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$\mathcal{R}$-CRITICAL WEYL STRUCTURES

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ABSTRACT. Weyl structures can be viewed as generalizations of Riemannian metrics. We study Weyl structures which are critical points of the squared L^2 norm functional of the full curvature tensor, defined on the space of Weyl structures on a compact 4-manifold. We find some relationship between these critical Weyl structures and the critical Riemannian metrics. Then in a search for homogeneous critical structures we study left-invariant metrics on some solv-manifolds and prove that they are not critical.

1. Introduction

In this paper we study the critical points of the squared L^2 norm functional of the full curvature tensor, denoted by $\mathcal{R}^w$, defined on the space $\mathcal{W}$ of Weyl structures on a 4-dimensional oriented compact manifold. This is a continuation of the work initiated in [7] where we defined two Weyl functionals related to scalar curvature and discussed about its critical points.

Let us recall that a Weyl structure on a smooth manifold M consists of a conformal class $[g]$ of Riemannian metrics and a torsion-free connection D preserving $[g]$; i.e., for any metric g in $[g]$, $Dg = \omega \otimes g$ for a 1-form ω. This structure may be viewed as a generalization of a Riemannian metric because given a Riemannian metric one can associate its conformal class and Levi-Civita connection. More precisely, the space of Riemannian metrics of unit volume is canonically embedded in the space of Weyl structures, see Section 2.

The study of Weyl structures stemmed from E. Cartan's work on 3-dimensional *Einstein-Weyl* structures [3]. Weyl geometry has been

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explored much to understand Einstein-Weyl structures [4-6, 10-13] recently. In [11], Pedersen-Poon-Swann observed two classes of Weyl structures which are absolute minima of $\mathcal{R}^w$. These two classes are generalizations of the corresponding two classes of metrics: Einstein metrics and half-conformally-flat zero scalar curved metrics.

This motivated us to study in this paper on critical points of $\mathcal{R}^w$. One should be aware that their classification might be very hard as in Riemannian case. What we do for now is to set up for the equation of critical points and then clarify some relationship between the $\mathcal{R}^w$-critical points and the critical points of the corresponding functional, to be denoted by $\mathcal{R}$, defined on the space of Riemannian metrics. This is done in Section 3. We prove in particular that any $\mathcal{R}^w$-critical Weyl structure is locally conformally a metric, that if the first deRham cohomology of M vanishes then every $\mathcal{R}^w$-critical Weyl structure is a (global) $\mathcal{R}$-critical metric, and that if the scalar curvature of the *Gauduchon* metric of the critical Weyl structure is non-positive, then the structure is a $\mathcal{R}$-critical metric.

In order to search for 4-dimensional $\mathcal{R}$-critical metrics or $\mathcal{R}^w$-Weyl structures, one naturally looks for compact (quotients of) homogeneous manifolds. Among the list of 4-dimensional geometric structures, the solvable cases remain most elusive after Lamontagne's work [9] where he proved that a 4-dimensional left-invariant metric on a unimodular, simply connected Lie group with non-trivial center is Einstein if it satisfies the $\mathcal{R}$-critical equation. So in Section 4, we study a class of left-invariant metrics on solvable Lie groups with *trivial* center and show that they are not $\mathcal{R}$-critical metrics and furthermore are not a Gauduchon metric of any $\mathcal{R}^w$-Weyl structure. We hope this computational argument can be extended to prove the general case in near future.

2. Preliminaries

In this section we shall review on curvatures of Weyl structures [11] and explain some properties of the $\mathcal{R}^w$ functional.

On an n-dimensional manifold M with a Weyl structure $([g], D)$, a choice of a metric g in $[g]$ induces a 1-form ω from the equation $Dg = \omega \otimes g$. Under a conformal change $g \mapsto f^2 g$, we have $\omega \mapsto \omega + 2d\ln(f)$. So $([g], D)$ may well be called closed if $d\omega = 0$ and exact if ω is exact. If $([g], D)$ is exact, then D is the Levi-Civita connection of some metric in $[g]$ and if closed, then D is locally a Levi-Civita connection of a metric. We will simply say that a structure $([g], D)$ is a metric or locally conformally a metric when it is exact or closed, respectively.

Any one-form ω together with the Levi-Civita connection ∇^g of a metric g determines a torsion-free connection D by $D_X Y = \nabla^g{}_X Y - \frac{1}{2}(\omega(X)Y + \omega(Y)X - g(X,Y)\omega^{\#})$, which preserves $[g]$, where $\omega^{\#}$ is the dual vector field to ω with respect to g.

A Weyl structure $([g], D)$ on a compact manifold has a unique, up to homothety, metric g in the conformal class such that its associated 1-form is co-closed [4]; i.e., $\delta_g \omega = 0$, where δ_g is the formal adjoint operator of d. We call this metric the *Gauduchon* metric of $([g], D)$.

From above discussion we may identify a Weyl structure $([g], D)$ with a pair (g, ω) of a metric g of unit volume and its co-closed 1-form ω. We shall use both expressions in this paper as convenient.

One can define the curvature tensors of a Weyl structure $([g], D)$ similarly to a Riemannian metric. The curvature R^D can be defined: $R^D_{X,Y}Z = D_{[X,Y]}Z - [D_X, D_Y]Z$, for $X, Y, Z \in TM$. And the Ricci curvature r^D is defined as $r^D(X,Y) = -g(R^D_{e_i,X}Y, e_i)$ for a metric $g \in [g]$ and g-orthonormal frame e_i, $i = 1, 2, \cdots, n$. This definition is well defined independent of the choice of a metric in $[g]$ but r^D is not necessarily symmetric. The conformal scalar curvature s^D is defined as the trace of r^D with respect to $[g]$. So s^D is so-called *of conformal weight* -2, which means that if one denotes the trace of r^D with respect to g by s^D_g, then $s^D_{f^2 g} = f^{-2} s^D_g$ holds.

From now on we will consider only 4-dimensional oriented compact smooth manifolds. We denote the symmetric part of a 2-tensor ϕ by $S(\phi)$ and the trace-free part of a symmetric 2-tensor ψ by $S_0(\psi)$. Let r, s, $z := r - \frac{1}{4}sg$, W and dv denote respectively the Ricci, scalar, trace-free Ricci, Weyl curvature tensor and volume form of a metric g.

Recall that there is a decomposition of Riemannian curvature tensor [2];

$$R = \frac{s}{24} g \odot g + \frac{1}{2} z \odot g + W, \tag{2.1}$$

where for 2-tensors α and g, $\alpha \odot g$ is the 4-tensor defined by

$$\begin{aligned} \alpha \odot g(x,y,z,t) =& \alpha(x,z)g(y,t) + \alpha(y,t)g(x,z) - \alpha(x,t)g(y,z) \\ & - \alpha(y,z)g(x,t). \end{aligned}$$

For a given metric g in $[g]$, its associated 1-form ω and its Levi-Civita connection ∇, the curvature tensor R^D of $([g], D)$ can be similarly decomposed as follows [11];

$$(2.2)\quad R^D = W + \frac{1}{2}S_0(r^D) \odot g + \frac{1}{24}s_g^D g \odot g + (\frac{1}{4}d\omega \odot g + \frac{1}{2}d\omega \otimes g).$$

We shall need that for a Gauduchon metric g, the following holds;

$$(2.3)\quad \begin{aligned} S(r^D) &= r_g + \frac{1}{2}(\omega \otimes \omega - |\omega|^2 g + 2\nabla\omega - d\omega), \\ s_g^D &= s_g - \frac{3}{2}|\omega|^2. \end{aligned}$$

Recall the following functional, which played an important role in Riemannian geometry and geometric analysis [2], defined on the space of all smooth Riemannian metrics on M;

$$\mathcal{R}(g) = \int_M |R_g|^2 dv.$$

This squared L^2 norm functional of the full curvature tensor can still be defined on the space of Weyl structures;

$$\mathcal{R}^w([g], D) = \int_M |R^D|^2 dv,$$

where we consider R^D as a $(3,1)$ tensor.

These two functionals has much to do with 4-dimensional topology via the generalized Gauss-Bonnet and the signature formulas which express the Euler characteristic χ and signature σ respectively as follows;

$$(2.4)\quad \begin{aligned} \chi &= \frac{1}{8\pi^2}\int_M (\frac{1}{24}s^2 - \frac{1}{2}|z|^2 + |W^+|^2 + |W^-|^2) dv, \\ \sigma &= \frac{1}{12\pi^2}\int_M (|W^+|^2 - |W^-|^2) dv, \end{aligned}$$

where W^+ and W^- are the positive and the negative, respectively, part of the Weyl curvature tensor.

A consequence of formulas (2.4) is that for $\mathcal{R}$ functional, Einstein metrics or half-conformally-flat metrics with zero-scalar-curvature are absolute minima and that similarly for the $\mathcal{R}^w$ functional [11] closed Einstein-Weyl structures or closed half-conformally-flat, zero conformal scalar curved Weyl structures are absolute minima.

Finally, recall [7] that a smooth metric [or a smooth Weyl structure] (g, ω) is a critical point of $\mathcal{F}$ functional ($\mathcal{F}$ is either $\mathcal{R}$ or $\mathcal{R}^w$) if it satisfies $(\mathcal{F}(g_t, \omega_t))'(0) = 0$ for any smooth curve of metrics [or Weyl structures] (g_t, ω_t) such that $(g_0, \omega_0) = (g, \omega)$.

3. $\mathcal{R}^w$-critical structures

In this section we set up and compute the equation for critical points of the $\mathcal{R}^w$ functional and discuss its consequences. First we recall from [7] that the space $\mathcal{W} := \{(g, \omega) | \delta_g \omega = 0\}$ of Weyl structures on M is a Banach submanifold of $\mathcal{M}_1 \times \Omega^1$ with $W^{k,p}$ Banach norm, where $\mathcal{M}_1$ is the space of Riemannian metrics on M with unit volume and Ω^1 the space of one-forms and in particular that for any pair (h, η) of a smooth symmetric 2-tensor h and a 1-form η satisfying $\delta_g\{\eta + h(\omega) + \frac{tr(h)}{2}\omega\} = 0$, i.e., $(h, \eta) \in T_{(g,\omega)}\mathcal{W}$, there exists a one-parameter family of smooth curve (g_t, ω_t) in $\mathcal{W}$ with $(g_0', \omega_0') = (h, \eta)$.

As a function defined on the Banach manifold of $W^{k,p}$ Weyl structures, $\mathcal{R}^w$ may have its derivative;

Lemma 3.1. *The functional $\mathcal{R}^w$ is differentiable and the derivative is as follows;*

$$
\begin{aligned}
&(\mathcal{R}^w)'_{g,\omega}(h, \eta) \\
&= \langle h, \ 2\delta^\nabla d^\nabla r_g - 2\frac{s_g}{3} z_g - 4\check{W} z_g + \frac{1}{2}|d\omega|^2 g - d\omega \circ d\omega \\
&\quad + \frac{1}{12}\{2\nabla d s^D + 2(\Delta s^D) g + \frac{1}{2}(s^D)^2 g - 2 s^D r_g - 3 s^D \omega \otimes \omega\} \\
&\quad - \frac{1}{12}\{2\nabla d s_g + 2(\Delta s_g) g + \frac{1}{2} s_g^2 g - 2 s_g r_g\}\rangle \\
&\quad + \langle \eta, \ 2\delta_g d\omega - \frac{1}{2} s^D \omega\rangle,
\end{aligned}
$$

where (h, η) satisfies $\delta_g\{\eta + h(\omega) + \frac{tr(h)}{2}\omega\} = 0$.

In the above $(d\omega \circ d\omega)_{ij} = (d\omega)_{ik}(d\omega)_{kj}$ for g-orthonormal frame e_i, $i = 1, \cdots, 4$ and the differential operator d^∇ acts on symmetric 2-tensors ψ in local coordinates by $(d^\nabla \psi)_{ijk} = \nabla_i \psi_{jk} - \nabla_j \psi_{ik}$. δ^∇ is the dual operator of d^{∇} and so in local coordinates $(\delta^{\nabla}\phi)_{jk} = -2\nabla^i\phi_{ijk}$. $(\breve{W}z)_{ij} = W_{ipjq}z^{pq}$. Also, Δ denotes the Laplace operator.

Proof. From (2.2), being careful that R^D is a (3,1) tensor, we compute

$$\mathcal{R}^w(g,\omega) = \int |R^D|^2 dv = \int 4|W|^2 + 2|S_0(r^D)|^2 + \frac{1}{6}(s^D)^2 + 3|d\omega|^2 dv.$$

We also have from [11]

$$\int |R^D|^2 dv = \int 4|S_0(r^D)|^2 dv + \int 2|d\omega|^2 dv + 32\pi^2\chi.$$

From these two formulas and (2.4), we deduce

$$\int |R^D|^2 dv = \int |R_g|^2 + |d\omega|^2 + \frac{1}{12}\{(s^D)^2 - (s_g)^2\} dv. \tag{3.1}$$

Denoting the Riemannian functional $\int |R_g|^2 dv$ by $\mathcal{R}(g)$, we have [2, p.134]:

$$\mathcal{R}_g'(h) = \langle 2\delta^{\nabla} d^{\nabla} r_g - 2\frac{s_g}{3} z_g - 4\breve{W} z_g, h\rangle.$$

Derivative of $\int_M (s^D)^2 dv$ is [7, Lemma 5.1]:

$$\langle -6s^D\omega, \eta\rangle + \langle 2\nabla ds^D + 2(\Delta s^D)g + \frac{1}{2}(s^D)^2 g - 2s^D r_g - 3s^D\omega\otimes\omega, h\rangle.$$

Derivative of $\int_M |d\omega|^2 dv$ is [7, Lemma 5.2]:

$$\langle \frac{1}{2}|d\omega|^2 g - d\omega\circ d\omega, h\rangle + \langle 2\delta_g d\omega, \eta\rangle.$$

Derivative of $\int_M (s_g)^2 dv$ is from [2, p.133]:

$$\langle 2\nabla ds_g + 2(\Delta s_g)g + \frac{1}{2}{s_g}^2 g - 2s_g r_g, h\rangle.$$

Putting above last 4 formulas into (3.1) we can finish the proof of Lemma 3.1. $\square$

THEOREM 3.2. *If a Weyl structure (g, ω) on a smooth oriented compact 4-manifold M is $\mathcal{R}^w$-critical, then it is closed. Moreover, it holds that $\int_M s^D|\omega|^2 dv = 0$.*

Proof. From Lemma 3.1, (g, ω) is $\mathcal{R}^w$-critical if and only if it satisfies the following equation:
(3.2)

$$\begin{aligned} 0 =&\langle h, 2\delta^\nabla d^\nabla r_g - 2\frac{s_g}{3} z_g - 4\check{W} z_g + \frac{1}{2}|d\omega|^2 g - d\omega \circ d\omega \\ &+ \frac{1}{12}\{2\nabla ds^D + 2(\Delta s^D)g + \frac{1}{2}(s^D)^2 g - 2s^D r_g - 3s^D \omega \otimes \omega\} \\ &- \frac{1}{12}\{2\nabla ds_g + 2(\Delta s_g)g + \frac{1}{2}s_g^2 g - 2s_g r_g\}\rangle \\ &+ \langle \eta, 2\delta_g d\omega - \frac{1}{2}s^D\omega\rangle \end{aligned}$$

for any smooth pair (h, η) satisfying $\delta_g(\eta + h(\omega) + \frac{tr(h)}{2}\omega) = 0$.

We analyze this by considering first the pair $(h, \eta) = (0, \omega)$. Then from above (3.2) we get

$$\int_M 4|d\omega|^2 dv = \int_M s^D|\omega|^2 dv. \tag{3.3}$$

Next, consider $(h, \eta) = (g, 0)$ to get the following from (3.2)

$$\begin{aligned} 0 &= \int_M 2\Delta s_g + 4|d\omega|^2 + \frac{1}{2}\Delta s^D + \frac{1}{6}\{(s^D)^2 - (s^D)s_g\} - \frac{1}{4}s^D|\omega|^2 \\ &\qquad - \frac{1}{2}\Delta s_g dv \\ &= \int_M \Delta(\frac{3s_g}{2} + \frac{s^D}{2}) - \frac{s^D}{2}|\omega|^2 + 4|d\omega|^2 dv \\ &= \int_M -\frac{s^D}{2}|\omega|^2 + 4|d\omega|^2 dv, \end{aligned}$$

where we used (2.3) in the second equality. So using (3.3) together we get

$$\int_M \frac{s^D}{2}|\omega|^2 dv = \int_M 4|d\omega|^2 dv = \int_M s^D|\omega|^2 dv.$$

It follows that $d\omega = 0$ and $\int_M s^D|\omega|^2 dv = 0$. This finishes the proof of Theorem 3.2. □

From Theorem 3.2 a $\mathcal{R}^w$-critical Weyl structure always has its one form ω to be harmonic, so the study of $\mathcal{R}^w$-critical Weyl structure has to do with the first deRham cohomology of the manifold. In particular, when there is no harmonic one form on a manifold, we have

COROLLARY 3.3. *If the first deRham cohomology $H^1_{dR}(M)$ of M vanishes, then every $\mathcal{R}^w$-critical Weyl structure is a $\mathcal{R}$-critical Riemannian metric.*

REMARK 3.1. Many examples of 4-dimensional non-exact Einstein-Weyl structures on compact manifolds were constructed in [12, 13]. Theorem 3.2 implies that none of these are $\mathcal{R}^w$-critical except only one type of examples: see below Example 3.1.

EXAMPLE 3.1. On $S^1 \times S^3$, let g_t be the product of the metric $t^2 d\theta^2$on $S^1 = \{\exp(i\theta) | \theta \in [0, 2\pi)\}$ and the canonical metric g_{can} on S^3 with sectional curvature one. Let ω_t be the 1-form $2td\theta$, which is harmonic with respect to g_t. For each t, (g_t, ω_t) is an Einstein-Weyl structure with $s^{D_t} = 0$. Any closed non-exact 4-dimensional Einstein-Weyl structures is locally equivalent to this structure [6].

Now we may point out another interest as follows;

PROPOSITION 3.4. *A $\mathcal{R}^w$-critical Weyl structure (g, ω) is a $\mathcal{R}$-critical Riemannian metric if one of the followings holds:*

(1) *the scalar curvature s_g is non-positive.*
(2) *the conformal scalar curvature s^D is either non-positive or non-negative, but nonzero somewhere.*

Proof. We only need to show that ω vanishes on M. From Theorem 3.2 and (2.3), we have that $\int_M s^D |\omega|^2 dv = \int_M s_g |\omega|^2 - \frac{3}{2}|\omega|^4 = 0$. So if s_g is non-positive, then $\omega = 0$.

In the case s^D is either non-positive or non-negative, if it is nonzero somewhere, then it is nonzero on an open set. The form ω has to vanish on that open set. As ω is harmonic, it is zero on M, see Remark 3.2 below. □

REMARK 3.2. It is well known that harmonic forms satisfy *weak* unique continuation property as above; or one may refer to [1].

REMARK 3.3. If g is critical for $\mathcal{R}$, then the scalar curvature is constant [2]. We ask if a $\mathcal{R}^w$-critical Weyl structure always has s^D and s_g constant.

Now consider Weyl structures (g, ω) such that the Gauduchon metrics g have non-positive sectional curvature. Lamontagne has proved [8] that any metric of non-positive sectional curvature is Einstein if it is critical with respect to $\mathcal{R}$ functional. Therefore from Proposition 3.4, we conclude;

COROLLARY 3.5. *Any $\mathcal{R}^w$ critical Weyl structure (g, ω) with non positively curved metric g is an Einstein metric.*

4. Critical Weyl structures on a class of solv-manifolds

It is currently very hard to classify $\mathcal{R}$-critical metrics. Indeed there are no examples found yet except the two classes of metrics mentioned in the introduction. Similarly as in the study of Einstein metrics, one may look for (compact quotients of) homogeneous manifolds, to find any. We are here interested in solv-manifolds because most of these are yet to be analyzed after Lamontagne's work.

In this section we consider solv-manifolds which are compact quotients of 4-dimensional simply connected solvable Lie groups $\mathrm{Sol}^4_{m,n}$ [14]. The Lie algebra has a basis $\{e_i\}$ with

$$[e_1, e_2] = ae_2, \qquad [e_1, e_3] = be_3, \qquad [e_1, e_4] = ce_4, \tag{4.1}$$

and $[e_i, e_j] = 0$ for any other pair $i, j = 1, 2, 3, 4$. Here, $a > b > c$ are real numbers, $a + b + c = 0$, and e^a, e^b, e^c are the roots of $x^3 - mx^2 + nx - 1 = 0$ with m and n distinct positive integers. Then the Lie algebra is unimodular and has *trivial* center. The compact quotient manifolds are T^3-bundles over S^1 whose monodromy is a linear map with characteristic polynomial $x^3 - mx^2 + nx - 1 = 0$. For an example, we may choose the 3×3 matrix of determinant 1:

$$\begin{pmatrix} 2 & 1 & 0 \\ -1 & 6 & 1 \\ 0 & 6 & 1 \end{pmatrix}.$$

From the roots of its characteristic polynomial, we compute using Mathematica software program;

$$a \approx 1.92, \qquad b \approx 0.75, \qquad c \approx -2.67.$$

We consider the left-invariant metrics with orthonormal basis $\{e_i\}$ as in (4.1), and compute their sectional curvatures. Note the sign convention of curvature tensor; $R(x,y)z = \nabla_{[x,y]}z - \nabla_x\nabla_y z + \nabla_y\nabla_x z$,

$$R_{1212} = -a^2, \qquad R_{1313} = -b^2, \qquad R_{1414} = -c^2,$$
$$R_{2323} = -ab, \qquad R_{2424} = -ac, \qquad R_{3434} = -bc,$$
$$R_{ijkl} = 0 \text{ for any other } i,j,k,l = 1,2,3,4.$$

The Ricci curvature tensor is as follows.

$$r_{11} = -a^2 - b^2 - c^2, \qquad r_{22} = r_{33} = r_{44} = 0,$$
$$r_{ij} = 0 \text{ for any other } i,j = 1,2,3,4.$$

The trace-free Ricci curvature tensor is as follows.

$$z_{11} = -\tfrac{3}{4}(a^2 + b^2 + c^2),$$
$$z_{22} = z_{33} = z_{44} = \tfrac{1}{4}(a^2 + b^2 + c^2),$$
$$z_{ij} = 0 \text{ for any other } i,j = 1,2,3,4.$$

The scalar curvature is $s = -a^2 - b^2 - c^2$.

Now we check the criticality of metrics and Weyl structures. From the $\mathcal{R}$-critical metric equation

$$2\delta^{\nabla} d^{\nabla} r_g - 2\frac{s_g}{3} z_g - 4\breve{W} z_g = 0,$$

we get

$$2\int_M |d^{\nabla} r_g|^2 dv = \int_M (2\frac{s_g}{3} z_g + 4\breve{W} z_g, z_g) dv. \tag{4.2}$$

The integrand on the right hand side of (4.2) is $(2\frac{s_g}{3} z_g + 4\breve{W} z_g, z_g) = 2\sum_{i,j}^4 (\lambda_i + \lambda_j)^2 \sigma_{ij}$, where $z = \sum_i^4 \lambda_i e_i \otimes e_i$ and $\sigma_{ij} = R_{ijij}$; the sectional curvature [8]. The integrand on the right hand side of (4.2) is now computed:

$$\begin{aligned} 2\sum_{i,j}^4 (\lambda_i + \lambda_j)^2 \sigma_{ij} &= 2\{\frac{(a^2 + b^2 + c^2)}{2}\}^2 \sum_{i,j}^4 \sigma_{ij} \\ &= \frac{1}{2}(a^2 + b^2 + c^2)^2 s_g = -\frac{1}{2}(a^2 + b^2 + c^2)^3. \end{aligned}$$

But the left hand side of (4.2) is non-negative, which is a contradiction. So this homogeneous metric is not $\mathcal{R}$ critical. As the scalar curvature is negative, we may apply Proposition 3.4 to conclude that there does not exist even a critical $\mathcal{R}^w$ Weyl structure associated to g. Now we summarize the above discussion in the following

THEOREM 4.1. *The left-invariant metrics on* $\mathrm{Sol}^4_{m,n}$ *with orthonormal basis as in* (4.1) *are not* $\mathcal{R}$*-critical and furthermore are not associated to* $\mathcal{R}^w$*-critical Weyl structures.*

REMARK 4.1. We ask if all 4-dimensional left-invariant metrics on compact (quotients of) homogeneous manifolds are not $\mathcal{R}$-critical unless they are Einstein or half-conformally-flat zero-scalar-curved.

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