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THE EXPECTED INDEPENDENT DOMINATION NUMBER OF TWO TYPES OF TREES

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ABSTRACT. We derive formulas for the expected values $\mu(n)$ of the independent domination numbers of a random planted plane tree and a random trivalent tree with n vertices, respectively, and we determine the asymptotic behavior of $\mu(n)$ as n goes to infinity.

1. Introduction

Let D be a digraph. A subset S of vertices of D is a *dominating set* of D if for each vertex v not in S there exists a vertex u in S such that (u, v) is an arc of D. The *domination number* of D is the number $\alpha(D)$ of vertices in any smallest dominating subset of vertices in D. A subset I of vertices of D is an *independent set* of D if no two vertices of I are joined by an arc in D. The *independence number* of D is the number $\beta(D)$ of vertices in any largest independent subset of vertices in D. An *independent dominating set* of D is an independent and dominating set of D. The *independent domination number* of D is the number $\alpha'(D)$ of vertices in any smallest independent dominating subset of vertices in D. A *directed rooted tree* is an oriented rooted tree in which every direction is led away from the root. In this article, *rooted trees are regarded as directed rooted trees* in the sense above. For definitions not given here, see [2].

There are $\binom{2n}{n}/(n+1)$ binary trees T with $2n+1$ vertices. Let $\mu(2n+1)$ denote the expected value of the independent domination number $\alpha'(T)$ over the set of such binary trees. Lee showed in [5] that

$$\mu(2n+1) = \sum (k+1)2^k \frac{\langle n\rangle_k}{\langle 2n\rangle_k}$$

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for $n = 0, 1, 2, \dots$, where the inner sum is over all even integers k with $1 \le k \le n$ and $\langle n \rangle_k$ denotes the falling factorial $\langle n \rangle_k = n(n-1)\cdots(n-k+1)$, and that

$$\frac{\mu(2n+1)}{2n+1} \longrightarrow \frac{1}{2}$$

as $n \to \infty$.

The goal of this article is to do similar work for "planted plane trees" and "trivalent trees": For planted plane trees with $n+2$ vertices,

$$\mu(n+2) = 1 + \frac{n+1}{\binom{2n}{n}} \sum_{l=1}^{n} \sum_{k=1}^{l} \frac{(-1)^{l-k} k}{2n+k} \binom{l}{k} \binom{2n+k}{n}$$

for $n \ge 1$ and

$$\frac{\mu(n+2)}{n+2} \longrightarrow \frac{1}{2}$$

as $n \to \infty$. For trivalent trees with $2n+2$ vertices,

$$\begin{aligned} \mu(2n+2) \quad = \quad & \frac{n+1}{\binom{2n}{n}} \left\{ \sum_{k=0}^{n} \frac{k 2^{k+1}}{2n-k} \binom{2n-k}{n} \right. \\ & \left. - \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(2k+1)4^k}{2n-2k+1} \binom{2n-2k+1}{n+1} \right\} \end{aligned}$$

for $n \ge 1$ and

$$\frac{\mu(2n+2)}{2n+2} \longrightarrow \frac{1}{2}$$

as $n \to \infty$.

2. Preliminaries

An *oriented tree* is a tree in which each edge is assigned a unique direction. A digraph might have no independent dominating sets as we can see in 3-cycles. However, every oriented tree has a unique independent dominating set [5]. Therefore, we have the following lemma.

LEMMA 1. *Every rooted tree has a unique independent dominating set.*

If T is a tree with root r and v is a vertex of T, then the *level number* of v is the length of the unique path from r to v in T. If a vertex v of a rooted tree T has level number l, we say that v is at level l. It is, thus, easy to see that the set of vertices at even levels of T is actually the unique independent dominating set of T. Let $N_e(T)$ and $N_o(T)$ denote the number of vertices at even and odd levels of T, respectively. Then, we obtain that

$$N_e(T) = \alpha'(T) \quad \text{and} \quad N_e(T) + N_o(T) = |V(T)|.$$

3. Planted plane trees

A *planted plane tree* is a tree that is embedded in the plane and rooted at an end-vertex (or a vertex of degree one). Two such trees are equivalent if there exists a one-to-one correspondence between their vertices such that

(a) the roots correspond,
(b) adjacency of vertices is preserved,
(c) the cyclic ordering of the vertices adjacent to each vertex is preserved.

Let y_n denote the number of planted plane trees with $n+2$ vertices for $n \geq 0$. Clearly, $y_0 = 1$. If $n \geq 1$, consider an ordered set of j planted plane trees $T_1, \ldots, T_j$ that have $n+j$ vertices altogether. If the roots of these j trees are identified and joined to a new vertex r, the resulting configuration may be regarded as a planted plane tree T with $n+2$ vertices that is rooted at the end-vertex r. The vertices are not labeled and different orderings of the subtrees $T_1, \ldots, T_j$ yield different trees T in general. It follows, therefore, that

$$y_n = \sum_{j=1}^{n} \sum y_{a_1} \cdots y_{a_j}$$

for $n \geq 1$, where the inner sum is over all solutions in integers to the equation $a_1 + \cdots + a_j = n - j$. Thus if

$$y = y(x) = \sum_{n=0}^{\infty} y_n x^n,$$

then

$$y = 1 + xy + x^2y^2 + \cdots = \frac{1}{1-xy}. \tag{3.1}$$

Hence, we obtain

$$y = \frac{1}{2x}\left(1 - \sqrt{1-4x}\right) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1} \tag{3.2}$$

since $y(0) = 1$. This, of course, is well-known argument. See [3]. In the argument above, observe that

$$N_e(T) = 1 + \sum_{k=1}^{j} N_o(T_k), \tag{3.3}$$

$$N_o(T) = 1 - j + \sum_{k=1}^{j} N_e(T_k). \tag{3.4}$$

For $1 \leq k \leq n+1$, let $f_{n,k}$ and $g_{n,k}$ denote the number of planted plane trees T with $n+2$ vertices such that $N_e(T) = k$ and $N_o(T) = k$, respectively. Let

$$F = F(x,z) = \sum_{n=0}^{\infty} \Big(\sum_{k=1}^{n+1} f_{n,k} z^k \Big) x^n, \tag{3.5}$$

$$G = G(x,z) = \sum_{n=0}^{\infty} \Big(\sum_{k=1}^{n+1} g_{n,k} z^k \Big) x^n. \tag{3.6}$$

It follows by a slight extension of the argument used to establish equation (3.1) that

$$F = z(1 + xG + x^2G^2 + \cdots) = \frac{z}{1 - xG}, \tag{3.7}$$

$$G = z + xF + z^{1-2}x^2F^2 + \cdots = \frac{z^2}{z - xF}. \tag{3.8}$$

The factor z is present in the equation (3.7) because of (3.3), and the factor z^{1-j} of the term $z^{1-j}x^jF^j$ in the equation (3.8) is present because of (3.4). Notice that

$$F(x,1) = y(x) \quad \text{and} \quad G(x,1) = y(x). \tag{3.9}$$

THEOREM 2. *Let $\mu(n+2)$ denote the expected independent domination number of the $\binom{2n}{n}\frac{1}{n+1}$ planted plane trees with $n+2$ vertices and define*

$$M(x) = \sum_{n=0}^{\infty} \mu(n+2) \binom{2n}{n} \frac{x^n}{n+1}. \tag{3.10}$$

Then we have

$$M(x) = y - 1 + \frac{1}{2-y}. \tag{3.11}$$

Proof. It is easy to see that

$$M(x) = F_z(x, 1). \tag{3.12}$$

If we differentiate both sides of equations (3.7) and (3.8) with respect to z, set $z = 1$, and use equations (3.12), (3.9), and (3.1), we obtain

$$M(x) = y + (y-1)G_z(x,1), \tag{3.13}$$

$$G_z(x,1) = -y^2 + 2y + (y-1)M(x). \tag{3.14}$$

If we substitute (3.14) for $G_z(x,1)$ in (3.13), solve the resulting equation for $M(x)$, and use (3.1) again, we obtain the required result. □

We know that $M(x)$ is the generating function for the total sums of the independent domination numbers of planted plane trees. Therefore, using power series expansion of $M(x)$ in x, we could find directly the expected value $\mu(n+2)$ of the independent domination numbers of planted plane trees for small n. Actually, it follows from (3.2), (3.11), and the routine use of *Mathematica* that

$$\begin{aligned} y(x) &= 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 \\ &\quad +1430x^8 + 4862x^9 + 16796x^{10} + \cdots \end{aligned}$$

and

$$\begin{aligned} M(x) &= 1 + 2x + 5x^2 + 15x^3 + 49x^4 + 168x^5 + 594x^6 \\ &\quad +2145x^7 + 7865x^8 + 29172x^9 + 109174x^{10} + \cdots . \end{aligned}$$

Table 1 shows the values of $\mu(n+2)$ and $\mu(n+2)/(n+2)$. The entries for $n \le 4$ were verified using the diagrams in [7] for planted plane trees with up to 6 vertices.

Furthermore, we can derive a reasonably explicit formula for $\mu(n+2)$ as follows.

TABLE 1. $\mu(n+2)$ and $\mu(n+2)/(n+2)$

n	0	1	2	3	4
$\mu(n+2)$	1	2	2.5	3	3.5
$\mu(n+2)/(n+2)$	.5000	.6666	.6250	.6000	.5833

n	5	6	7	8	9
$\mu(n+2)$	4	4.5	5	5.5	6
$\mu(n+2)/(n+2)$	.5714	.5625	.5555	.5500	.5454

THEOREM 3. *The expected value $\mu(n+2)$ of the independent domination numbers of planted plane trees with $n+2$ vertices is*

$$\mu(n+2) = 1 + \frac{n+1}{\binom{2n}{n}} \sum_{l=1}^{n} \sum_{k=1}^{l} \frac{(-1)^{l-k} k}{2n+k} \binom{l}{k} \binom{2n+k}{n}$$

for $n \geq 1$.

Proof. The following identity appears in [8]:

$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{k=0}^{\infty} \frac{n(2k+n-1)!}{k!(k+n)!} x^k \tag{3.15}$$

for integer $n \geq 1$. Let $f = \sum a_n x^n$ be a formal power series and let $[x^n]f$ denote the coefficient a_n of x^n in f. It is easy to see from (3.10) and (3.2) that

$$[x^n]M(x) = \mu(n+2)\binom{2n}{n}\frac{1}{n+1} \tag{3.16}$$

and

$$[x^n](y-1) = \binom{2n}{n}\frac{1}{n+1}. \tag{3.17}$$

To find $[x^n](1/(2-y))$, recall that (3.15) is not valid for $n=0$ and that $y-1$ has a power series expansion with zero constant coefficient. For $n \geq 1$, it follows from (3.15) that

$$\begin{aligned} [x^n]\left(\frac{1}{2-y}\right) &= [x^n]\left(\frac{1}{1-(y-1)}\right) \\ &= [x^n]\sum_{l=0}^{\infty}(y-1)^l = [x^n]\sum_{l=1}^{n}(y-1)^l \\ &= [x^n]\sum_{l=1}^{n}\sum_{k=1}^{l}(-1)^{l-k}\binom{l}{k}y^k \\ &= [x^n]\sum_{l=1}^{n}\sum_{k=1}^{l}(-1)^{l-k}\binom{l}{k} \\ &\qquad \times \sum_{m=0}^{\infty}\frac{k}{2m+k}\binom{2m+k}{m}x^m \end{aligned} \tag{3.18}$$

$$= \sum_{l=1}^{n}\sum_{k=1}^{l} \frac{(-1)^{l-k}k}{2n+k}\binom{l}{k}\binom{2n+k}{n}.$$

It follows from (3.11) that

$$[x^n]M(x) = [x^n]\Big(y-1+\frac{1}{2-y}\Big) = [x^n](y-1)+[x^n]\Big(\frac{1}{2-y}\Big)$$

and thus from (3.16), (3.17), and (3.18) that

$$\mu(n+2)\binom{2n}{n}\frac{1}{n+1} = \binom{2n}{n}\frac{1}{n+1} + \sum_{l=1}^{n}\sum_{k=1}^{l}\frac{(-1)^{l-k}k}{2n+k}\binom{l}{k}\binom{2n+k}{n}.$$

Therefore, we obtain the required result. □

COROLLARY 4. *The expected value $\mu(n+2)$ of the numbers of vertices at even levels of planted plane trees with $n+2$ vertices is*

$$\mu(n+2) = 1 + \frac{n+1}{\binom{2n}{n}}\sum_{l=1}^{n}\sum_{k=1}^{l}\frac{(-1)^{l-k}k}{2n+k}\binom{l}{k}\binom{2n+k}{n}$$

for $n \geq 1$.

To determine the asymptotic behavior of $\mu(n+2)/(n+2)$, we need the following lemma [5].

LEMMA 5. *Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be power series with radii of convergence $\rho_1 \geq \rho_2$, respectively. Suppose that $A(x)$ converges absolutely at $x = \rho_1$. Suppose that $b_n > 0$ for all n and that b_{n-1}/b_n approaches a limit b as $n \to \infty$. If $\sum_{n=0}^{\infty} c_n x^n = A(x)B(x)$, then $c_n \sim A(b)b_n$.* □

Now we can state the main result of this section.

THEOREM 6. *The expected value $\mu(n+2)$ of the independent domination numbers of planted plane trees with $n+2$ vertices is*

$$\mu(n+2) \sim \frac{1}{2}(n+2).$$

Proof. Recall the equation (3.11) and consider the second term $1/(2-y)$ in (3.11). It follows from (3.2) that

$$\frac{1}{2-y} = \frac{2x}{1-\sqrt{1-4x}} \frac{1}{\sqrt{1-4x}}. \tag{3.19}$$

Let

$$A(x) = \frac{2x}{1-\sqrt{1-4x}}$$

and

$$B(x) = \frac{1}{\sqrt{1-4x}}$$

so that

$$\frac{1}{2-y} = A(x)B(x).$$

Then, we obtain

$$A(x) = \frac{1+\sqrt{1-4x}}{2},$$

which has a power series expansion in x with radius of convergence $1/4$. Moreover, this power series converges absolutely at $x = 1/4$ (see, for example, [4, p. 426]). On the other hand, we obtain

$$B(x) = \frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n,$$

which converges for $|x| < 1/4$. If we let

$$b_n = \binom{2n}{n},$$

it is easily checked that $b_{n-1}/b_n \to 1/4$ as $n \to \infty$ and that $b_n > 0$ for all n. If we let

$$\frac{1}{2-y} = \sum_{n=0}^{\infty} c_n x^n,$$

we obtain from Lemma 5 that

$$c_n \sim A(1/4)b_n = \frac{1}{2}\binom{2n}{n}$$

and hence from (3.10) that

$$\begin{aligned} \mu(n+2)\frac{\binom{2n}{n}}{n+1} &= \frac{\binom{2n}{n}}{n+1} + c_n \\ &\sim \frac{\binom{2n}{n}}{n+1} + \frac{1}{2}\binom{2n}{n}, \end{aligned}$$

which implies that

$$\mu(n+2) \sim \frac{1}{2}(n+2).$$

This completes the proof. □

COROLLARY 7. *The expected value $\mu(n+2)$ of the numbers of vertices at even levels of planted plane trees with $n+2$ vertices is*

$$\mu(n+2) \sim \frac{1}{2}(n+2).$$

We know [6] that the expected independence number $\nu(n+2)$ of planted plane trees with $n+2$ vertices is

$$\nu(n+2) \sim .6180\cdots(n+2).$$

It is easy to see that

$$\alpha'(T) \leq \beta(T)$$

for any planted plane tree T. Our result

$$\mu(n+2) \sim .5(n+2)$$

is consistent with these two facts.

4. Trivalent trees

A *trivalent tree* is a planted plane tree in which each vertex has degree one or three. This restriction on the degrees implies that there must be an even number of vertices in such a tree.

Let y_n denote the number of trivalent trees with n vertices of degree three (and thus, $2n+2$ vertices altogether) for $n \geq 0$. Clearly, $y_0 = 1$. If $n \geq 1$, consider an ordered pair of trivalent trees T_1 and T_2 with n_1 and n_2 vertices of degree three, respectively. If the roots of these trees are identified and joined to a new vertex r, the resulting configuration may be regarded as a trivalent tree T with n_1+n_2+1 vertices of degree three that is rooted at the vertex r. Notice that the vertices are not labeled and that different orderings of the subtree T_1 and T_2 yield different trees T in general. It follows, therefore, that

$$y_n = \sum_{j=0}^{n-1} y_j y_{n-1-j}$$

for $n \geq 1$. Thus if

$$y = y(x) = \sum_{n=0}^{\infty} y_n x^n,$$

then

$$y = 1 + xy^2. \tag{4.1}$$

Hence, we obtain

$$y = \frac{1}{2x}\left(1 - \sqrt{1-4x}\right) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1} \tag{4.2}$$

since $y(0) = 1$. This, of course, is well-known argument. See [1] or [3]. In the argument above, observe that

$$N_e(T) = 1 + N_o(T_1) + N_o(T_2), \tag{4.3}$$

$$N_o(T) = -1 + N_o(T_1) + N_o(T_2). \tag{4.4}$$

For $1 \leq k \leq 2n+1$, let $f_{n,k}$ and $g_{n,k}$ denote the number of trivalent trees T with n vertices of degree three such that $N_e(T) = k$ and $N_o(T) = k$, respectively. Let

$$F = F(x,z) = \sum_{n=0}^{\infty} \Big(\sum_{k=1}^{2n+1} f_{n,k} z^k \Big) x^n, \tag{4.5}$$

$$G = G(x,z) = \sum_{n=0}^{\infty} \Big(\sum_{k=1}^{2n+1} g_{n,k} z^k \Big) x^n. \tag{4.6}$$

It follows by a slight extension of the argument used to establish the equation (4.1) that

$$F = z + zxG^2, \tag{4.7}$$

$$G = z + z^{-1}xF^2. \tag{4.8}$$

The factor z is present in the equation (4.7) because of (4.3), and the factor z^{-1} in the equation (4.8) is present because of (4.4). Notice that

$$F(x,1) = y(x) \quad \text{and} \quad G(x,1) = y(x). \tag{4.9}$$

THEOREM 8. *Let $\mu(2n+2)$ denote the expected independent domination number of the $\binom{2n}{n}\frac{1}{n+1}$ trivalent trees with $2n+2$ vertices and define*

$$M(x) = \sum_{n=0}^{\infty} \mu(2n+2) \binom{2n}{n} \frac{x^n}{n+1}. \tag{4.10}$$

Then we have

$$M(x) = \frac{2}{1-2xy} - \frac{y}{1-4x^2y^2}. \tag{4.11}$$

Proof. It is easy to see that

$$M(x) = F_z(x,1). \tag{4.12}$$

If we differentiate both sides of equations (4.7) and (4.8) with respect to z, set $z = 1$, and use equations (4.12), (4.9), and (4.1), we obtain

$$M(x) = y + 2xyG_z(x,1), \tag{4.13}$$

$$G_z(x,1) = 2 - y + 2xyM(x). \tag{4.14}$$

If we substitute (4.14) for $G_z(x,1)$ in (4.13), solve the resulting equation for $M(x)$, and use (4.1) again, we obtain the required result. □

We know that $M(x)$ is the generating function for the total sums of the independent domination numbers of trivalent trees. Therefore, using power series expansion of $M(x)$ in x, we could find directly the expected value $\mu(2n+2)$ of the independent domination numbers of trivalent trees for small n. Actually, it follows from (4.2), (4.11), and the routine use of *Mathematica* that

$$\begin{aligned} y(x) &= 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 \\ &\quad +1430x^8 + 4862x^9 + 16796x^{10} + \cdots \end{aligned}$$

and

$$\begin{aligned} M(x) &= 1 + 3x + 6x^2 + 23x^3 + 74x^4 + 270x^5 + 972x^6 \\ &\quad +3599x^7 + 13410x^8 + 50474x^9 + 191124x^{10} + \cdots . \end{aligned}$$

Table 2 shows the values of $\mu(2n+2)$ and $\mu(2n+2)/(2n+2)$. The entries for $n \leq 4$ were verified using the diagrams in [7] for trivalent trees with up to 10 vertices.

Furthermore, we can derive a reasonably explicit formula for $\mu(2n+2)$ as follows.

TABLE 2. $\mu(2n+2)$ and $\mu(2n+2)/(2n+2)$

n	0	1	2	3	4
$\mu(2n+2)$	1	3	3	4.6	5.2857
$\mu(2n+2)/(2n+2)$	.5000	.7500	.5000	.5750	.5285

n	5	6	7	8	9
$\mu(2n+2)$	6.4285	7.3636	8.3892	9.3776	10.3813
$\mu(2n+2)/(2n+2)$	.5357	.5259	.5243	.5209	.5190

THEOREM 9. *The expected value $\mu(2n+2)$ of the independent domination numbers of trivalent trees with $2n+2$ vertices is*

$$\begin{aligned}\mu(2n+2) &= \frac{n+1}{\binom{2n}{n}}\Big\{\sum_{k=0}^{n}\frac{k2^{k+1}}{2n-k}\binom{2n-k}{n} \\ &\qquad - \sum_{k=0}^{\lfloor n/2\rfloor}\frac{(2k+1)4^k}{2n-2k+1}\binom{2n-2k+1}{n+1}\Big\}\end{aligned}$$

for $n \geq 1$.

Proof. It follows from (4.11) and (3.15) that

$$\begin{aligned}M(x) &= \frac{2}{1-2xy} - \frac{y}{1-4x^2y^2} \\ &= 2\sum_{m=0}^{\infty}(2xy)^m - y\sum_{m=0}^{\infty}(4x^2y^2)^m \\ &= \sum_{n=0}^{\infty}\Big\{\sum_{k=0}^{n}\frac{k2^{k+1}}{2n-k}\binom{2n-k}{n} \\ &\qquad - \sum_{k=0}^{\lfloor n/2\rfloor}\frac{(2k+1)4^k}{2n-2k+1}\binom{2n-2k+1}{n+1}\Big\}x^n.\end{aligned}$$

Therefore, by equating the coefficients of x^n in both sides of the equality above, we have

$$\mu(2n+2)\frac{\binom{2n}{n}}{n+1} = \sum_{k=0}^{n}\frac{k2^{k+1}}{2n-k}\binom{2n-k}{n} - \sum_{k=0}^{\lfloor n/2\rfloor}\frac{(2k+1)4^k}{2n-2k+1}\binom{2n-2k+1}{n+1}$$

and obtain the required result. □

COROLLARY 10. *The expected value $\mu(2n+2)$ of the numbers of vertices at even levels of trivalent trees with $2n+2$ vertices is*

$$\mu(2n+2) = \frac{n+1}{\binom{2n}{n}}\Big\{\sum_{k=0}^{n}\frac{k2^{k+1}}{2n-k}\binom{2n-k}{n} - \sum_{k=0}^{\lfloor n/2\rfloor}\frac{(2k+1)4^k}{2n-2k+1}\binom{2n-2k+1}{n+1}\Big\}$$

for $n \geq 1$.

Now we can state the main result of this section.

THEOREM 11. *The expected value $\mu(2n+2)$ of the independent domination numbers of trivalent trees with $2n+2$ vertices is*

$$\mu(2n+2) \sim \frac{1}{2}(2n+2).$$

Proof. Consider the first term $2/(1-2xy)$ of $M(x)$ in (4.11). It is easy to see from (4.2) that

$$\frac{2}{1-2xy} = \frac{2}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} 2\binom{2n}{n}x^n. \tag{4.15}$$

Next, consider the second term $y/(1-4x^2y^2)$ of $M(x)$ in (4.11). It is easy to see from (4.2) that

$$\frac{y}{1-4x^2y^2} = \frac{1-\sqrt{1-4x}}{2x}\,\frac{2+\sqrt{1-4x}}{3+4x}\,\frac{1}{\sqrt{1-4x}}. \tag{4.16}$$

Let

$$A(x) = \frac{1-\sqrt{1-4x}}{2x}\,\frac{2+\sqrt{1-4x}}{3+4x}$$

and

$$B(x) = \frac{1}{\sqrt{1-4x}},$$

so that

$$\frac{y}{1-4x^2y^2} = A(x)B(x).$$

Since both factors of $A(x)$ have power series expansions which converge for $|x| < 1/4$ and converge absolutely at $x = 1/4$ (see [4, p. 426]), $A(x)$

converges for $|x| < 1/4$ and converges absolutely at $x = 1/4$. On the other hand, we obtain

$$B(x) = \frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n,$$

which converges for $|x| < 1/4$. If we let

$$b_n = \binom{2n}{n},$$

it is easily checked that $b_{n-1}/b_n \to 1/4$ as $n \to \infty$ and that $b_n > 0$ for all n. If we let

$$\frac{y}{1-4x^2y^2} = \sum_{n=0}^{\infty} c_n x^n,$$

we obtain from Lemma 5 that

$$c_n \sim A(1/4)b_n = \binom{2n}{n}$$

and hence from (4.11), (4.15), and (4.16) that

$$\begin{aligned} \mu(2n+2)\frac{\binom{2n}{n}}{n+1} &= 2\binom{2n}{n} - c_n \\ &\sim 2\binom{2n}{n} - \binom{2n}{n}, \end{aligned}$$

which implies that

$$\mu(2n+2) \sim \frac{1}{2}(2n+2).$$

This completes the proof. □

COROLLARY 12. *The expected value* $\mu(2n+2)$ *of the numbers of vertices at even levels of trivalent trees with* $2n+2$ *vertices is*

$$\mu(2n+2) \sim \frac{1}{2}(2n+2).$$

We know [6] that the expected independence number $\nu(2n+2)$ of trivalent trees with $2n+2$ vertices is

$$\nu(2n+2) \sim .5857\cdots(2n+2).$$

It is easy to see that

$$\alpha'(T) \leq \beta(T)$$

for any trivalent tree T. Our result

$$\mu(2n+2) \sim .5(2n+2)$$

is consistent with these two facts.

References

[1] A. Cayley, *On the analytical forms called trees*, Philos. Mag. **28** (1858), 374–378. [Collected Mathematical Papers, Cambridge **4** (1891), 112–115.]

[2] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Wadsworth & Brooks, Monterey, 1986.

[3] F. Harary, G. Prins, and W. T. Tutte, *The number of plane trees*, Proc. Kon. Ned. Akad. v. Wetensch. **67** (1964), 319–329.

[4] K. Knopp, *Theory and Application of Infinite Series*, Dover, Mineola, 1990.

[5] C. Lee, *The Expectation of Independent Domination Number over Random Binary Trees*, Ars Combin. **56** (2000), 201–209.

[6] A. Meir and J. W. Moon, *The Expected Node-Independence Number of Random Trees*, Proc. Kon. Ned. v. Wetensch. **76** (1973), 335–341.

[7] R. C. Read, *An Atlas of Graphs*, Clarendon Press, Oxford, 1998.

[8] H. S. Wilf, *Generatingfuctionology*, Academic, San Diego, 1990.

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