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HAUSDORFF DIMENSION OF GENERALIZED MARKOV ATTRACTORS FOR ITERATED FUNCTION SYSTEMS

Jung Ju Park, Hung Hwan Lee,
Hun Ki Baek, and Hyun Jae Yoo

Abstract. We construct lots of non-self similar fractal sets called generalized Markov attractors for a given (hyperbolic) iterated function system and calculate bounds of their Hausdorff dimensions.

1. Introduction

Given a hyperbolic iterated function system, we can have a fractal set called its attractor. The fractal sets obtained in this way are necessarily self-similar [6]. Ellis and Branton have discussed the Hausdorff dimension of the attractors of disjoint hyperbolic iterated function systems [3].

One is also attempted to consider the non self-similar fractal sets and to study the fractal dimensions of those sets. This problem has been investigated by several authors ([1], [3], [9] and [10]) who studied the attractors called the Markov attractors associated with the Markov transition matrices. In [3], Ellis and Branton obtained an upper bound for the Hausdorff dimension of the Markov attractor and have left with a conjecture for a lower bound. This conjecture has been fully proved to be true by Yin in [10].

In this paper, we extend the concept of Markov attractors and thereby we obtain lots of non self-similar attractors for a given hyperbolic iterated function system. We calculate the Hausdorff dimension of those

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sets. It turns out that the Markov attractor discussed in [3] and [10] belongs to a category of the generalized Markov attractors defined in this paper and we will see that the Hausdorff dimension is not perturbed by that kind of generalizations. That is, the Hausdorff dimension of the Markov attractors is determined only by the given Markov transition matrix.

In Section 2, we give the necessary notation and preliminaries. In Section 3, we state and prove the main results of this paper.

2. Preliminaries

DEFINITION 2.1. Let (X, d) be a compact metric space and T_i's be continuous maps from X to X for $i = 1, 2, \cdots, n$. Then $(X; T_1, \cdots, T_n)$ is called an *iterated function system.*

We say $(X; T_1, \cdots, T_n)$ is *hyperbolic* if there exists a constant $0 < s < 1$ such that

$$d(T_i x, T_i y) \leq s d(x, y) \quad \text{for } x, y \in X \text{ and } 1 \leq i \leq n.$$

DEFINITION 2.2. Let $(X; T_1, \cdots, T_n)$ be a hyperbolic iterated function system. Then a subset A of X is called the *attractor* of the system if

(1) A is not empty and closed,
(2) $T_i(A) \subset A$ for $1 \leq i \leq n$,
(3) A is minimal with respect to (1) and (2).

Hutchinson [6] proved that there exists the attractor A for every hyperbolic iterated function system and $A = \bigcup_{i=1}^{n} T_i(A)$. Moreover, for each $a \in A$, there exists a sequence $(i_1, i_2, \cdots)$ such that

$$\lim_{j \to \infty} T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_j}(x) = a$$

for all $x \in X$.

The attractor A of a hyperbolic iterated function system $(X; T_1, \cdots, T_n)$ is said to be *disjoint* if $T_i(A) \cap T_j(A) = \varnothing$ whenever $i \neq j$.

EXAMPLE 2.3 ([3]). Let

$$\Sigma_n = \{(i_1, i_2, \cdots) \mid 1 \le i_j \le n, j = 1, 2, \cdots\}$$

and define maps $\sigma_i : \Sigma_n \to \Sigma_n$ by

$$\sigma_i(i_1, i_2, \cdots) = (i, i_1, i_2, \cdots), \ 1 \le i \le n.$$

Define a metric d on Σ_n by $d(\mathbf{i}, \mathbf{j}) = 2^{-k}$, if $i_1 = j_1, \cdots, i_k = j_k$, $i_{k+1} \neq j_{k+1}$ where $\mathbf{i} = (i_1, i_2, \cdots)$ and $\mathbf{j} = (j_1, j_2, \cdots)$.
Since

$$d(\sigma_i(\mathbf{i}), \sigma_i(\mathbf{j})) = \frac{1}{2} d(\mathbf{i}, \mathbf{j}), \quad 1 \le i \le n,$$

$((\Sigma_n, d); \sigma_1, \sigma_2, \cdots, \sigma_n)$ becomes a disjoint hyperbolic iterated function system with $A = (\Sigma_n, d)$ as its attractor.

DEFINITION 2.4 ([3, 10]). A square matrix M is called a *Markov transition matrix* if all of its entries are 1 or 0.

DEFINITION 2.5. A non-negative square matrix M (all entries of M are non-negative, written by $M \ge 0$) is called *primitive* if $M^k > 0$ (all entries of $M^k > 0$) for some positive integer k.

DEFINITION 2.6. An $n \times n$ matrix M is said to be *reducible* if there is a permutation that puts it into the form

$$\widetilde{M} = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

where M_{11} and M_{22} are square matrices. Otherwise, M is said to be *irreducible*.

Clearly, a primitive matrix is irreducible. For an irreducible nonnegative matrix M, we have the following Perron-Frobenius Theorem.

THEOREM 2.7 ([8]) (Perron-Frobenius Theorem). *Let $M \ge 0$ be an irreducible square matrix. Then $\|M\|$, the maximal modulus of eigenvalues of M, is an eigenvalue of M and has strictly positive eigenvector y (i.e, all components of $y > 0$).*

Now we are going to define the generalized Markov attractors for a given hyperbolic iterated function system. Let $\Omega = \{w_1, w_2, \cdots, w_m\}$ be any finite sequence of (different) objects and $\mathbf{P}(\Omega)$ be the set of all permutations $(w_{i_1}, w_{i_2}, \cdots, w_{i_m})$ of Ω. And let $\mathcal{T}$ be the set of all sequences $(\tau_1, \tau_2, \cdots)$, where $\tau_i \in \mathbf{P}(\Omega)$, $i = 1, 2, \cdots$. Note that for a sequence $(i_1, i_2, \cdots) \in \Sigma_n$ in Example 2.3, we may re-enumerate $(i_1, i_2, \cdots.)$ as $(i_{11}, i_{12}, \cdots, i_{1m}, i_{21}, i_{22}, \cdots, i_{2m}, \cdots)$.

The following definition is a new and generalized concept of M- admissible sequence for a Markov transition matrix M given in [3] and [10].

We denote by $M(i, j)$ for the (i, j)-entry of M.

DEFINITION 2.8. Let M be a Markov transition matrix. For a given $\tau = (\tau_1, \tau_2, \cdots) \in \mathcal{T}$, a sequence $(i_{11}, i_{12}, \cdots, i_{1m}, i_{21}, i_{22}, \cdots, i_{2m}, \cdots)$ $\in \Sigma_n$ is said to be *(M, τ)-admissible*, if

$$M(i_{k\tau_k(j)}, i_{(k+1)\tau_{k+1}(j)}) = 1$$

for $j = 1, 2, \cdots, m$, $k = 1, 2, 3, \cdots$, where $\tau_k(j)$ is the position of w_j in τ_k.

Let

$$\Sigma_{(M,\tau)} = \{(i_1, i_2, \cdots) \mid (i_1, i_2, \cdots) \quad \text{is} \quad (M, \tau) - \text{admissible}\}.$$

DEFINITION 2.9. Let $(X; T_1, T_2, \cdots, T_n)$ be a hyperbolic iterated function system with attractor A. Let M be a Markov transition matrix and let $\tau \in \mathcal{T}$ be any sequence (of permutations) given above. We say that a point a in A is *(M, τ)-attractive* if there exists an (M, τ)-admissible sequence $(i_1, i_2, \cdots)$ such that

$$a = \lim_{j \to \infty} T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_j}(x)$$

for all $x \in X$. The set of all (M, τ)-attractive points of A, denoted by $A_{(M,\tau)}$, is called the *generalized Markov attractor* of the system associated with M and τ.

EXAMPLE 2.10. Let

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then $M^2 > 0$. So M is primitive. If $m = 1$, then Ω is a one point set and so $\mathcal{T}$ is a one point set. Hence, for $\tau \in \mathcal{T}$, $\Sigma_{(M,\tau)} = \Sigma_M$ and $A_{(M,\tau)} = A_M$. where $\Sigma_M = \{(i_1, i_2, \cdots) \mid (i_1, i_2, \cdots)$ is M- admissible$\}$ and A_M is the Markov attractor of the system associated with M given in [3] and [10]. Actually,

$$\Sigma_M = \Sigma_{(M,\tau)} = \{(i_1, i_2, \cdots) \mid M(i_k, i_{k+1}) = 1 \quad \text{for all} \quad k\}.$$

On the other hand, if $m \geq 2$, we may have different Markov attractors according to τ. For example, if $m = 2$ then Ω is a two point set and we choose, $\tau = (\tau_1, \tau_2, \cdots) \in \mathcal{T}$ so that τ_i is identity permutation of (w_1, w_2) for all i. Then

$$\Sigma_{(M,\tau)} = \{(i_1, i_2, \cdots) \mid M(i_k, i_{k+2}) = 1 \quad \text{for all } k\}.$$

Note that $(1, 3, 2, 3, 1, 3, 2, 3, \cdots)$ belongs to Σ_M but not to $\Sigma_{(M,\tau)}$ and $(1, 2, 3, 2, 1, 2, 3, 2, \cdots)$ belongs to $\Sigma_{(M,\tau)}$ but not to Σ_M.

3. Main results

From now on, we consider a disjoint hyperbolic iterated function system $(X; T_1, \cdots, T_n)$ satisfying $s_i d(x, y) \leq d(T_i(x), T_i(y)) \leq \overline{s}_i d(x, y)$ for all $x, y \in X$ and $i = 1, 2, \cdots, n$, where $0 < s_i \leq \overline{s_i} < 1$, d is a metric on X, and M is an irreducible Markov transition matrix.

Let $l, u > 0$ be such that $\|MS^l\| = 1$ and $\|M\overline{S}^u\| = 1$, where

$$S = \begin{pmatrix} s_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_n \end{pmatrix}, \quad \overline{S} = \begin{pmatrix} \overline{s}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \overline{s}_n \end{pmatrix}.$$

Now, we define another metric d_l on Σ_n. For sequences $\mathbf{i} = (i_1, i_2, \cdots) \in \Sigma_n$ and $\mathbf{j} = (j_1, j_2, \cdots) \in \Sigma_n$,

$$d_l(\mathbf{i}, \mathbf{j}) = \begin{cases} (s_{i_1} \cdots s_{i_k})^l, & i_1 = j_1, \cdots, i_k = j_k, i_{k+1} \neq j_{k+1} \\ 0, & i_1 = j_1, i_2 = j_2, \cdots \\ 1, & i_1 \neq j_1. \end{cases}$$

It is easy to show that d_l is a metric on Σ_n which is different from d in Example 2.3. We see that $((\Sigma_n, d_l); \sigma_1, \cdots, \sigma_n)$ is also a disjoint hyperbolic iterated function system with attractor (Σ_n, d_l) and

$\dim(\Sigma_n, d_l) = k$, where $\dim F$ is the Hausdorff dimension of the set F and $\sum_{i=1}^{n} (s_i^l)^k = 1$. For $\tau \in \mathcal{T}$, $(\Sigma_{(M,\tau)}, d_l)$ is also a generalized Markov attractor of $((\Sigma_n, d_l); \sigma_1, \cdots, \sigma_n)$. For a finite sequence $I = (i_1, i_2, \cdots, i_k)$ we say that I has length k.

THEOREM 3.1. *For all* $\tau \in \mathcal{T}$, $\dim((\Sigma_{(M,\tau)}, d_l)) = 1$.

Proof. The idea of the following proof is based on that of Proposition 1 of [10]. Let $\mathbf{v} = (v_1, v_2, \cdots, v_n)^t$ with $\sum_{i=1}^{n} v_i = 1$ be the eigenvector of MS^l associated with $\|MS^l\| = 1$ in Perron-Frobenius Theorem. Let $[i_{11}, \cdots, i_{1m}, i_{21}, \cdots, i_{2m}, \cdots, i_{k1}, \cdots, i_{km}]$ be the set of all sequences of $\sum_{(M,\tau)}$ with $(i_{11}, \cdots, i_{1m}, i_{21}, \cdots, i_{2m}, \cdots, i_{k1}, \cdots, i_{km})$ as their first mk entries and call it a block. Let $\beta_k = \{[i_{11}, \cdots, i_{1m}, \cdots, i_{k1}, \cdots, i_{km}] : (i_{11}, \cdots, i_{km})$ is (M, τ)-admissible$\}$. Then β_k is a cover of $\Sigma_{(M,\tau)}$ and $d_l(B) = (s_{i_{11}} \cdots s_{i_{km}})^l$ for $B = [i_{11}, \cdots, i_{km}] \in \beta_k$, where $d_l(B)$ means the diameter of the set B. Clearly, $d_l(B) \to 0$ as $k \to \infty$ for each $B \in \beta_k$. We notice that if $\sum_{B \in \beta_k} d_l(B) \le L < \infty$ for some constant L and any k, then $\dim(\Sigma_{(M,\tau)}, d_l) \le 1$. Denoting the sum $\sum_{(i_{11}, \cdots, i_{km}):(M,\tau)-\text{admissible}}$ by $\sum'$, we have

$$\begin{aligned}
\beta_k(1) &= \sum{}' d_l([i_{11}, \cdots, i_{km}]) = \sum{}' (s_{i_{11}} \cdots s_{i_{km}})^l \\
&= \sum{}' (s_{i_{1\tau_1(1)}} \cdots s_{i_{k\tau_k(1)}})^l \cdots (s_{i_{1\tau_1(m)}} \cdots s_{i_{k\tau_k(m)}})^l \\
&\le c^m \sum{}' (s_{i_{1\tau_1(1)}} \cdots s_{i_{k\tau_k(1)}})^l v_{i_{k\tau_k(1)}} \cdots \\
&\qquad (s_{i_{1\tau_1(m)}} \cdots s_{i_{k\tau_k(m)}})^l v_{i_{k\tau_k(m)}} \\
&\le c^m \Big(\sum_{(i_{11}, \cdots, i_{km})} \prod_{r=1}^{m} (s_{i_{1\tau_1(r)}})^l (MS^l)(i_{1\tau_1(r)}, i_{2\tau_2(r)}) \cdots \\
&\qquad\qquad (MS^l)(i_{(k-1)\tau_{k-1}(r)}, i_{k\tau_k(r)}) v_{i_{k\tau_k(r)}} \Big) \\
&= c^m \prod_{r=1}^{m} \sum_{i_{1\tau_1(r)}} (s_{i_{1\tau_1(r)}})^l v_{i_{1\tau_1(r)}} \\
&\le c^m \prod_{r=1}^{m} \sum_{i_{1\tau_1(r)}} (s_{i_{1\tau_1(r)}})^l \\
&= c^m (\sum_{i=1}^{n} s_i^l)^m < \infty,
\end{aligned}$$

where $c = \dfrac{1}{\min_i\{v_i\}}$.

On the other hand, we will show that $\sum_{B\in\beta} d_l(B)$ have a positive lower bound independent of the choice of covers β of $\Sigma_{(M,\tau)}$. Thus we get $\dim(\Sigma_{(M,\tau)}, d_l) \geq 1$. Let $\beta = \{B_j\}$ be a cover of $\Sigma_{(M,\tau)}$. Since $(\Sigma_{(M,\tau)}, d_l)$ is compact, we may assume that β is finite . For any $B_j \in \beta$, we can find a block $[i_1, \cdots, i_{k_j}]$ with the maximal length such that $B_j \subset [i_1, \cdots, i_{k_j}]$ and $d_l(B_j) = (s_{i_1} \cdots s_{i_{k_j}})^l = d_l([i_1 \cdots i_{k_j}])$. Let $\overline{k}_j$ be the first multiple number of multiples of m greater than or equal to k_j and let $\overline{k} = \max_j\{\overline{k}_j\}$. Then $\overline{k}_j = mp$ and $\overline{k} = mt$ for some $p, t \in \mathbb{N}$ and $k_j \leq \overline{k}_j \leq \overline{k}$ for all j. For each j, let $\overline{B}_j = [i_1, \cdots, i_{\overline{k}_j}]$. Consider the cover $\alpha_{\overline{k}}(B_j)$ of B_j by blocks of length $\overline{k}$. Let us use the notation $\beta(p) = \sum_j (d_l(B_j))^p$ for each $p > 0$ and for any cover $\beta = \{B_j\}$ of a set. Then

$$\begin{aligned}\alpha_{\overline{k}}(B_j)(1) &= \sum_{(q_{k_j+1},\cdots,q_{\overline{k}})}' d_l([i_1, \cdots, i_{k_j}, q_{k_j+1}, \cdots, q_{\overline{k}}])\\ &= \sum_{(q_{k_j+1},\cdots,q_{\overline{k}})}' (s_{i_1} \cdots s_{i_{k_j}} s_{q_{k_j+1}} \cdots s_{q_{\overline{k}}})^l,\end{aligned}$$

where the sum $\sum'$ is (M,τ)-admissible sequences with $M(i_{k_j-m+r}, q_{k_j+r}) = 1$ for $r = 1, \cdots, m$. We may re-enumerate $(q_{\overline{k}_j+1}, \cdots, q_{\overline{k}})$ as $(q_{(p+1)1}, \cdots, q_{(p+1)m}, \cdots, q_{t1}, \cdots, q_{tm})$. Then

$$\begin{aligned}\alpha_{\overline{k}}(B_j)(1) \leq{}& (s_{i_1} \cdots s_{i_{k_j}})^l \sum_{(i_{k_j+1},\cdots,i_{\overline{k}_j})} (s_{i_{k_j+1}} \cdots s_{i_{\overline{k}_j}})^l\\ &\sum_{(q_{(p+1)\tau_{p+1}(1)},\cdots,q_{t\tau_t(m)})}' \prod_{r=1}^{m} (s_{q_{(p+1)\tau_{p+1}(r)}} \cdots s_{q_{t\tau_t(r)}})^l\\ \leq{}& (nc)^m (s_{i_1} \cdots s_{i_{k_j}})^l\\ &\sum_{(q_{(p+1)\tau_{p+1}(1)},\cdots,q_{t\tau_t(m)})}' \prod_{r=1}^{m} (MS^l)(i_{p\tau_p(r)}, q_{(p+1)\tau_{p+1}(r)})\\ &\cdots (MS^l)(q_{(t-1)\tau_{t-1}(r)}, q_{t\tau_t(r)}) v_{q_{t\tau_t(r)}})\\ ={}& (nc)^m (s_{i_1} \cdots s_{i_{k_j}})^l \prod_{r=1}^{m} v_{i_{p\tau_p(r)}}\\ \leq{}& (nc)^m (s_{i_1} \cdots s_{i_{k_j}})^l = (nc)^m d_l(B_j),\end{aligned}$$

where $c = \dfrac{1}{\min_i\{v_i\}}$. Hence

$$\begin{aligned}
\beta(1) &= \sum_j d_l(B_j) \geq (nc)^{-m} \sum_j \alpha_{\overline{k}}(B_j)(1) \\
&= (nc)^{-m} \sum_j \sum_{(q_{k_j+1},\cdots,q_{\overline{k}})}' (s_{i_1} \cdots s_{i_{k_j}} s_{q_{k_j+1}} \cdots s_{q_{\overline{k}}})^l \\
&\geq (nc)^{-m} \sum_{(i_1,\cdots,q_{\overline{k}}):(M,\tau)-admissible} (s_{i_1} \cdots s_{q_{\overline{k}}})^l \\
&\geq (nc)^{-m} \sum_{(i_1,\cdots,i_m)} (s_{i_1} \cdots s_{i_m})^l \sum_{(i_{2\tau_2(1)},\cdots,i_{t\tau_t(m)})} \prod_{r=1}^{m} \\
&\qquad (MS^l)(i_{1\tau_1(r)}, i_{2\tau_2(r)}) \cdots (MS^l)(i_{(t-1)\tau_{t-1}(r)}, i_{t\tau_t(r)}) v_{i_{t\tau_t(r)}} \\
&= (nc)^{-m} \sum_{(i_1,\cdots,i_m)} (s_{i_1} \cdots s_{i_m})^l (v_{i_1} \cdots v_{i_m}) \\
&= (nc)^{-m} (\sum_i s_i^l v_i)^m > 0
\end{aligned}$$

□

LEMMA 3.2 ([7]). *Let X, Y be two metric spaces, $f : X \to Y$ be a map and $\delta, \gamma > 0$. Then*
(a) *if $d(f(x), f(y)) \geq \gamma d(x,y)^\delta$, then $\dim(Y) \geq \frac{1}{\delta}\dim(X)$,*
(b) *if $f(X) = Y$ and $d(f(x), f(y)) \leq \gamma d(x,y)^\delta$, then $\dim(Y) \leq \frac{1}{\delta}\dim(X)$.*

THEOREM 3.3. *Suppose that $(X; T_1, \cdots, T_n)$ is a disjoint hyperbolic iterated function system satisfying $s_i d(x,y) \leq d(T_i(x), T_i(y)) \leq \overline{s}_i d(x,y)$ for all $x, y \in X$ and $i = 1, 2, \cdots, n$, where $0 < s_i \leq \overline{s_i} < 1$, M is an irreducible Markov transition matrix. Then, for any $\tau \in \mathcal{T}$, $l \leq \dim(A_{(M,\tau)}) \leq u$ where $A_{(M,\tau)}$ is the Markov attractor associated with M and τ, $\|MS^l\| = 1$ and $\|M\overline{S}^u\| = 1$, where*

$$S = \begin{pmatrix} s_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_n \end{pmatrix} \quad \text{and} \quad \overline{S} = \begin{pmatrix} \overline{s}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \overline{s}_n \end{pmatrix}.$$

Proof. Define $f : \Sigma_{(M,\tau)} \to A_{(M,\tau)}$ by $f(i_1, i_2, \cdots) = \lim_{j\to\infty} T_{i_1} \circ T_{i_2} \circ \cdots T_{i_j}(x)$ for any $x \in X$. Clearly, f is well-defined. For any $\mathbf{i} = (i_1, i_2, \cdots)$, $\mathbf{j} = (j_1, j_2, \cdots) \in \Sigma_{(M,\tau)}$, we can easily show that there exists $\gamma > 0$ such that $d(x,y) \geq \gamma(d_l(\mathbf{i},\mathbf{j}))^{l^{-1}}$ by using the disjointness, where $x = f(\mathbf{i})$, $y = f(\mathbf{j})$. (See the argument used in the proof of Theorem 1 in [10]). By Lemma 3.2(a), we have $\dim(A_{(M,\tau)}) \geq l$. On the other hand, we can also show $\dim(A_{(M,\tau)}) \leq u$ using Lemma 3.2(b) and the result of Theorem 3.1 obtained by replacing s_i and l by $\overline{s}_i$ and u respectively. □

REMARK 3.4. If we take $m = 1$, we obtain Theorem 1 in [10] as a Corollary to Theorem 3.3.

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DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702–701, KOREA

E-mail: hhlee@knu.ac.kr
hkbaek@math.knu.ac.kr