq 0 \quad \text{on} \ \ \mathcal{S}_\Delta \cap \Omega. \end{aligned} \tag{3}$$

We further assume that (3) holds at each stage of prolongation. In this paper we study the cases that the total derivatives of (1) of sufficiently high order, say order r, can be solved for all the partial derivatives of u of certain order, say k, as functions of the lower order derivatives:

$$u^a_K = H^a_K(x, u^{(k-1)}) \quad \forall K, \quad \text{with } |K| = k, \quad \forall a = 1, \dots, m. \tag{4}$$

We will call (4) a complete system of finite order k and (1) is said to admit prolongation to a complete system (4) (see Definition 1.3). An easy calculation shows that as we differentiate (1) r times

$$\frac{\text{number of the equations}}{\text{number of the variables in } u^{(q+r)}}$$

tends to l/m as $r \to \infty$. Hence, for a sufficiently large r it happens generically that the hypothesis of the implicit function theorem holds so that

$$D_J \Delta_\lambda = 0, \qquad |J| \le r, \quad \lambda = 1, \dots, l$$

is solvable for all the k-th order partial derivatives of u, for some k, in terms of the lower order derivatives as in (4). In this paper, however, we consider only the cases where H^a_K are differential polynomials. Every ordinary differential equation of order n

$$y^{(n)} = F(x, y, y', y'', \dots, y^{(n-1)})$$

is obviously a complete system of order n.

EXAMPLE 1. Consider the following system for one unknown function $u(x, y)$ in two independent variables:

$$\left\{ \begin{array}{l} u_x + uu_y = a(x, y) \\ u_{yy} + u^2 = b(x, y). \end{array} \right. \tag{5}$$

We shall show that (5) admits prolongation to a complete system of order 2. Differentiate the first equation of (5) with respect to x and y, respectively, to obtain

$$\begin{aligned} u_{xx} + u_x u_y + uu_{xy} &= a_x, \\ u_{xy} + u_y^2 + uu_{yy} &= a_y, \\ u_{yy} + u^2 &= b. \end{aligned} \tag{6}$$

By solving (6) for all the second order derivatives we have

$$\begin{aligned} u_{xx} &= H_{11}(x, u^{(1)}) := -u_x u_y + u{u_y}^2 - u^4 + bu^2 - a_y u + a_x, \\ u_{xy} &= H_{12}(x, u^{(1)}) := -{u_y}^2 + u^3 - ub + a_y, \\ u_{yy} &= H_{22}(x, u^{(1)}) := -u^2 + b, \end{aligned} \tag{7}$$

which is a complete system of order 2.

Once a complete prolongation is attained, the problems of existence, uniqueness and regularity of solutions of (1) reduce to those of a Pfaffian differential system in the jet space, which are essentially problems of ordinary differential equations: On the jet space $J^{(k-1)}(X,U)$ we consider a Pfaffian differential system

$$(8)\qquad \begin{cases} \theta^a := du^a - \sum_{\lambda=1}^{n} u^a_\lambda dx^\lambda = 0, \\ \theta^a_I := du^a_I - \sum_{\lambda=1}^{n} u^a_{I,\lambda} dx^\lambda = 0, \qquad |I| \leq k-2, \end{cases}$$

$$(9)\qquad \theta^a_J := du^a_J - \sum_{\lambda=1}^{n} H^a_{J,\lambda} dx^\lambda = 0, \qquad |J| = k-1$$

with the independence condition

$$(10)\qquad \Omega := dx^1 \wedge \cdots \wedge dx^n \neq 0,$$

where $H^a_{J,\lambda}$ are as in (4). Observe that (9) is defined only for those points $(x, u^{(k-1)})$ that satisfy (1) and its prolongations. Let $u = f(x)$ be a C^k mapping of an open subset of $\mathbb{R}^n$ into $\mathbb{R}^m$. If $u = f(x)$ is a solution of (1) then

$$x \mapsto (x, f^{(k-1)}(x))$$

is an integral manifold of the n-dimensional distribution defined by (8)–(10). Thus we have

THEOREM 1. (UNIQUENESS AND REGULARITY OF SOLUTIONS). *Suppose that* (1) *admits prolongation to a complete system of order* k. *Then a* C^k *solution is uniquely determined by its* $(k-1)$*-jet at a point, that is, if* $u = f(x)$ *and* $u = g(x)$ *are* C^k *solutions of* (1) *and if* $g^{(k-1)}(x_0) = f^{(k-1)}(x_0)$ *then* $f = g$. *Furthermore, a* C^k *solution* f *is indeed* C^∞. *If the coefficients of each* Δ_λ *in* (1) *is real analytic then* f *is real analytic.*

Now suppose that a complete system (4) is obtained from the r-th prolongation of (1). We consider a subset $\mathcal{S}^r_{k-1}$ of $J^{k-1}(X,U)$ where the initial data may vary (Definition 1.4). Then we apply the Frobenius theorem to the Pfaffian system (8)–(10) on $\mathcal{S}^r_{k-1}$ to obtain necessary and sufficient conditions for a unique solution to exist for any initial data that lies on $\mathcal{S}^r_{k-1}$. Let $\mathcal{D}$ be the distribution defined along $\mathcal{S}^r_{k-1}$ by (8)–(10). We prove

THEOREM 2. *Suppose that (4) is a complete system of order k obtained from the r-th prolongation of (1) and that $\mathcal{S}_{k-1}^r$ is the associated initial data set. Then the following are equivalent:*

i) *(1) has a unique smooth solution for every initial condition $(x_0, u_0^{(k-1)}) \in \mathcal{S}_{k-1}^r$.*

ii) *$\mathcal{D}$ is tangent to $\mathcal{S}_{k-1}^r$ and integrable in the sense of Frobenius.*

iii) *$\mathcal{D}$ is tangent to $\mathcal{S}_{k-1}^r$ and (4) satisfies the compatibility conditions on $\mathcal{S}_{k-1}^r$, namely,*

$$D_\lambda H_{J,\mu}^a = D_\mu H_{J,\lambda}^a \quad \text{on } \mathcal{S}_{k-1}^r \tag{11}$$

for each $a = 1, \dots, m$, each multi-index J with $|J| = k-1$ and each $\mu, \lambda = 1, \dots, n$.

To relate our theorems to the theory of exterior differential system we recall first that a smooth mapping $u = f(x)$ is a solution of (1) if and only if the q-th jet graph $j^q f(x)$ is an integral manifold of the Pfaffian differential system (8) with $|I| \le q-1$ and the independence condition (10) on $\mathcal{S}_\Delta$. Then the Cartan-Kähler theorem asserts that if all the data are real analytic and the system is involutive (see [1] for definition) then there exist integral manifolds. The existence of integral manifolds is proved by repeated applications of the Cauchy-Kowalevski theorem. The Cartan-Kuranishi theorem [10] asserts that given an analytic Pfaffian differential system $\theta = 0, \Omega \neq 0$, under certain regularity assumptions the system converts by a finite number of prolongations either to an involutive system or to a system that has no solutions. For the Pfaffian differential system (8)–(10) given by a complete system of order k the 1-forms $\{\theta_I^a : |I| \le k-1\}$ and $dx^1, \dots, dx^n$ form a coframe of $J^{k-1}(X, U)$ along $\mathcal{S}_{k-1}^r$ and the involutivity is equivalent to the conditions as in Theorem 2. Complete prolongation has twofold merits: Firstly, all the terms of order $\ge k$ are reduced to the lower orders so that our argument stays in the $(k-1)$st jet. Secondly, the existence of solutions can be discussed in smooth (C^∞) category, where the problem is finding an appropriate submanifold $\mathcal{S}$ of $J^{k-1}(X, U)$ having the following properties:

i) the initial data up to order $(k-1)$ take values in $\mathcal{S}$ and the distribution $\mathcal{D}$ is uniquely defined along $\mathcal{S}$,

ii) $\mathcal{D}$ is tangent to $\mathcal{S}$,

iii) $\mathcal{D}$ satisfies the Frobenius conditions on $\mathcal{S}$.

The submanifold $\mathcal{S}^r_{k-1}$ in Theorem 2 is the largest such $\mathcal{S}$. Recently, several authors have constructed complete systems in various mapping problems, namely, [8] for CR mappings between real hypersurfaces of finitely degenerated Levi form, [6] for mappings of CR manifolds of nondegenerate Levi form into a higher dimensional CR manifold, [9] for pseudohermitian embeddings. Similar construction is possible for the mappings preserving Riemannian or conformal structures, see [7]. Determination of mappings by finite jet that are studied in [3] and [13] seems to be closely related to the complete prolongation. We attempt to develop existence theory for such mappings that are determined by finite jet.

For the definitions and notations that are not defined in this paper we refer to our main references [1], [4] and [11]. The algebraic setting is to be chosen appropriately depending on the problems. In this paper we restrict our interest solely to differential polynomials. This paper was written in the fall of 2000 when the authors were visiting the University of Illinois. We thank J. P. D'Angelo and the faculty of the Math Department for their interest and hospitality.

1. Complete prolongation and the Frobenius integrability

Overdetermined PDE systems generically admit prolongation to complete systems of finite order as we have observed. However, actual calculation of prolongation to a complete system is usually very complicated. The crux of the method of prolongation is in the reduction of order by eliminating the highest order terms using the symmetry of the system. In the cases of embedding equations knowing the geometric local invariants are often helpful in finding the right symmetry (cf. [2], [5], [7]). In this section we discuss the compatibility conditions and the existence of solution of (1) that admits prolongation to a complete system (4) of order k. Our strategy is to find a submanifold $\mathcal{S}^r_{k-1}$ of $J^{k-1}(X, U)$ where the initial condition may vary and then check the Frobenius integrability conditions. Let

$$\Delta = (\Delta_1, \cdots, \Delta_l)$$

be a system of differential polynomials of order q as in (1). The common zero set $\mathcal{S}_\Delta \subset J^q(X, U)$ of $\Delta_\lambda, \quad \lambda = 1, \ldots, l$, shall be called the solution submanifold of (1). A smooth mapping $u = f(x)$ of X into U is a solution of (1) if and only if its q-jet graph $(x, f^{(q)}(x))$ is contained in $\mathcal{S}_\Delta$. The set A of all differential polynomials forms a commutative algebra over the ring of smooth functions in x. For each nonnegative integer p let A_p be the subalgebra of differential polynomials of order $\leq p$. Let $\bar{\Delta}$ be the set of all the differential polynomials that are raised up to order q by differentiating Δ_λ's. For each nonnegative integer r the r-th prolongation of Δ, denoted by $\Delta^{(r)}$, is the ideal (algebraic ideal) in the algebra A_{q+r} generated by all the total derivatives of $\bar{\Delta}$ of order up to r. For each pair (r, s) of nonnegative integers let

$$\Delta_s^r = \Delta^{(r)} \cap A_s.$$

Then Δ_s^r is an ideal in A_s. The elements of $\Delta_s^r \setminus \Delta_s^{r-1}$ with $s < q+r$ occur when the highest order terms in $\Delta^{(r)}$ cancel out by the algebra operations of A_{q+r} as the following two examples show. These elements play important roles in prolongation.

EXAMPLE 1.1. Let u be an unknown function in two variables (x_1, x_2) Let $\Delta = (u_{11} + u_2, \quad u_{12} + u_1)$. Then

$$D_2(u_{11} + u_2) - D_1(u_{12} + u_1) = u_{22} - u_{11} \ \in \Delta_2^1 \setminus \Delta_2^0.$$

EXAMPLE 1.2. Let M^n be a smooth manifold with a smooth Riemannian metric g. Let $x = (x_1, \dots, x_n)$ be local coordinates of M. A smooth mapping $u = (u^1, \dots, u^m)$ of M into $\mathbb{R}^m$ is an isometric embedding if u satisfies

$$\Delta_{ij} := \sum_{a=1}^{m} u_i^a u_j^a - g_{ij}(x) = 0. \tag{1.1}$$

By differentiating (1.1) two times we get the Gauss equations: For each 4-tuple of integers $i, j, k, l = 1, \dots, n$ we have

$$\begin{aligned} &\sum_{a=1}^{m} (u_{ik}^a u_{jl}^a - u_{il}^a u_{jk}^a) \\ &= \frac{1}{2} \left[\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} + \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} \right]. \end{aligned} \tag{1.2}$$

In the process of addition and subtraction of second derivatives of (1.1) all the third order derivatives of u cancel out due to the symmetry in (1.1). Each equation of (1.2) belongs to $\Delta_2^2 \setminus \Delta_2^1$. See [2] for the details.

Let $S_s^r \subset J^s(X, U)$ be the set of common zeros of all elements of Δ_s^r.

DEFINITION 1.3. Given a system (1) of order q let r and k be a pair of nonnegative integers having the following properties: the r-th prolongation $\Delta^{(r)}$ contains differential polynomials of the form

$$b_K^a(x, u^{(k)}) := u_K^a - H_K^a(x, u^{(k-1)}) \tag{1.3}$$

for each $a = 1, \dots, m$, each multi-index K with $|K| = k$.

If such a pair of integers (r, k) exists we say that (1) admits prolongation to a complete system of order k and $b_K^a(x, u^{(k)}) = 0$ is called a complete prolongation of (1).

DEFINITION 1.4. Let (4) be a complete prolongation of (1) that is obtained from the r-th prolongation $\Delta^{(r)}$. Then the common zero set $\mathcal{S}_{k-1}^r \subset J^{k-1}(X, U)$ of Δ_{k-1}^r is called the initial data set associated with the complete prolongation.

THEOREM 1.5. (UNIQUENESS OF THE COMPLETE PROLONGATION). *Suppose that the complete system* (4) *is attained by the r-th prolongation of* (1). *Then* (4) *is the unique complete system of order k on* $\mathcal{S}_{k-1}^r$.

Proof. Let

$$u_K^a = G_K^a(x, u^{(k-1)}), \quad a = 1, \dots, m, \quad |K| = k$$

be a complete system of order k obtained from the r-th prolongation $\Delta^{(r)}$. Then the difference between these two complete systems $H_K^a - G_K^a$ belongs to the ideal $\Delta^{(r)}$ and hence belongs to Δ_{k-1}^r. Therefore, on $\mathcal{S}_{k-1}^r$ we have

$$H_K^a - G_K^a = 0$$

for each $a = 1, \dots, m$, each multi-index K with $|K| = k$ and each $a = 1, \dots, m$. $\square$

Now let $\mathcal{D}$ be the n-dimensional distribution defined along $\mathcal{S}_{k-1}^r$ by (8)–(10). The main theorem of this paper is the following

THEOREM 2. *Suppose that* (4) *is a complete system of order* k, $k \geq q$, *obtained from the r-th prolongation of* (1) *and that* $\mathcal{S}^r_{k-1}$ *is the associated initial data set. Then the following are equivalent:*

i) (1) *has a unique smooth solution for every initial condition* $(x_0, u_0^{(k-1)}) \in \mathcal{S}^r_{k-1}$.

ii) $\mathcal{D}$ *is tangent to* $\mathcal{S}^r_{k-1}$ *and integrable in the sense of Frobenius.*

iii) $\mathcal{D}$ *is tangent to* $\mathcal{S}^r_{k-1}$ *and* (4) *satisfies the compatibility conditions on* $\mathcal{S}^r_{k-1}$, *namely,*

$$D_\lambda H^a_{J,\mu} = D_\mu H^a_{J,\lambda} \quad \text{on } \mathcal{S}^r_{k-1} \tag{11}$$

for each $a = 1, \ldots, m$, *each multi-index* J *with* $|J| = k-1$ *and each* $\mu, \lambda = 1, \ldots, n$.

Proof. i) $\Longleftrightarrow$ ii)

Suppose that i) holds. For any point $(x_0, u_0^{(k-1)}) \in \mathcal{S}^r_{k-1}$ let $u = f(x)$ be the solution with the initial condition $f^{(k-1)}(x_0) = u^{(k-1)}(x_0)$. Then for every x in a neighborhood of x_0 in $\mathbb{R}^n$ $(x, f^{(k-1)}(x))$ annihilates every differential polynomial of Δ^r_{k-1}. Hence, the $(k-1)$-jet graph $x \mapsto (x, f^{(k-1)}(x))$ is a submanifold of $\mathcal{S}^r_{k-1}$ and also an integral manifold of $\mathcal{D}$. This implies that $\mathcal{D}$ is tangent to $\mathcal{S}^r_{k-1}$ at $(x_0, u_0^{(k-1)})$ and $\mathcal{S}^r_{k-1}$ is foliated by integral manifolds of $\mathcal{D}$, which is the Frobenius integrability of $\mathcal{D}$. Conversely, by the Frobenius theorem ii) implies that for every $(x_0, u_0^{(k-1)}) \in \mathcal{S}^r_{k-1}$ there exists a unique integral manifold $(x, f^{(k-1)}(x))$ of $\mathcal{D}$ such that $f^{(k-1)}(x_0) = u_0^{(k-1)}$. Then $u = f(x)$ is a solution of (1).

ii) $\Longleftrightarrow$ iii)

Extend $\mathcal{D}$ by extending (9) to a neighborhood of $\mathcal{S}^r_{k-1}$ in $J^{k-1}(X, U)$. We use the summation convention under the agreement that the indices in Greek letters vary 1 through n and the indices in the Roman letters vary 1 through m. Now we compute the exterior derivative of each 1-form: By (8)

$$\begin{aligned} d\theta^a &= d(du^a - u^a_\lambda dx^\lambda) \\ &= -du^a_\lambda \wedge dx^\lambda \\ &= -(\theta^a_\lambda + u^a_{\lambda\mu} dx^\mu) \wedge dx^\lambda \\ &\equiv -u^a_{\lambda\mu} dx^\mu \wedge dx^\lambda \\ &\equiv -\sum_{\mu<\nu} (u^a_{\lambda\mu} - u^a_{\mu\lambda}) dx^\mu \wedge dx^\lambda, \qquad \bmod \theta. \end{aligned} \tag{1.4}$$

Since $u^a_{\lambda\mu} = u^a_{\mu\lambda}$ we have

$$d\theta^a \equiv 0, \quad \text{mod } \theta. \tag{1.5}$$

Similarly, for the multi-indices I with $|I| \le k-3$ we have

$$d\theta^a_I \equiv 0, \qquad \text{mod } \theta. \tag{1.6}$$

For the multi-indices I with $|I| = k-2$ we have

$$\begin{aligned} d\theta^a_I &= -du^a_{I,\lambda} \wedge dx^\lambda \\ &= -(\theta^a_{I,\lambda} + H_{I,\lambda,\mu})dx^\mu \wedge dx^\lambda \\ &\equiv -\sum_{\mu<\lambda}(H^a_{I,\lambda,\mu} - H^a_{I,\mu,\lambda})dx^\mu \wedge dx^\lambda, \qquad \text{mod } \theta. \end{aligned} \tag{1.7}$$

Since $H^a_{I,\lambda,\mu} = H^a_{I,\mu,\lambda}$ we have

$$d\theta^a_I \equiv 0, \quad \text{mod } \theta.$$

Now for the multi-indices J with $|J| = k-1$ we have

$$\begin{aligned} -d\theta^a_J &= dH^a_{J,\lambda} \wedge dx^\lambda \\ &= (H^a_{J,\lambda})_{x^\mu} dx^\mu \wedge dx^\lambda + (H^a_{J,\lambda})_{u^j} du^j \wedge dx^\lambda \\ &\quad + \sum_{|I|\le k-2} (H^a_{J,\lambda})_{u^j_I} du^j_I \wedge dx^\lambda \\ &\quad + \sum_{|I|=k-1} (H^a_{J,\lambda})_{u^j_I} du^j_I \wedge dx^\lambda. \end{aligned} \tag{1.8}$$

Substituting $du^j = \theta^j + u^j_\mu dx^\mu$, $\quad du^j_I = \theta^j_I + u^j_{I,\mu}dx^\mu \quad$ for $|I| \le k-2$ and $du^j_I = \theta^j_I + H^j_{I,\mu}dx^\mu \quad$ for $|I| = k-1$ the right hand side of (1.8) becomes, mod θ,

$$\begin{aligned} &\sum_{\lambda,\mu}\Big\{(H^a_{J,\lambda})_{x^\mu} + (H^a_{J,\lambda})_{u^j}u^j_\mu + \sum_{|I|\le k-2}(H^a_{J,\lambda})_{u^j_I}u^j_{I,\mu} \\ &\qquad + \sum_{|I|=k-1}(H^a_{J,\lambda})_{u^j_I}H^j_{I,\mu}\Big\}dx^\mu \wedge dx^\lambda \\ &= \sum_{\mu,\lambda} D_\mu(H^a_{J,\lambda})dx^\mu \wedge dx^\lambda \\ &= \sum_{\mu<\lambda}(D_\mu H^a_{J,\lambda} - D_\lambda H^a_{J,\mu})dx^\mu \wedge dx^\lambda. \end{aligned}$$

Therefore, $d\theta^a_J \equiv 0$, mod θ, on S^r_{k-1} if and only if (11) holds. This completes the proof. $\square$

2. Examples

In this section we present two examples of overdetermined PDE systems and discuss the compatibility conditions and the existence of solutions by using Theorem 2.

EXAMPLE 2.1. For one unknown function of two variables $u(x,y)$ let

$$\left\{\begin{array}{l}\Delta_1 := u_x + u_y - a(x,y) = 0\\ \Delta_2 := u_{yy} + c(x,y)u^2 - b(x,y) = 0.\end{array}\right. \tag{2.1}$$

Differentiate the first equation of (2.1) with respect to x and y, respectively, to obtain

$$\begin{aligned} u_{xx} + u_{xy} &= a_x,\\ u_{xy} + u_{yy} &= a_y,\\ u_{yy} + cu^2 &= b. \end{aligned} \tag{2.2}$$

Solving for all the second order derivatives of u we obtain

$$\begin{aligned} u_{xx} &= a_x - a_y - cu^2 + b := H_{11},\\ u_{xy} &= a_y + cu^2 - b := H_{12},\\ u_{yy} &= -cu^2 + b := H_{22}. \end{aligned} \tag{2.3}$$

Notice that (2.3) was obtained from the ideal Δ^0 of A_2 generated by

$$\bar{\Delta} := \{\Delta_1, D_x\Delta_1, D_y\Delta_1, \Delta_2\}.$$

Thus, we take the initial data set $\mathcal{S}_1^0$, the zero set of Δ_1^0. It is easy to see that $\mathcal{S}_1^0$ is the subset of $J^1(X,U)$ defined by the first equation of (2.1), namely, $\Delta_1 = 0$. The distribution $\mathcal{D}$ is given by 1-forms

$$\begin{aligned} \theta &:= du - u_x dx - u_y dy,\\ \theta_x &:= du_x - H_{11}dx - H_{12}dy,\\ \theta_y &:= du_y - H_{12}dx - H_{22}dy. \end{aligned}$$

To see whether $\mathcal{D}$ is tangent to $\mathcal{S}_1^0$ we check whether the exterior derivative of the defining function Δ_1 belongs to the algebraic ideal generated by $\{\theta, \theta_x, \theta_y\}$ on $\mathcal{S}_1^0$. A straightforward calculation shows that

$$d\Delta_1 = d(u_x + u_y - a) = \theta_x + \theta_y,$$

which implies that $\mathcal{D}$ is tangent to $\mathcal{S}_1^0$.

As coordinates of $\mathcal{S}_1^0$ we use (x, y, u, u_x). Compatibility conditions are

$$\begin{aligned} D_y H_{11} &= D_x H_{12}, \\ D_y H_{12} &= D_x H_{22}. \end{aligned} \tag{2.4}$$

Restriction (2.4) to $\mathcal{S}_1^0$ gives

$$a_{yy} - (b_x + b_y) + 2acu + (c_x + c_y)u^2 = 0. \tag{2.5}$$

By Theorem 2 the system (2.1) has a unique solution for every initial condition $(x_0, y_0, u_0, (u_x)_0, (u_y)_0)$ with $(u_x)_0 + (u_y)_0 = a(x_0, y_0)$ if and only if (2.5) holds on $\mathcal{S}_1^0$. Now (2.5) is equivalent to

$$\begin{aligned} &a_{yy} - (b_x + b_y) = 0, \\ &ac = 0, \\ &c_x + c_y = 0, \end{aligned} \tag{2.6}$$

which holds if and only if either

$$\begin{aligned} &a = 0, \\ &b_x + b_y = 0, \\ &c_x + c_y = 0 \end{aligned} \tag{2.7}$$

or

$$\begin{aligned} &c = 0, \\ &a_{yy} - (b_x + b_y) = 0. \end{aligned} \tag{2.8}$$

Therefore, (2.1) has a unique solution for any initial condition that lies on $\mathcal{S}_1^0$ if and only if the coefficients satisfy either (2.7) or (2.8).

Our second example is the "test case" (see [12]) of the equivalence problem of Riemannian structures. We present another proof of the classical theorem which states that a Riemannian n-manifold is locally isometric to $\mathbb{R}^n$ if and only if the curvature tensor is zero.

EXAMPLE 2.2. Let M be a smooth manifold of dimension n with a smooth Riemannian metric g. Let $x = (x^1, \dots, x^n)$ be a local coordinate system of M. A mapping $u = (u^1, \dots, u^n) : M \to \mathbb{R}^n$ is an isometric equivalence if u satisfies

$$\sum_{\alpha=1}^{n} u_i^\alpha u_j^\alpha = g_{ij} \quad \text{for each } i, j = 1, \dots, n, \tag{2.9}$$

where $u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}$ and $g_{ij} = g(\partial_i, \partial_j)$. Differentiating (2.9) with respect to x^k we have

$$\sum_{\alpha=1}^{n} u_{ik}^\alpha u_j^\alpha + \sum_{\alpha=1}^{n} u_i^\alpha u_{jk}^\alpha = \frac{\partial g_{ij}}{\partial x^k} \quad \text{for each} \quad i, j, k = 1, \cdots, n.$$

A linear summation after permuting the indices in the above yields

$$\sum_{\alpha}^{n} u_{jk}^\alpha u_i^\alpha = \frac{1}{2}\left[\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right].$$

Denoting the right hand side by $[jk, i]$ we have

$$\sum_{\alpha=1}^{n} u_{jk}^\alpha u_i^\alpha = [jk, i] \quad \text{for each} \quad i, j, k = 1, \dots, n. \tag{2.10}$$

Since the matrix $[u_i^\alpha]_{\alpha,i}$ is nonsingular, we can solve (2.10) for all the second order derivatives of u^α in terms of $(x, u^{(1)})$ by the implicit function theorem. To work in the category of differential polynomials we consider the following dual expressions of (2.9):

$$\sum_{i,j=1}^{n} g^{ij} u_i^\alpha u_j^\beta = \delta^{\alpha\beta}, \tag{2.11}$$

where the matrix $\left[g^{ij}\right]$ is the inverse matrix of $[g_{ij}]$ and $\delta^{\alpha\beta}$ is the Kronecker delta. By multiplying (2.10) by $u_l^\beta g^{il}$ and summation over repeated indices we obtain

$$\sum_{\alpha.i.l} u_{jk}^\alpha u_i^\alpha u_l^\beta g^{il} = \sum_{il} [jk, i] u_l^\beta g^{il}. \tag{2.12}$$

Let $\Gamma^l_{jk} := \sum_{i=1}^n g^{il}[jk, i]$ be the Christoffel symbols. Substituting (2.11) for $u^\alpha_i u^\beta_l g^{il}$ in the left hand side of (2.12) we have for each $\beta, j, k = 1, \dots, n$

$$u^\beta_{jk} = \sum_{l=1}^n \Gamma^l_{jk}(x) u^\beta_l, \tag{2.13}$$

which is a complete prolongation of order 2 of (2.9). (2.13) is obtained from the first prolongation of (2.9) and the associated initial data set is $\mathcal{S}^1_1 \subset J^1(X, U)$, which is defined by (2.9). For each $\alpha, j = 1, \dots, n$ let

$$\theta^\alpha_j := du^\alpha_j - \Gamma^l_{j\lambda} u^\alpha_l dx^\lambda. \tag{2.14}$$

Then the exterior derivative of the defining functions of $\mathcal{S}^1_1$ is

$$\begin{aligned} & d(u^\alpha_i u^\alpha_j - g_{ij}) \\ &= du^\alpha_i u^\alpha_j + u^\alpha_i du^\alpha_j - dg_{ij} \\ &\equiv (\Gamma^l_{i\lambda} u^\alpha_l u^\alpha_j + \Gamma^l_{j\lambda} u^\alpha_l u^\alpha_i - \frac{\partial g_{ij}}{\partial x^\lambda}) dx^\lambda, \quad \mathrm{mod}\ \theta := \{\theta^\alpha_j\}, \end{aligned} \tag{2.15}$$

by (2.14). Substituting $\Gamma^l_{i\lambda} := \sum_{t=1}^n g^{tl}[i\lambda, t]$ and $\Gamma^l_{j\lambda} := \sum_{t=1}^n g^{tl}[j\lambda, t]$ we see that for each λ the quantity in the parenthesis of the last equation of (2.15) is zero on $\mathcal{S}^1_1$, which implies that the distribution $\mathcal{D}$ defined by $\theta^\alpha_j = 0$ is tangent to $\mathcal{S}^1_1$. Therefore, by Theorem 2, (2.9) has a unique solution for any initial condition $(x_0, u^{(1)}_0) \in \mathcal{S}^1_1$ if and only if the compatibility conditions

$$D_i \left(\sum_{l=1}^n \Gamma^l_{jk} u^\beta_l \right) = D_j \left(\sum_{l=1}^n \Gamma^l_{ik} u^\beta_l \right) \tag{2.16}$$

holds on $\mathcal{S}^1_1$ for all $i, j, k, \beta = 1, \dots, n$. Then by (2.13) the left hand side of (2.16) is

$$\sum_{\lambda=1}^n \left(\frac{\partial \Gamma^\lambda_{jk}}{\partial x^i} + \sum_{l=1}^n \Gamma^l_{jk} \Gamma^\lambda_{li} \right) u^\beta_\lambda$$

and the right hand side of (2.16) is similarly

$$\sum_{\lambda=1}^n \left(\frac{\partial \Gamma^\lambda_{ik}}{\partial x^j} + \sum_{l=1}^n \Gamma^l_{ik} \Gamma^\lambda_{lj} \right) u^\beta_\lambda.$$

Therefore, (2.16) holds if and only if

$$\sum_{\lambda=1}^{n}\left[\frac{\partial\Gamma_{jk}^{\lambda}}{\partial x^{i}}-\frac{\partial\Gamma_{ik}^{\lambda}}{\partial x^{j}}+\sum_{l=1}^{n}\left(\Gamma_{jk}^{l}\Gamma_{li}^{\lambda}-\Gamma_{ik}^{l}\Gamma_{lj}^{\lambda}\right)\right]u_{\lambda}^{\beta}=0. \tag{2.17}$$

The quantity in the bracket in (2.17) is the component R_{ijk}^{λ} of the curvature tensor. Since the matrix $\left(u_{\lambda}^{\beta}\right)$ is nonsingular on $\mathcal{S}_1^1$, (2.17) holds if and only if R_{jkl}^{λ} is equal to zero for each j, k, l, λ.

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