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ITERATIVE APPROXIMATION OF FIXED POINTS FOR STRONGLY PSEUDO-CONTRACTIVE MAPPINGS

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ABSTRACT. The aim of this paper is to prove a convergence theorem of a generalized Ishikawa iteration sequence for two multi-valued strongly pseudo-contractive mappings by using an approximation method in real uniformly smooth Banach spaces. We generalize and extend the results of Chang and Chang, Cho, Lee, Jung, and Kang.

1. Introduction and Preliminaries

Let X be a real Banach space, X^* be the duality space of X and $(\cdot,\cdot)$ be the pairing between X and X^*. For $1 < p < \infty$, the mapping $J_p : X \to 2^{X^*}$ defined by

$$J_p(X) = \{f \in X^* : (x, f) = \|f\|\|x\|, \quad \|f\| = \|x\|^{p-1}\}$$

is called the duality mapping with gauge function $\phi(t) = t^{p-1}$. In particular, for $p = 2$, the duality mapping J_2 with gauge function $\phi(t) = t$ is called the normalized duality mapping.

The following proposition gives some basic properties of duality mappings:

PROPOSITION 1. *Let X be a real Banach space. For $1 < p < \infty$, the duality mapping $J_p : X \to 2^{X^*}$ has the following basic properties:*

(i) $J_p(x) \neq \phi$ *for all* $x \in X$ *and* $D(J_p)$ (: *the domain of* J_p) $= X$,
(ii) $J_p(X) = \|x\|^{p-2} J_2(x)$ *for all* $x \in X$ $(x \neq 0)$,
(iii) $J_p(\alpha x) = \alpha^{p-1} J_p(x)$ *for all* $\alpha \in [0, \infty)$,
(iv) $J_p(-x) = -J_p(x)$,

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(v) J_p *is bounded, i.e., for any bounded subset* $A \subset X$, $J_p(A)$ *is a bounded subset in* X^*,

(vi) J_p *can be equivalently defined as the subdifferential of the functional*

$\psi(x) = p^{-1}\|x\|^p$ (Asplund [1]), i.e.,
$J_p(x) = \partial\psi(x) = \{f \in X^* : \psi(y) - \psi(x) \geq (f, y - x) \text{ for all } y \in X\}$

(vii) X *is a uniformly smooth Banach space (equivalently,* X^* *is a uniformly convex Banach space) if and only if* J_p *is single-valued and uniformly continuous on any bounded subset of* X *(Xu and Roach [21]).*

DEFINITION 1. Let X be a real normed space and let K be a nonempty subset of X. Let $T : K \to 2^X$ be a multivalued mapping.

(i) T is said to be accretive if, for any $x, y \in K$, $u \in Tx$ and $v \in Ty$, there exists $j_2 \in J_2(x - y)$ such that

$$(u - v, j_2) \geq 0,$$

or equivalently, there exists $j_p \in J_p(x - y)$, $1 < p < \infty$ such that

$$(u - v, j_p) \geq 0.$$

(ii) T is said to be strongly accretive if, for any $x, y \in K, u \in Tx$ and $v \in Ty$, there exists $j_2 \in J_2(x - y)$ such that

$$(u - v, j_2) \geq k\|x - y\|^2,$$

or equivalently, there exists $j_p \in J_p(x - y)$, $1 < p < \infty$, such that

$$(u - v, j_p) \geq k\|x - y\|^p$$

for some constant $k > 0$. Without loss of generality, we can assume that $k \in (0, 1)$ and such a number k is called the strong accretive constant of T.

(iii) T is said to be (strongly) pseudo-contractive if $I - T$ (where I denotes the identity mapping) is a (strongly) accretive mapping.

The concept of a single-valued accretive mapping was introduced independently by Browder [2] and Kato [13] in 1967. An early fundamental result in the theory of accretive mappings, which is due to Browder states that the following initial value problem,

$$\frac{du(t)}{dt} + Tu(t) = 0, \quad u(0) = u_0,$$

is solvable if T is locally Lipschitzian and accretive on X.

DEFINITION 2. Let X be a real Banach space, let K be a nonempty convex subset of X and let $T : K \to 2^K$ be a multivalued mapping. Given $x_0 \in K$, the sequence $\{x_n\}$ defined by

$$\text{(A)} \qquad \begin{cases} x_{n+1} \in (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n \in (1-\beta_n)x_n + \beta_n T x_n \end{cases}$$

for all $n = 0, 1, 2, \dots$ is called the Ishikawa iteration sequence of T, where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0, 1]$ satisfying some conditions. Especially, if $\beta_n = 0$ for all $n = 0, 1, 2, \dots$, then $\{x_n\}$ is called the Mann iteration sequence.

If $T_1, T_2 : K \to 2^K$ be two multivalued mappings. Given $x_0 \in K$, the sequence $\{x_n\}$ defined by

$$\text{(B)} \qquad \begin{cases} x_{n+1} \in (1-\alpha_n)x_n + \alpha_n T_1 y_n, \\ y_n \in (1-\beta_n)x_n + \beta_n T_2 x_n \end{cases}$$

for all $n = 0, 1, 2, \dots$. (B) is generalization of Ishikawa iteration sequence.

Obviously, if $T_1 = T_2 = T$, (B) reduces to (A).

The convergence problems of Ishikawa and Mann iteration sequences were studied extensively by many authors Chidume [5]-[7], Tan and Xu [18], Reich [16], Ishikawa [11, 12], Mann [14], Deng [8]-[10], Morales [15], Rhoades [17], Xu [20], Zhou and Jia [23], Chang [3], and Chang, Cho, Lee, Jung, and Kang [4].

Chang [3] generalized the result of Chidume [6]. He also extended and improved the results of Deng [9], Zhou [22] and Tan and Xu [18]. Recently Chang, Cho, Lee, Jung and Kang [4] proved a result for single-valued mapping T. They generalized Chidume [6]. They improved and extended the results of Chidume [5, 6, 7], Tan and Xu [18] and Zhou and Zia [23].

In this paper, by using an approximation method, we study the convergence problem of a generalized Ishikawa iteration sequence for two multi-valued strongly pseudo-contractive mappings in real uniformly smooth Banach spaces. We generalize and extend the results of Chang [3] and Chang, Cho, Lee, Jung, and Kang [4].

LEMMA 1 ([4]). *Let X be a real Banach space and let $J_p : X \to 2^{X^*}$, $1 < p < \infty$, be a duality mapping. Then for any given $x, y \in X$, we have*

$$\|x + y\|^p \leq \|x\|^p + p(y, j_p)$$

for all $j_p \in J_p(x + y)$.

LEMMA 2 ([19]). *Let $\{\nu_n\}$ be a nonnegative real sequence and let $\{\lambda_n\}$ be a real sequence in $[0, 1]$ such that*

$$\sum_{n=0}^{\infty} \lambda_n = \infty.$$

(i) *For any given $\varepsilon > 0$, if there exists a positive integer n_o such that*

$$\nu_{n+1} \leq (1 - \lambda_n)\nu_n + \varepsilon\lambda_n$$

for all $n \geq n_0$, then we have $0 \leq \lim_{n\to\infty} \sup \nu_n \leq \varepsilon$.

(ii) *If there exists a positive integer n_1 such that*

$$\nu_{n+1} \leq (1 - \lambda_n)\nu_n + \lambda_n\sigma_n$$

for all $n \geq n_1$ where $\sigma_n \geq 0$ for all $n = 0, 1, 2, \ldots$ and $\sigma_n \to 0$ as $n \to \infty$, then we have

$$\lim_{n\to\infty} \nu_n = 0.$$

LEMMA 3 ([4]). *Let X be a Banach space and let $T : X \to 2^X$ be a multivalued strongly pseudo-contractive mapping. Then for any given $x, y \in X$, $u \in Tx$ and $v \in Ty$, there exists $\widetilde{j}_p \in J_p(x - y)$, $1 < p < \infty$ such that*

$$(u - v, \widetilde{j}_p) \leq (1 - k)\|x - y\|^p,$$

where $k \in (0, 1)$ is the strongly accretive constant of $I - T$.

For the rest of this paper, $F(T)$ denotes the set of fixed points of the mapping T.

2. Main results

THEOREM. *Let X be a real uniformly smooth Banach space and let K be a non empty bounded closed convex subset of X. Let $T_1, T_2 : K \to 2^K$ be two multi-valued strongly pseudo-contractive mappings with non empty closed values. Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences of real numbers satisfying the following conditions :*

(i) $0 \leq \alpha_n,\ \beta_n \leq 1$ *for all* $n = 0, 1, 2, \ldots,$

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty,\ \alpha_n \to 0$ *and* $\beta_n \to 0$ *as* $n \to \infty$.

If $F(T_1) \cap F(T_2) \neq \emptyset$ then for any given $x_0 \in K$, a generalized Ishikawa iteration sequence $\{x_n\}$ defined by

$$\left\{ \begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n u_n, \\ y_n &= (1-\beta_n)x_n + \beta_n v_n \end{aligned} \right. \tag{1}$$

where $u_n \in T_1 y_n$ and $v_n \in T_2 x_n$ for all $n = 0, 1, 2, \ldots$ converges strongly to the unique common fixed point of T_1 and T_2 in K.

Proof. Take $q \in F(T_1) \cap F(T_2)$ and so $q = T_1 q$ and $q = T_2 q$. By (1) and Lemma 1, we have for $1 < p < \infty$,

$$\begin{aligned} \|x_{n+1} - q\|^p &= \|(1-\alpha_n)x_n + \alpha_n u_n - q\|^p \\ &= \|(1-\alpha_n)(x_n - q) + \alpha_n(u_n - q)\|^p \\ &\leq (1-\alpha_n)^p \|x_n - q\|^p + p\alpha_n (u_n - q, J_p(x_{n+1} - q)) \\ &= (1-\alpha_n)^p \|x_n - q\|^p + p\alpha_n (u_n - q, J_p(y_n - q)) \\ &\quad + p\alpha_n a_n, \end{aligned} \tag{2}$$

where $a_n = (u_n - q, J_p(x_{n+1} - q) - J_p(y_n - q))$.

(I) From Lemma 3, it follows that

$$(u_n - q, J_p(y_n - q)) \leq (1 - k_1)\|y_n - q\|^p, \tag{3}$$

where $k_1 \in (0, 1)$ is strongly accretive constant of $I - T_1$. From (2) and (3), we have

$$\|x_{n+1} - q\|^p \leq (1-\alpha_n)^p \|x_n - q\|^p + p\alpha_n(1-k_1)\|y_n - q\|^p + p\alpha_n a_n. \tag{4}$$

(II) Next we prove that $a_n \to 0$ as $n \to \infty$. In fact, because K is a bounded set in X and x_n, u_n, v_n and $q \in K$, then $\{u_n - q\}$, $\{v_n\}$, $\{u_n\}$

and $\{x_n\}$ all are bounded sequences in X. It follows from the conditions (i) and (ii) that as $n \to \infty$

$$x_{n+1} - q - (y_n - q) = (\beta_n - \alpha_n)x_n + u_n\alpha_n - \beta_n v_n \to 0.$$

In view of uniform continuity of J_p on any bounded subset of X, we have

$$J_p(x_{n+1} - q) - J_p(y_n - q) \to 0$$

and so $a_n \to 0$ as $n \to \infty$.

(III) Now, we estimate $\|y_n - q\|^p$. From (1) and Lemma 1, we have

$$\begin{aligned} \|y_n - q\|^p &= \|(1-\beta_n)(x_n - q) + \beta_n(v_n - q)\|^p \\ &\le (1-\beta_n)^p\|x_n - q\|^p + p\beta_n(v_n - q, J_p(y_n - q)) \\ &= (1-\beta_n)^p\|x_n - q\|^p + p\beta_n(v_n - q, J_p(x_n - q)) \\ &\quad + p\beta_n(v_n - q, J_p(y_n - q) - J_p(x_n - q)) \\ &= (1-\beta_n)^p\|x_n - q\|^p \\ &\quad + p\beta_n(v_n - q, J_p(x_n - q)) + p\beta_n b_n, \end{aligned} \tag{5}$$

where $b_n = (v_n - q, J_p(y_n - q) - J_p(x_n - q))$. We can show that $b_n \to 0$ as $n \to \infty$.

(IV) From Lemma 3, it follows that

$$(v_n - q, J_p(x_n - q)) \le (1 - k_2)\|x_n - q\|^p, \tag{6}$$

where $k_2 \in (0,1)$ is strongly accretive constant of $I - T_2$. Substituting (6) into (5), we have

$$\begin{aligned} \|y_n - q\|^p &\le (1-\beta_n)^p\|x_n - q\|^p + p\beta_n(1-k_2)\|x_n - q\|^p + p\beta_n b_n \\ &= \{(1-\beta_n)^p + p\beta_n(1-k_2)\}\|x_n - q\|^p + p\beta_n b_n \\ &\le \|x_n - q\|^p + p\beta_n(1-k_2)M + p\beta_n b_n, \end{aligned} \tag{7}$$

where $M = \sup_{n\ge 0}\|x_n - q\|^p < \infty$. Let $k = \min(k_1, k_2)$. Substituting (7) into (4), we have

$$\begin{aligned} \|x_{n+1} - q\|^p &\le (1-\alpha_n)^p\|x_n - q\|^p + p\alpha_n(1-k)\{\|x_n - q\|^p \\ &\qquad + p\beta_n(1-k)M + p\beta_n b_n\} + p\alpha_n a_n \\ &= \{(1-\alpha_n)^p + p\alpha_n(1-k)\}\|x_n - q\|^p + \alpha_n c_n, \end{aligned} \tag{8}$$

where $c_n = p(1-k)\{p\beta_n(1-k)M + p\beta_n b_n\} + pa_n$. Because we have

$$\begin{aligned} 0 &\le (1-\alpha_n)^p + p\alpha_n(1-k) \\ &= 1 - p\alpha_n + \frac{p(p-1)\alpha_n{}^2}{2!} - \frac{p(p-1)(p-2)\alpha_n{}^3}{3!} + \dots \\ &\quad + (-\alpha_n)^p + p\alpha_n(1-k) \\ &\le 1 - k\alpha_n + \alpha_n e_n, \end{aligned}$$

where $e_n = \frac{p(p-1)\alpha_n}{2!} - \frac{p(p-1)(p-2)\alpha_n{}^2}{3!} + \dots + (-\alpha_n)^{p-1}$. Thus (8), can be written as follows

$$\begin{aligned} \|x_{n+1} - q\|^p &\le (1-k\alpha_n)\|x_n - q\|^p + \alpha_n e_n \|x_n - q\|^p + \alpha_n c_n \\ &\le (1-k\alpha_n)\|x_n - q\|^p + \alpha_n e_n M + \alpha_n c_n \\ &= (1-k\alpha_n)\|x_n - q\|^p + \alpha_n(e_n M + c_n). \end{aligned}$$

Taking

$$\begin{aligned} r_n &= \|x_n - q\|^p, \\ \lambda_n &= k\alpha_n, \\ \sigma_n &= \frac{e_n M + c_n}{k}, \end{aligned}$$

we have

$$r_{n+1} \le (1-\lambda_n) r_n + \lambda_n \sigma_n$$

for $n = 0, 1, 2, \dots$. From Lemma 2, it follows that $x_n \to q$ as $n \to \infty$.

(V) Finally, we prove that q is the unique common fixed point of T_1 and T_2 in K. If $q_1 \in F(T_1) \cap F(T_2)$, by the same method as proved above we can also prove that $x_n \to q_1$ and so $q = q_1$. This completes the proof. □

REMARK 1.
(i) Our Theorem generalizes and extends Theorem 3.3 of Chang, Cho, Lee, Jung, and Kang [4].
(ii) Theorem 3.2 of Chang [3] is a special case of our Theorem.

If we put $\beta_n = 0$ for all $n = 0, 1, 2, \dots$ and $T_1 = T_2 = T$ in our Theorem, we obtain the following:

COROLLARY. *Let X be a real uniformly smooth Banach space, let K be a nonempty, bounded closed convex subset of X and let $T : K \to 2^K$*

be a multi-valued strongly pseudo-contractive mapping with non empty closed values. Let $\{\alpha_n\}$ *be a real sequence in* $[0,1]$ *satisfying* $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\alpha_n \to 0$ *as* $n \to \infty$. *If* $F(T) \neq \phi$, *then for any given* $x_0 \in K$, *the Mann iteration sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n,$$

where $u_n \in Tx_n$, *for* $n = 0, 1, 2, \ldots$ *converges strongly to the unique fixed point of* T *in* K.

REMARK 2. Above Corollary, generalizes Theorem 4.1 of Chang, Cho, Lee, Jung, and Kang [4].

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