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MODULI OF SELF-DUAL METRICS ON COMPLEX HYPERBOLIC MANIFOLDS

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ABSTRACT. On compact complex hyperbolic manifolds of complex dimension two, we show that the dimension of the space of infinitesimal deformations of self-dual conformal structures is smaller than that of the deformation obstruction space and that every self-dual metric with covariantly constant Ricci tensor must be a standard one upto rescalings and diffeomorphisms.

1. Introduction

Let (M, g) be an oriented Riemannian four-manifold. The bundle of 2-forms over M splits as the Whitney sum $\bigwedge^2 M = \bigwedge^+ + \bigwedge^-$, $\bigwedge^\pm$ being the eigenspace bundle of the Hodge star operator $* \in$ End $\bigwedge^2 M$. The Weyl tensor $W \in$ End $\bigwedge^2 M$ leaves $\bigwedge^\pm$ invariant, and the restriction $W^\pm$ of W to $\bigwedge^\pm$ may be viewed as a $(0,4)$ tensor, operating trivially on $\bigwedge^\mp$ [3]. We say (M, g) is self-dual or anti-self-dual if $W^- = 0$ or $W^+ = 0$. This is a property of the underlying conformal structure. As explained in [1], the self-dual equation, $W^- = 0$, is the integrability condition of a natural complex structure on the unit sphere bundle in $\bigwedge^-$. This gives rise to the Penrose correspondence between self-dual conformal structures on four-manifolds and certain complex three-manifolds called twistor spaces. Not every manifold allows a self-dual metric. For instance $S^2 \times S^2$ does not allow any self-dual metric, since this manifold has signature zero, which would force any putative self-dual metric to be conformally flat-whereas the only simply connected conformally flat

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manifold is S^4 [8]. A conformally flat four-manifold is both self-dual and anti-self-dual. Examples of conformally flat four-manifolds which are well known are manifolds of constant curvature and a product four-manifold $\Sigma_k \times CP^1$ with metrics of opposite constant curvatures. Here Σ_k denotes a genus $k(\geq 2)$ compact Riemann surface. From now on the moduli of self-dual metrics (resp., conformally flat metrics) on M means the set of all self-dual metrics (resp., conformally flat metrics) on M modulo the action of the conformal transformation group of M. For S^4 the moduli of conformally flat metrics consists of a single point, the standard conformally flat structure [7]. In fact each conformally flat structure has by making use of the developing map a holonomy correspondence $\pi_1(M) \longrightarrow \mathrm{SO}(5,1)$, the conformal group of S^4 with the standard metric, so that the moduli of conformally flat structures is mapped into the representation space $R(\pi_1(M);\ \mathrm{SO}(5,1))$, the space of conjugacy classes of representations $\pi_1(M) \longrightarrow \mathrm{SO}(5,1)$. For a compact real hyperbolic four-manifold, $(M_R,\ h_R)$, Johnson and Milson have shown that the moduli of conformally flat structures can be arbitrarily large [6]. Although the metric does not deform as a hyperbolic metric by Mostow rigidity, the conformally flat structure can be deformed by bending along totally geodesic hypersurfaces.

The local structure of moduli space of self-dual metrics is controlled by an elliptic deformation complex. The zeros of the Kuranishi map then give charts for the moduli space. If there are no conformal isometries and no obstructions to deformation, the moduli space is a smooth manifold of dimension $\frac{1}{2}(15\chi - 29\tau)$, where χ is the Euler number and τ is the signature [7]. In Section 2 we show that on a compact complex hyperbolic manifold of complex dimension two, $(M_C,\ h_C)$, the dimension of the space of infinitesimal deformations of self-dual conformal structures at h_C is smaller than that of the deformation obstruction space at h_C. In Section 3 we review briefly the fundamental properties of the Weyl tensor W. In Section 4 we show that on M_C any self-dual metric with covariantly constant Ricci tensor is a standard complex hyperbolic metric, h_C, upto rescalings and diffeomorphisms.

2. An inequality between $\dim H^1_{h_C}$ and $\dim H^2_{h_C}$

Consider a compact oriented smooth four-manifold M. A smooth Riemannian metric g on M is a smooth section of the bundle S^2T^*M of symmetric 2-tensors which is positive definite everywhere. The space $\mathcal{E}$ of all Riemannian metrics on M is a convex open cone in $\Gamma(S^2T^*M)$.

Thus the tangent space $T_g\mathcal{E}$ is canonically identified with $\Gamma(S^2T^*M)$. The group $\mathcal{D}$ of orientation-preserving diffeomorphisms of M is an infinite-dimensional Lie group acting on $\mathcal{E}$ by pullback. We now take the quotient of the space of metrics by a larger group, called a conformal transformation group. As a manifold, this conformal transformation group is $\mathcal{F} = \mathcal{D} \times \mathcal{C}^\infty_+$, where the second factor is the space of positive smooth functions on M. $\mathcal{F}$ acts smoothly on $\mathcal{E}$ on the right:

$$\mathcal{F} \times \mathcal{E} \to \mathcal{E},$$
$$((\rho, f), g) \to f\rho^*(g).$$

The stabilizer C_g of g is called the conformal isometry group of g.

The curvature tensor $\mathcal{R}$ of a Riemannian metric, considered as an endomorphism of the bundle of 2-forms, has the following block decomposition with respect to the orthogonal splitting $\bigwedge^2 = \bigwedge^+ + \bigwedge^-$ induced by the Hodge *-operator [2]:

$$\mathcal{R} = \begin{array}{c} \begin{array}{cc} \bigwedge^+ & \bigwedge^- \end{array} \\ \left(\begin{array}{c|c} W^+ + \frac{s}{12}Id & \mathrm{ric}_o \\ \hline {}^t\mathrm{ric}_o & W^- + \frac{s}{12}Id \end{array}\right) \end{array} \begin{array}{c} \bigwedge^+ \\ \bigwedge^- \end{array}$$

where s is the scalar curvature, ric_o is the traceless Ricci tensor, W^+ is the self-dual Weyl tensor, and W^- is the anti-self-dual Weyl tensor. We are interested in the solution space to the self-dual equation for the Weyl tensor, $W = W^+ + W^-$:

$$W^- = 0 \Leftrightarrow *W = W. \tag{2.1}$$

A metric or conformal structure satisfying (2.1) is called self-dual. This is invariant under the action of $\mathcal{F}$, so we will consider the solution space as a subvariety of $\frac{\mathcal{E}}{\mathcal{F}}$. The Chern-Weil formula shows that $W^- = 0$ implies $\tau(M) \geq 0$, with equality if and only if $W = 0$, i.e. the metric is conformally flat. Note that a change of orientation interchanges W^+ and W^-. Consider a fixed self-dual metric g on a compact oriented four-manifold M of nonnegative signature. We want to describe all nearby solutions of the self-dual equation. Infinitesimally, they give rise to tensors in the kernel of the linearized operator $T_gW^- : T_g\mathcal{E} \to \Gamma(S^2_0 \bigwedge^-)$. This kernel certainly contains the tangent space to the family $\mathcal{F}$ of self-dual metrics, but these are the uninteresting deformations. The interesting infinitesimal deformations are represented by the first cohomology of the complex [7]:

$$\Gamma(TM) \oplus \Gamma(R) \xrightarrow{t^*} \Gamma(S^2T^*M) \xrightarrow{T_gW^-} \Gamma(S^2_0 \bigwedge^-). \tag{2.2}$$

We have written R for the trivial R-bundle over M, so that $\Gamma(R)$ is the space of functions $\Gamma(M,R)$. In this complex $\Gamma(S^2T^*M)$ stands for $T_g\mathcal{E}$. This complex has the following cohomology groups; $H^0_g = \ker t^*$, $H^1_g = \ker(t \oplus T_gW^-)$, $H^2_g = \mathrm{coker} T_gW^-$, where H^0_g, H^1_g and H^2_g are the space of conformal Killing vector fields for g, the space of infinitesimal deformations of self-dual conformal structures at g and the space of obstruction for local deformations at g, respectively. Furthermore, the complex (2.2) is elliptic, with indices equal to

$$\frac{1}{2}(15\chi(M) - 29\tau(M)) = \dim H^0_g - \dim H^1_g + \dim H^2_g, \tag{2.3}$$

where χ and τ are the Euler characteristic of M and signature of M, respectively [7]. In order to show the main result in this section, we shall need the following fact [4]:

LEMMA 2.1. *There are no non-trivial global 1-parameter groups of conformal transformations on a compact oriented Riemannian manifold M of dimension $n \geq 2$ with negative definite Ricci tensor r.*

Proof. Let X be the infinitesimal conformal transformation induced by a given 1-parameter group of conformal transformations of M and ξ the 1-form defined by X by duality. By the Ricci identity and the conformal killing vector field condition, we obtain

$$\triangle\xi + (1 - \frac{2}{n})d\delta\xi - 2\,Q\xi = 0, \tag{2.4}$$

where δ, $\triangle$ and Q is the co-differential operator, the Laplace-Beltrami operator $d\delta + \delta d$ and the operator on 1-forms defined by $(Q\alpha)_i = r_i^j\alpha_j$, respectively. Taking the inner product with ξ and integrating, this tells us that

$$0 = \int_M \left(|d\xi|^2 + 2\,(1 - \frac{1}{n})|\delta\xi|^2 - 2\,Q(\xi,\xi) \right) dv. \tag{2.5}$$

Since Q is negative definite, it follows that $\xi = 0$, that is X vanishes. This completes the proof. □

Now we show the main result in this section:

THEOREM 2.2. *On a compact complex hyperbolic manifold (M_C, h_C) of complex dimension two,*

$$\dim H^1_{h_C} < \dim H^2_{h_C}$$

holds.

Proof. The given manifold has the following topological properties [9]:

$$\tau(M_C) > 0, \chi(M_C) = 3\tau(M_C). \tag{2.6}$$

On the other hand, we have $\dim H^0_{h_C} = 0$ by Lemma 2.1. Consequently, by (2.3), $\dim H^1_{h_C} < \dim H^2_{h_C}$ holds and this completes the proof. □

3. Basic facts

In this section we review briefly the fundamental properties of the Weyl tensor W. In an oriented four-dimensional Riemannian manifold (M, g), the endomorphisms W and $*$ of $\bigwedge^2(M)$ commute and consequently, W leaves the subbundles $\bigwedge^{\pm}$ invariant. The restrictions $W^{\pm}$ of W to $\bigwedge^{\pm}$ satisfy the relation [3]:

$$\mathrm{Trace} W^{\mp} = 0. \tag{3.7}$$

Furthermore, the second Bianchi identity implies the following well known divergence formula [3]: In local coordinates,

$$\nabla^p W_{pkij} = \nabla_i P_{kj} - \nabla_j P_{ki}, \tag{3.8}$$

where

$$P = \frac{1}{2}(r - \frac{1}{6}sg). \tag{3.9}$$

Suppose now that M is a compact oriented four-dimensional manifold. The following formula

$$g \longrightarrow \int_M |W|^2 dV \tag{3.10}$$

defines a conformally invariant functional in the space of all Riemannian metrics on M. It is known that the critical points of (3.10) are characterized as follows [3]: A metric g on a compact oriented four-manifold M is a critical point of (3.10) if and only if its Bach tensor B, given by the local coordinate formula

$$B_{ij} = \nabla^q \nabla^p W_{pijq} + \frac{1}{2} r^{pq} W_{pijq} \tag{3.11}$$

vanishes identically. For $x \in M$, we can choose an oriented orthogonal basis ω, η, θ with length $\sqrt{2}$ (resp., ω^-, η^-, θ^- with length $\sqrt{2}$) of $\bigwedge^+_x$

(resp., $\bigwedge_x^-$), consisting of eigenvectors of W such that we have, at x, a relation of the form

$$W = \frac{1}{2}(\lambda\omega\otimes\omega+\mu\eta\otimes\eta+\nu\theta\otimes\theta) + \frac{1}{2}(\lambda^-\omega^-\otimes\omega^-+\mu^-\eta^-\otimes\eta^-+\nu^-\theta^-\otimes\theta^-), \tag{3.12}$$

where λ, μ and ν (resp., λ^-, μ^- and ν^-) are the eigenvalues of W_x^+ (resp., W_x^-). Thus (3.7) implies

$$\lambda+\mu+\nu=0, \quad \lambda^-+\mu^-+\nu^-=0. \tag{3.13}$$

Since $\bigwedge^\pm$ are invariant under parallel displacements, in a neighbourhood of any $x\in M$ we have

$$\begin{aligned} \bigtriangledown\omega &= c\otimes\eta-b\otimes\theta, \\ \bigtriangledown\eta &= -c\otimes\omega+a\otimes\theta, \\ \bigtriangledown\theta &= b\otimes\omega-a\otimes\eta \end{aligned} \tag{3.14}$$

for some 1 forms a, b, c defined near x (Clearly, similar formulae hold for $\bigtriangledown\omega^-, \bigtriangledown\eta^-, \bigtriangledown\theta^-$). W^+ satisfies the condition $\delta W^+=0$, i.e., the vanishing of its divergence if and only if relations

$$\begin{aligned} d\lambda &= (\lambda-\mu)\theta c+(\lambda-\nu)\eta b, \\ d\mu &= (\mu-\lambda)\theta c+(\mu-\nu)\omega a, \\ d\nu &= (\nu-\lambda)\eta b+(\nu-\mu)\omega a \end{aligned} \tag{3.15}$$

hold [3]. Consequently, for any oriented Riemannian four-manifold such that $\delta W^+=0$, we have the following expression for $\triangle\lambda=\delta d\lambda$ [3]:

$$\triangle\lambda = 2\lambda^2+4\mu\nu-\frac{1}{2}s\lambda+2(\nu-\lambda)|b|^2+2(\mu-\lambda)|c|^2. \tag{3.16}$$

4. A rigidity theorem

In this section we shall prove that on a compact four-manifold that does not admit a conformally flat metric, any self-dual metric with covariantly constant Ricci tensor must be Einstein. Consequently, on M_C, we show that any self-dual metric with covariantly constant Ricci tensor is a standard hyperbolic metric, h_C, upto rescalings and diffeomorphism. Here a smooth Riemannian metric g is said to be Einstein if its Ricci tensor r is a constant multiple of the metric g. For any compact oriented Riemannian four-manifold (M,g), the signature $\tau(M)$ of M can

be expressed as follows:

$$12\pi^2\tau(M) = \int_M \left(|W^+|^2 - |W^-|^2\right) dv \leq \int_M |W|^2 dv, \tag{4.17}$$

equality holds if and only if (M, g) is self-dual [1]. Thus, every self-dual metric over M provides an absolute minimum for the functional (3.10). Consequently, the Bach tensor of any compact self-dual Riemannian four-manifold vanishes identically. The following Lemma is the crucial ingredient to prove the main result in this section.

LEMMA 4.1. *On a compact four-manifold that does not admit a conformally flat metric, any self-dual metric with covariantly constant Ricci tensor must be Einstein.*

Proof. The above conditions together with (3.8), (3.9) and (3.11) yield $r^{pq}W_{pijq} = 0$ which implies that the symmetric tensor $T = \lambda\omega r\omega + \mu\eta r\eta + \nu\theta r\theta$ vanishes identically. If $r - \frac{s}{4}g$ does not vanish identically, then it is non-zero at all points of some connected open subset U of M_C. Fix $x \in U$ and find $Y \in T_xM_C$ which is not an eigenvector of r_x. Since Y is orthogonal to ωY, ηY, and θY, rY is not orthogonal to one of them; for instance, $r(Y, \omega Y) \neq 0$. Relation $T = 0$ gives, for any tangent vector Z, $0 = g(Z, T\omega Z) = \lambda r(Z, \omega Z) + (\mu - \nu)r(\eta Z, \theta Z)$. Substituting here $Z = Y$ and $Z = \eta Y$, we obtain, respectively, $\lambda r(Y, \omega Y) + (\mu - \nu)r(\eta Y, \theta Y) = 0$ and $(\mu - \nu)r(Y, \omega Y) + \lambda r(\eta Y, \theta Y) = 0$. Since $r(Y, \omega Y) \neq 0$, this linear system must satisfy the determinant relation $0 = \lambda^2 - (\mu - \nu)^2 = (\lambda - \mu + \nu)(\lambda + \mu - \nu)$, i.e., by $\lambda + \mu + \nu = 0$, $\mu\nu = 0$. Therefore $\det W^+ = \lambda\mu\nu = 0$ everywhere in U. Suppose now that $W = W^+ \neq 0$ at all points of some connected open subset U_1 of U. Taking U_1 sufficiently small, we may assume, without loss of generality, that $\lambda = 0$ and, by $\lambda + \mu + \nu = 0$, $\nu = -\mu \neq 0$ everywhere in U_1. In U_1, (3.15) and $d\lambda = 0$ yield $|b| = |c|$, and (3.16) implies $0 = \triangle\lambda = 4\mu\nu$. This contradiction shows that $W = 0$ in U. However the covariantly constant Ricci tensor implies either that $r - \frac{s}{4}g$ vanishes identically or that $r - \frac{s}{4}g$ is non-vanishing, so non-conformally flat condition forces our given metric to be Einstein. This completes the proof. □

We shall now prove the following:

THEOREM 4.2. *On M_C, any self-dual metric, g_o, with covariantly constant Ricci tensor is a standard complex hyperbolic metric, h_C, upto rescalings and diffeomorphisms.*

Proof. According to Lemma 4.1, g_o is Einstein. On the other hand, by Seiberg-Witten theory, the Einstein metric g_o on M_C must be kähler [11]

and hence (M_C, g_o) is locally symmetric [9]. From these facts it follows that (M_C, g_o) is a compact complex hyperbolic manifold (M_C, h_C) upto diffeomorphisms and rescalings by Mostow rigidity. This completes the proof. □

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References

[1] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, *Self duality in four dimensional Riemannian geometry*, Pro. R. Soc. Lond., Ser. A. **362** (1978), 425–461.
[2] A. L. Besse, *Einstein manifolds*, Springer Verlag, 1986.
[3] A. Derdzinski, *Self-dual kähler manifolds and Einstein manifolds of dimension four*, Compositio Matematica. **49** (1983), 405–433.
[4] S. I. Goldberg, *Curvature and Homology*, Dover, 1982.
[5] M. Itoh, *Moduli of half conformally flat structures*, Math. Ann. **296** (1993), 687–708.
[6] D. Johnson and J. Millson, *Deformation spaces associated to compact hyperbolic manifolds*, Discrete groups in geometry and analysis. Birkhäser, 1987.
[7] A. D. King and D. Kotschick, *The deformation theory of anti-self-dual conformal structures*, Math. Ann. **294** (1992), 591–609.
[8] H. N. Kuiper, *On Conformally Flat spaces in the Large*, Ann. Math. **50** (1949), 916–924.
[9] C. LeBrun, *Einstein Metrics and Mostow Rigidity*, Math. Res. Lett. **2** (1995), 1–8.
[10] G. D. Mostow, *Quasi-Conformal Mappings in n-Space and the Rigidity of hyperbolic space Forms*, Publ. IHES **34** (1968), 53–104.
[11] E. Witten, *Monopoles and Four-Manifolds*, Journal of Mathematics and Mechanics. **18** (1994), 779–786.

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