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REMARKS ON k-FOLD INTEGER-VALUED POLYNOMIALS

VICHIAN LAOHAKOSOL AND ANGKANA SRIPAYAP

ABSTRACT. A polynomial is k-fold integer-valued if itself and its derivatives up to order k are integer-valued. Necessary and sufficient conditions are established for
(i) polynomials with rational coefficients to be k-fold integer-valued, and for
(ii) a quotient of two k-fold integer-valued polynomials to be k-fold integer-valued.

1. Introduction

By an integer-valued polynomial we mean a polynomial with rational coefficients which takes integral values at the integers. Denote by $\text{Int}(\mathbb{Z})$ the set of integer-valued polynomials, and by $\text{Int}^{(k)}(\mathbb{Z})$ the set of polynomials which together with their derivatives up to order k are integer-valued, called *k-fold integer-valued polynomials.* It is well-known, see e.g. Pólya and Szegö [7] or Cahen and Chabert [3] or Narkiewicz [6], that the set $\left\{\binom{x}{n}; n \geq 0\right\}$ forms a basis over $\mathbb{Z}$ for $\text{Int}(\mathbb{Z})$. Let us call such basis a *binomial basis* for short. Similar precise results for higher $\text{Int}^{(k)}(\mathbb{Z})$ are known only for $\text{Int}^{(1)}(\mathbb{Z})$, see Brizolis and Straus [2]. However certain related results are known, e.g. Carlitz [4] in 1959 derived a necessary and sufficient condition for polynomials to be in $\text{Int}^{(1)}(\mathbb{Z})$ based upon the shape of their coefficients written with respect to the binomial basis. Our first aim is to extend this result to general $\text{Int}^{(k)}(\mathbb{Z})$. As these conditions are rather complicated, simpler (yet only) sufficient conditions for $\text{Int}^{(k)}(\mathbb{Z})$ are derived. As another consequence we deduce a necessary and sufficient condition for polynomials with *shifted* binomial basis to be in $\text{Int}^{(k)}(\mathbb{Z})$ given that the former ones are. In another

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direction, Lind [5] in 1971, also using the binomial basis, proved that for two polynomials in Int($\mathbb{Z}$) the values at the integers of one divide that of the other if and only if one polynomial divides the other in Int($\mathbb{Z}$). Brizolis [1], using different approach, viz. asymptotic consideration, furthered this result to algebraic number fields. Our second objective is to extend Lind's result to general $\mathrm{Int}^{(k)}(\mathbb{Z})$. Since $\mathbb{Z}$ is a subring of the ring of algebraic integers, our result carries over to any algebraic number field.

2. Coefficients

Using the binomial basis for Int($\mathbb{Z}$), Carlitz [4] in 1959 derived a necessary and sufficient condition for polynomials to be in $\mathrm{Int}^{(1)}(\mathbb{Z})$. His result reads:

CARLITZ'S THEOREM. *Let* $f(x) = \sum_{s=0}^{n} a_s \binom{x}{n-s}$, $a_s \in \mathbb{Z}$. *Then* $f(x) \in \mathrm{Int}^{(1)}(\mathbb{Z})$ *if and only if the numbers*

$$b_t = \sum_{s=0}^{t-1} (-1)^{t-s-1} \frac{a_s}{t-s} \quad (t = 1, \ldots, n) \quad \text{are all integral.}$$

Our first result is a generalization of this theorem to $\mathrm{Int}^{(k)}(\mathbb{Z})$.

THEOREM 1. *Let* $f(x) = \sum_{s=0}^{n} a_s \binom{x}{n-s}$, *and* k *a nonnegative integer. Then* $f(x) \in \mathrm{Int}^{(k)}(\mathbb{Z})$ *if and only if all the numbers*

(0) $a_s \qquad (s = 0, 1, \ldots, n)$

(1) $$b_t^{(1)} = \sum_{s=0}^{t-1} (-1)^{t-s-1} \frac{a_s}{t-s} \qquad (t = 1, 2, \ldots, n)$$

(2) $$b_t^{(2)} = \sum_{s=0}^{t-2} (-1)^{t-s-2} \frac{2a_s H_{t-s-1}}{t-s} \qquad (t = 2, 3, \ldots, n)$$

(3) $$b_t^{(3)} = \sum_{s=0}^{t-3} (-1)^{t-s-3} 2a_s \sum_{j=s+2}^{t-1} \frac{H_{j-s-1}}{(t-j)(j-s)} \qquad (t = 3, 4, \ldots, n)$$

$$(k) \quad b_t^{(k)} = \sum_{s=0}^{t-k} (-1)^{t-s-k} 2a_s \sum_{j_{k-2}=s+k-1}^{t-1} \sum_{j_{k-3}=s+k-2}^{j_{k-2}-1} \cdots \times \sum_{j_1=s+2}^{j_2-1} \frac{H_{j_1-s-1}}{(t-j_{k-2})(j_{k-2}-j_{k-3})\cdots(j_1-s)}$$

$$(t = k, k+1, \ldots, n)$$

are all integral, where $H_r = 1 + \frac{1}{2} + \ldots + \frac{1}{r}$.

Proof. The case $k = 0$ is trivial and Carlitz's theorem above gives the result for the case $k = 1$. Consider the case $k = 2$. Writing

$$f^{(j)}(x) = \sum_{t=j}^{n} b_t^{(j)} \binom{x}{n-t} \quad (j = 1, 2),$$

then, using the identity in Carlitz's Theorem, for $t = 2, 3, \ldots, n$, we have

$$\begin{aligned} b_t^{(2)} &= \sum_{j=1}^{t-1} \frac{(-1)^{t-j-1}}{t-j} b_j^{(1)} \\ &= \sum_{j=1}^{t-1} \frac{(-1)^{t-j-1}}{t-j} \sum_{s=0}^{j-1} \frac{(-1)^{j-s-1}}{j-s} a_s \\ &= \sum_{s=0}^{t-2} \frac{(-1)^{t-s-2}}{t-s} a_s \sum_{j=s+1}^{t-1} \left(\frac{1}{t-j} + \frac{1}{j-s} \right) \\ &= \sum_{s=0}^{t-2} (-1)^{t-s-2} \frac{2a_s H_{t-s-1}}{t-s}, \end{aligned}$$

which yields the result for this case. Assume that the result holds for $0, 1, \ldots, k-1$ and writing

$$f^{(j)}(x) = \sum_{t=j}^{n} b_t^{(j)} \binom{x}{n-t} \quad (j = 1, 2, \ldots, k),$$

then, using Carlitz's Theorem we have

$$b_t^{(k)} = \sum_{r=k-1}^{t-1} \frac{(-1)^{t-r-1}}{t-r} b_r^{(k-1)} \quad (t = k, k+1, \ldots, n).$$

We only need to determine $b_t^{(k)}$. For $t = k, k+1, \ldots, n$ by the induction hypothesis, we have

$$\begin{aligned} b_t^{(k)} = & \sum_{r=k-1}^{t-1} \frac{(-1)^{t-r-1}}{t-r} \\ & \times \Big(\sum_{s=0}^{r-k+1} (-1)^{r-s-k+1} 2a_s \sum_{j_{k-3}=s+k-2}^{r-1} \sum_{j_{k-4}=s+k-3}^{j_{k-3}-1} \cdots \\ & \quad \times \sum_{j_1=s+2}^{j_2-1} \frac{H_{j_1-s-1}}{(r-j_{k-3})(j_{k-3}-j_{k-4})\cdots(j_1-s)} \Big) \\ = & \sum_{s=0}^{t-k} (-1)^{t-s-k} 2a_s \\ & \times \Big(\sum_{r=s+k-1}^{t-1} \sum_{j_{k-3}=s+k-2}^{r-1} \cdots \\ & \quad \times \sum_{j_1=s+2}^{j_2-1} \frac{H_{j_1-s-1}}{(t-r)(r-j_{k-3})(j_{k-3}-j_{k-4})\cdots(j_1-s)} \Big), \end{aligned}$$

and the result follows. □

A simpler, yet only sufficient, condition for $\mathrm{Int}^{(k)}\mathbb{Z}$ can easily be deduced. This can be regarded as a partial analogue to the result in the discrete setting due to Carlitz [4], the lemma on p. 298, which says briefly as follows: Let $M_r(x)$ $(r \geq 1)$ be the set of all integer-valued polynomials $f(x)$ such that their differences $\Delta^k f(x)$ are integer-valued for $1 \leq k \leq r$. Put $L_n^{(r)} = \mathrm{lcm}(s_1 s_2 \ldots s_r)$, where the least common multiple is taken over all products of integers s_j subject to $s_1 + \ldots + s_r \leq n,\ s_j \geq 1$. Carlitz's result states that for $a \in \mathbb{Z}$, we have $a \binom{x}{n} \in M_r(x)$ if and only if $L_n^{(r)} | a$.

THEOREM 2. *Let* $f(x) = \sum_{s=0}^{n} a_s \binom{x}{n-s}$, *$k$ a nonnegative integer,*

$$f^{(k)}(x) = \sum_{t=k}^{n} b_t^{(k)} \binom{x}{n-t}$$

where

$$b_t^{(k)} = \sum_{s=0}^{t-k}(-1)^{t-s-k}2a_s \sum_{j_{k-2}=s+k-1}^{t-1} \sum_{j_{k-3}=s+k-2}^{j_{k-2}-1} \cdots \times \sum_{j_1=s+2}^{j_2-1} \frac{H_{j_1-s-1}}{(t-j_{k-2})(j_{k-2}-j_{k-3})\cdots(j_1-s)}$$

and $L_t = \operatorname{lcm}(1,2,\ldots,t)$. *If*

(0) $a_0 \in \mathbb{Z}$,

$t \mid a_0 \qquad (t = 1,2,\ldots,n)$,

$tL_{t-1} \mid 2a_0 \qquad (t = 2,3,\ldots,n)$,

$(t-2)2L_1, (t-3)3L_2, \ldots, 1(t-1)L_{t-2} \mid 2a_0 \qquad (t = 3,4,\ldots,n)$,

$\vdots$

$(t-j_{k-2})(j_{k-2}-j_{k-3})\ldots(j_2-j_1)j_1L_{j_1-1} \mid 2a_0 \quad (t = k, k+1, \ldots, n)$,

$(k-1 \leq j_{k-2} \leq t-1, k-2 \leq j_{k-3} \leq j_{k-2}-1, \ldots, 2 \leq j_1 \leq j_2-1)$.

(1) $a_1 \in \mathbb{Z}$,

$t-1 \mid a_1 \qquad (t = 2,3,\ldots,n)$,

$(t-1)L_{t-2} \mid 2a_1 \qquad (t = 3,4,\ldots,n)$,

$(t-3)2L_1, (t-4)3L_2, \ldots, 1(t-2)L_{t-3} \mid 2a_1 \qquad (t = 4,5,\ldots,n)$,

$\vdots$

$(t-j_{k-2})(j_{k-2}-j_{k-3})\ldots(j_2-j_1)(j_1-1)L_{j_1-2} \mid 2a_1$

$(t = k+1, k+2, \ldots, n)$,

$(k \leq j_{k-2} \leq t-1, k-1 \leq j_{k-3} \leq j_{k-2}-1, \ldots, 3 \leq j_1 \leq j_2-1)$.

$\vdots$

(n) $a_n \in \mathbb{Z}$, *then* $f(x) \in \mathit{Int}^{(k)}(\mathbb{Z})$.

Proof. The result is trivial for $k = 0$. By Carlitz's Theorem, it is true for $k = 1$. Assume that the result holds for $0, 1, \ldots, k-1$. Since

$$f^{(k)}(x) = \sum_{t=k}^{n} b_t^{(k)} \binom{x}{n-t}$$

where

$$b_t^{(k)} = \sum_{s=0}^{t-k}(-1)^{t-s-k}2a_s \sum_{j_{k-2}=s+k-1}^{t-1} \sum_{j_{k-3}=s+k-2}^{j_{k-2}-1} \cdots$$
$$\times \sum_{j_1=s+2}^{j_2-1} \frac{H_{j_1-s-1}}{(t-j_{k-2})(j_{k-2}-j_{k-3})\cdots(j_1-s)}$$

and the denominator of H_j is L_j, then $b_t^{(k)} \in \mathbb{Z}$ if the following divisibility conditions are satisfied

- $(t-j_{k-2})(j_{k-2}-j_{k-3})\dots(j_2-j_1)j_1L_{j_1-1} \mid 2a_0$
  $(k-1 \le j_{k-2} \le t-1, k-2 \le j_{k-3} \le j_{k-2}-1, ..., 2 \le j_1 \le j_2-1)$,
- $(t-j_{k-2})(j_{k-2}-j_{k-3})\dots(j_2-j_1)(j_1-1)L_{j_1-2} \mid 2a_1$
  $(k \le j_{k-2} \le t-1, k-1 \le j_{k-3} \le j_{k-2}-1, \dots, 3 \le j_1 \le j_2-1)$,

$\vdots$

- $(t-j_{k-2})(j_{k-2}-j_{k-3})\dots(j_2-j_1)(j_1-t+k)L_{j_1-t+k-1} \mid 2a_{t-k}$
  $(t-1 \le j_{k-2} \le t-1,\ t-2 \le j_{k-3} \le j_{k-2}-1,$
  $\dots,\ t-k+2 \le j_1 \le j_2-1)$.

This set of conditions is equivalent to (0), (1), …, (n), and the proof is complete. □

The following example, for the case $k=1$, confirms that the conditions of Theorem 2 are merely sufficient. Let

$$f(x) = 30\binom{x}{5} + 6\binom{x}{4} + 3\binom{x}{3} + \binom{x}{2}.$$

Then

$$f'(x) = b_1^{(1)}\binom{x}{4} + b_2^{(1)}\binom{x}{3} + b_3^{(1)}\binom{x}{2} + b_4^{(1)}\binom{x}{1} + b_5^{(1)}\binom{x}{0},$$

where, by Carlitz's theorem, $b_1^{(1)} = 30, b_2^{(1)} = -9, b_3^{(1)} = 10, b_4^{(1)} = -6, b_5^{(1)} = 5$, and so $f(x) \in \mathrm{Int}^{(1)}(\mathbb{Z})$. Yet $a_0 = 30$ is not divisible by all t in {1,2,3,4,5} implying that $f(x)$ does not satisfy the condition(1) of Theorem 2.

As another application of Theorem 1, we deduce a necessary and sufficient condition for polynomials with *shifted* binomials to be in $\mathrm{Int}^{(k)}(\mathbb{Z})$, given that the former ones are. Results of this kind were used repeatedly in the work of Brizolis and Straus [2] in their construction of an explicit binomial basis for $\mathrm{Int}^{(1)}(\mathbb{Z})$.

COROLLARY *Let* $f(x) = \sum_{s=0}^{n} a_s \binom{x}{n-s} \in \mathrm{Int}^{(k)}(\mathbb{Z})$, *$k$ a nonnegative integer, and let* $g(x) = \sum_{s=0}^{n+1} a_s \binom{x}{n+1-s}$. *Then* $g(x) \in \mathrm{Int}^{(k)}(\mathbb{Z})$ *if and only if all the numbers*

$$(0) \quad g(0) = a_{n+1},$$

$$(1) \quad g'(0) = \sum_{s=0}^{n} (-1)^{n-s} \frac{a_s}{n+1-s},$$

$$(2) \quad g''(0) = \sum_{s=0}^{n-1} (-1)^{n-s-1} \frac{2a_s H_{n-s}}{n+1-s},$$

$$(3) \quad g'''(0) = \sum_{s=0}^{n-2} (-1)^{n-s-2} 2a_s \sum_{j=s+2}^{n} \frac{H_{j-s-1}}{(n+1-j)(j-s)},$$

$$\vdots$$

$$(k) \quad g^{(k)}(0) = \sum_{s=0}^{n+1-k} (-1)^{n+1-s-k} 2a_s \sum_{j_{k-2}=s+k-1}^{n} \sum_{j_{k-3}=s+k-2}^{j_{k-2}-1} \cdots \times \sum_{j_1=s+2}^{j_2-1} \frac{H_{j_1-s-1}}{(n+1-j_{k-2})(j_{k-2}-j_{k-3})\cdots(j_1-s)}$$

are integral, where $H_r = 1 + \frac{1}{2} + \ldots + \frac{1}{r}$.

Proof. The case $k = 0$ is trivial. Consider the case $k = 1$. Writing

$$f'(x) = b_1 \binom{x}{n-1} + b_2 \binom{x}{n-2} + \ldots + b_{n-1} \binom{x}{1} + b_n,$$

$$g'(x) = B_1 \binom{x}{n} + B_2 \binom{x}{n-1} + \ldots + B_n \binom{x}{1} + B_{n+1},$$

by the identities found in Theorem 1, we get, for $t = 1, 2, \ldots, n$ and $T = 1, 2, \ldots, n+1$ that

$$b_t = \sum_{s=0}^{t-1} (-1)^{t-s-1} \frac{a_s}{t-s}$$

and

$$B_T = \sum_{s=0}^{T-1} (-1)^{T-s-1} \frac{a_s}{T-s}.$$

Note that $b_t = B_t$ for $t = 1, 2, \ldots, n$, i.e. the coefficients of $f'(x)$ are the same as those of $g'(x)$ except $B_{n+1} = g'(0)$. If $g(x) \in \mathrm{Int}^{(1)}(\mathbb{Z})$, then $g'(0) \in \mathbb{Z}$. Conversely, assume that $g'(0) = B_{n+1} \in \mathbb{Z}$. Since $f'(x) \in \mathrm{Int}(\mathbb{Z})$, then $B_1 = b_1, \ldots, B_n = b_n$ are all integral. Observe that $g'(x)$ is a sum of terms of the form $B_i \binom{x}{n+1-i}$, with $B_i \in \mathbb{Z}$, then $g'(x) \in \mathrm{Int}(\mathbb{Z})$. Hence $g(x) \in \mathrm{Int}^{(1)}(\mathbb{Z})$, proving the result for $k = 1$. The proof of the general case follows readily by induction. □

Let us mention in passing that all results in this section are still valid if the ring of rational integers $\mathbb{Z}$ is replaced by the ring of p-adic integers.

3. Quotients

In 1971, Lind [4], see also Brizolis [1], proved the following interesting result to the effect that for two polynomials in $\mathrm{Int}(\mathbb{Z})$ the values at the integers of one divide that of the other if and only if one polynomial divides the other in $\mathrm{Int}(\mathbb{Z})$.

LIND'S THEOREM. *Suppose that $f(x), g(x) \in Int(\mathbb{Z})$. Then $f(n) \mid g(n)$ for all $n \in \mathrm{N}$ if and only if $\frac{g(x)}{f(x)} \in Int\ (\mathbb{Z})$.*

We now generalize this result.

THEOREM 3. *Suppose that $f(x), g(x) \in Int^{(k)}(\mathbb{Z})$, k a nonnegative integer. Then $\frac{g(x)}{f(x)} \in Int^{(k)}(\mathbb{Z})$ if and only if for all $n \in \mathrm{N}$, we have*

(0) $f(n) \mid g(n)$,

(1) $f(n) \mid \left(g'(n) - \frac{g(n)}{f(n)} f'(n) \right)$,

$\vdots$

(k) $f(n) \mid \left(g^{(k)}(n) - \sum_{s=0}^{k-1} \binom{k}{s} \left(\frac{g}{f}\right)^{(s)}(n) f^{(k-s)}(n) \right)$.

Proof. Let $H(x) = \frac{g(x)}{f(x)}$. We proceed by induction on k. By Lind's theorem, the result is true for $k = 0$. Suppose the result holds for 0, 1, $\ldots$, $k-1$. Assume that $H(x) \in \mathrm{Int}^{(k)}(\mathbb{Z})$. By induction hypothesis, we have $H^{(s)}(x)$ and $f^{(k-s)}(x)$ both belong to $\mathrm{Int}(\mathbb{Z})$ for $0 \leq s \leq k-1$. Thus

$$g^{(k)}(x) - \sum_{s=0}^{k-1} \binom{k}{s} H^{(s)}(x) f^{(k-s)}(x) \in \mathrm{Int}(\mathbb{Z}).$$

Since $H^{(k)}(n) = \dfrac{g^{(k)}(n) - \sum_{s=0}^{k-1} \binom{k}{s} H^{(s)}(n) f^{(k-s)}(n)}{f(n)}$ and $f(n)$ are both integral, then

$$f(n) \Big| \left[g^{(k)}(n) - \sum_{s=0}^{k-1} \binom{k}{s} H^{(s)}(n) f^{(k-s)}(n) \right].$$

Conversely, let the conditions $(0) - (k)$ be fulfilled and assume the result holds for 0, 1, $\ldots$,$k-1$. Thus $H(x) \in \mathrm{Int}^{(k-1)}(\mathbb{Z})$, and so $H^{(k-1)}(x) \in \mathrm{Int}(\mathbb{Z})$, yielding

$$H^{(k)}(x) f(x) = g^{(k)}(x) - \sum_{s=0}^{k-1} \binom{k}{s} H^{(s)}(x) f^{(k-s)}(x) \in \mathrm{Int}(\mathbb{Z}).$$

By (k) and Lind's Theorem, we deduce that $H^{(k)}(x) \in \mathrm{Int}(\mathbb{Z})$, as to be proved. □

The following example, for the case $k = 1$, shows that the converse of Theorem 3 is not generally true if one of the conditions, the condition (1) in this case, does not hold. Take $g(x) = 60 \binom{x}{6}$, $f(x) = 2 \binom{x}{2}$. Then

$$\frac{g(x)}{f(x)} = \frac{(x-2)(x-3)(x-4)(x-5)}{4 \cdot 3},$$

and so $\frac{g(n)}{f(n)} \in \mathbb{Z}$ for all $n \in \mathrm{N}$. But at $x = 2, g'(2) - \frac{g(2)}{f(2)} f'(2) = -1$ which is not divisible by $f(2) = 2$. Next, we show that $\frac{g(x)}{f(x)} \notin \mathrm{Int}^{(1)}(\mathbb{Z})$.

Suppose otherwise, then $\dfrac{g(x)}{f(x)}$ is polynomial of degree 4 in $\mathrm{Int}^{(1)}(\mathbb{Z})$. Using the binomial basis of $\mathrm{Int}^{(1)}(\mathbb{Z})$ in Brizolis and Straus[2], we can write

$$\frac{g(x)}{f(x)} = A\left[6\binom{x}{4} + \binom{x}{2}\right] + B \cdot 6\binom{x}{3} + C\binom{x}{2} + Dx + E,$$

where A, B, C, D and E are integral. Comparing the leading coefficients, we get $A = \frac{1}{3}$ which is a contradiction.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KASETSART UNIVERSITY, BANGKOK 10900, THAILAND
E-mail: fscivil@nontri.ku.ac.th
fscianr@nontri.ku.ac.th