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L_2-NORM ERROR ANALYSIS OF THE HP-VERSION WITH NUMERICAL INTEGRATION

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ABSTRACT. We consider the hp—version to solve non-constant coefficients elliptic equations with Dirichlet boundary conditions on a bounded, convex polygonal domain Ω in R^2. To compute the integrals in the variational formulation of the discrete problem we need the numerical quadrature rule scheme. In this paper we consider a family $G_p = \{I_m\}$ of numerical quadrature rules satisfying certain properties. When the numerical quadrature rules $I_m \in G_p$ are used for calculating the integrals in the stiffness matrix of the variational form we will give its variational form and derive an error estimate of $\|u - \tilde{u}_p^h\|_{0,\Omega}$.

1. Introduction

Let Ω be a bounded, convex polygonal domain in R^2 with boundary Γ. Let $\mathcal{M} = \{\mathcal{J}^h\}, h \geq 0$ be a quasi-uniform, regular family of meshes $\mathcal{J}^h = \{\Omega_k^h\}$ defined on Ω, where Ω_k^h is a closed quadrilateral, and

$$\max_{\Omega^h \in \mathcal{J}^h} \operatorname{diam}(\Omega^h) = h \quad \text{for all} \quad \Omega^h, \mathcal{J}^h \in \mathcal{M}. \tag{1.1}$$

Further we assume that for each $\Omega_k^h \in \mathcal{J}^h$ there exists an invertible mapping $T_k^h : \widehat{\Omega} \to \Omega_k^h$ with the following correspondence:

$$\widehat{x} \in \widehat{\Omega} \longleftrightarrow x = T_k^h(\widehat{x}) \in \Omega_k^h \tag{1.2}$$

and

$$\widehat{t} \in U_p(\widehat{\Omega}) \longleftrightarrow t = \widehat{t} \circ (T_k^h)^{-1} \in U_p(\Omega_k^h), \tag{1.3}$$

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where $\widehat{\Omega}$ denotes the reference element $\widehat{I} \times \widehat{I} = [-1,1]^2$ in R^2,

$$U_p(\widehat{\Omega}) = \{\widehat{t} : \widehat{t} \text{ is a polynomial of degree } \leq p \text{ in each variable on } \widehat{\Omega}\} \tag{1.4}$$

and

$$U_p(\Omega_k^h) = \{t : \widehat{t} = t \circ T_k^h \in U_p(\widehat{\Omega})\}. \tag{1.5}$$

We now consider the following model problem of elliptic equations :

$$\text{Find } u \in H_0^1(\Omega) \text{ such that} - \operatorname{div}(a\nabla u) = f \text{ in } \Omega \subset R^2, \tag{1.6}$$

where two functions a and f satisfy a compatibility condition to ensure a solution exists, and

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u \text{ vanishes on } \Gamma\}. \tag{1.7}$$

For the sake of simplicity, we assume that

$$0 < A_1 \leq a(x) \leq A_2 \quad \text{for all} \quad x \in \Omega \tag{1.8}$$

and

$$f \in L_2(\Omega). \tag{1.9}$$

In addition, we also assume that there exists a constant $M \geq 1$ such that

$$\|T_k^h\|_{m,\infty,\widehat{\Omega}}, \quad \|(T_k^h)^{-1}\|_{m,\infty,\Omega_k^h} \leq A \quad \text{for} \quad 0 \leq m \leq M, \tag{1.10}$$

$$\|\widehat{J_k^h}\|_{m,\infty,\widehat{\Omega}}, \quad \|(\widehat{J_k^h})^{-1}\|_{m,\infty,\Omega_k^h} \leq A \quad \text{for} \quad 0 \leq m \leq M-1, \tag{1.11}$$

where $\widehat{J_k^h}$ and $(\widehat{J_k^h})^{-1}$ denote the Jacobians of T_k^h and $(T_k^h)^{-1}$ respectively.
Then, as seen in [10, Theorem 3.12], we obtain the following correspondence: For any $\alpha \in [1,\infty]$, $0 \leq m \leq M$,

$$\widehat{t} \in W^{m,\alpha}(\widehat{\Omega}) \longleftrightarrow t = \widehat{t} \circ (T_k^h)^{-1} \in W^{m,\alpha}(\Omega_k^h) \tag{1.12}$$

with norm equivalence

$$C_1 h^{(m-\frac{2}{\alpha})}\|t\|_{m,\alpha,\Omega_k^h} \leq \|\widehat{t}\|_{m,\alpha,\widehat{\Omega}} \leq C_2 h^{(m-\frac{2}{\alpha})}\|t\|_{m,\alpha,\Omega_k^h}, \tag{1.13}$$

with the subscript α omitted when $\alpha = 2$. Namely, we have

$$C_1 h^{(m-1)}\|t\|_{m,\Omega_k^h} \leq \|\widehat{t}\|_{m,\widehat{\Omega}} \leq C_2 h^{(m-1)}\|t\|_{m,\Omega_k^h}. \tag{1.14}$$

Let us define

$$S_p^h(\Omega) = \{u \in H^1(\Omega) \,:\, u_{\Omega_k^h} \circ (T_k^h) \in U_p(\widehat{\Omega}) \text{ for all } \Omega_k^h \in \mathcal{J}^h\}, \tag{1.15}$$

where $u_{\Omega_k^h}$ denotes the restriction of $u \in H^1(\Omega)$ to $\Omega_k^h \in \mathcal{J}^h$ and

$$S_{p,0}^h(\Omega) = S_p^h(\Omega) \cap H_0^1(\Omega). \tag{1.16}$$

Then, using the $hp-$version of the finite element method with the mesh $\mathcal{J}^h = \{\Omega_k^h\}$ we obtain the following discrete variational form of (1.6):

$$\begin{aligned} &\text{Find} \quad u_p^h \in S_{p,0}^h(\Omega) \quad \text{satisfying} \\ &B(u_p^h, v_p^h) = (f, v_p^h)_\Omega \quad \text{for all} \quad v_p^h \in S_{p,0}^h(\Omega), \end{aligned} \tag{1.17}$$

where

$$B(u,v) = \int_\Omega a\nabla u \cdot \nabla v \, dx, \tag{1.18}$$

the usual inner product

$$(f,v)_\Omega = \int_\Omega f v \, dx. \tag{1.19}$$

Let us now give some approximation results which will be used later.

LEMMA 1.1. *For each integer $l \geq 0$, there exists a sequence of projections $\Pi_p^l \,:\, H^l(\widehat{\Omega}) \to U_p(\widehat{\Omega})$, $p = 1, 2, 3, \cdots$ such that*

$$\Pi_p^l \widehat{v_p} = \widehat{v_p} \quad \textit{for all} \quad \widehat{v_p} \in U_p(\widehat{\Omega}), \tag{1.20}$$

$$\begin{aligned} &\|\widehat{u} - \Pi_p^l \widehat{u}\|_{s,\widehat{\Omega}} \leq C\, p^{-(r-s)}\|\widehat{u}\|_{r,\widehat{\Omega}} \quad \textit{for all} \quad \widehat{u} \in H^r(\widehat{\Omega}) \\ &\textit{with} \quad 0 < s < l < r. \end{aligned} \tag{1.21}$$

Proof. See [11, Lemma 3.1]. □

LEMMA 1.2. *Let $\widehat{u} \in H^r(\widehat{\Omega})$ with $r \geq 2$. Then the projection Π_p^2 from Lemma 1.1 satisfies*

$$\|\widehat{u} - \Pi_p^2 \widehat{u}\|_{0,\infty,\widehat{\Omega}} \leq C p^{-(r-1)} \|\widehat{u}\|_{r,\widehat{\Omega}}. \tag{1.22}$$

Proof. By interpolation results ([9, Theorem 3.2] and [7, Theorem 6.2.4]) we have that for $0 < \varepsilon \leq \frac{1}{2}$,

$$\|\widehat{u} - \Pi_p^2 \widehat{u}\|_{0,\infty,\widehat{\Omega}} \leq C \|\widehat{u} - \Pi_p^2 \widehat{u}\|_{1+\varepsilon,\widehat{\Omega}}^{\frac{1}{2}} \|\widehat{u} - \Pi_p^2 \widehat{u}\|_{1-\varepsilon,\widehat{\Omega}}^{\frac{1}{2}}. \tag{1.23}$$

We also have from Lemma 1.1 that

$$\|\widehat{u} - \Pi_p^2 \widehat{u}\|_{r,\widehat{\Omega}} \leq C p^{-(s-r)} \|\widehat{u}\|_{s,\widehat{\Omega}} \quad \text{for} \quad 0 \leq r \leq 2 \leq s. \tag{1.24}$$

Hence, taking $r = 1 + \varepsilon$ and $r = 1 - \varepsilon$ in (1.24) we obtain

$$\|\widehat{u} - \Pi_p^2 \widehat{u}\|_{1+\varepsilon,\widehat{\Omega}}^{\frac{1}{2}} \|\widehat{u} - \Pi_p^2 \widehat{u}\|_{1-\varepsilon,\widehat{\Omega}}^{\frac{1}{2}} \leq C p^{-(s-1)} \|\widehat{u}\|_{s,\widehat{\Omega}}, \tag{1.25}$$

which completes the proof from (1.23). □

2. The *hp*–version with numerical integration

We consider numerical quadrature rules I_m defined on the reference element $\widehat{\Omega}$ by

$$I_m(\widehat{g}) = \sum_{i=1}^{n(m)} \widehat{\omega}_i^m \, \widehat{g}(\widehat{x}_i^m) \sim \int_{\widehat{\Omega}} \widehat{g}(\widehat{x}) \, d\widehat{x}, \tag{2.1}$$

where m is a positive integer. Let $G_p = \{I_m\}$ be a family of quadrature rules I_m with respect to $U_p(\widehat{\Omega})$, $p = 1, 2, 3, \cdots$, satisfying the following properties : For each $I_m \in G_p$,

(K1) $\widehat{\omega}_i^m > 0$ and $\widehat{x}_i^m \in \widehat{\Omega}$ for $i = 1, \cdots, n(m)$.

(K2) $I_m(\widehat{g}^2) \leq C_1 \|\widehat{g}\|_{0,\widehat{\Omega}}^2$ for all $\widehat{g} \in U_p(\widehat{\Omega})$.

(K3) $C_2 \|\widetilde{g}\|_{0,\widehat{\Omega}}^2 \leq I_m(\widetilde{g}^2)$ for all $\widetilde{g} \in \widetilde{U}_p(\widehat{\Omega})$,

where $\widetilde{U}_p(\widehat{\Omega}) = \{ \dfrac{\partial \widehat{g}}{\partial \widehat{x}_i} : \widehat{g} \in U_p(\widehat{\Omega}) \} \subset U_p(\widehat{\Omega})$.

(K4) $I_m(\widehat{g}) = \displaystyle\int_{\widehat{\Omega}} \widehat{g}(\widehat{x}) \, d\widehat{x}$ for all $\widehat{g} \in U_{d(m)}(\widehat{\Omega})$,

where $\widetilde{d}(p) > 0$ is a fixed integer relative to p and $d(m) \geq \widetilde{d}(p)$.

We also get a family $G_{p,\Omega} = \{I_{m,\Omega}\}$ of numerical quadrature rules with respect to $S_p^h(\Omega)$, defined by

$$
\begin{aligned}
I_{m,\Omega_k^h}(g_{\Omega_k^h}) &= \sum_{j=1}^{n(m)} w_j^k g_{\Omega_k^h}(x_j^m) = \sum_{j=1}^{n(m)} \widehat{w}_j^m \widehat{J_k^h}(\widehat{x}_j^m)(g_{\Omega_k^h} \circ T_k^h)(\widehat{x}_j^m) \\
&= \sum_{j=1}^{n(m)} \widehat{w}_j^m \widehat{J_k^h}(\widehat{x}_j^m)\widehat{g_{\Omega_k^h}}(\widehat{x}_j^m) = I_m(\widehat{J_k^h}\,\widehat{g_{\Omega_k^h}})
\end{aligned} \tag{2.2}
$$

and

$$
I_{m,\Omega}(g) = \sum_{\Omega_k^h \in \mathcal{J}^h} I_{m,\Omega_k^h}(g_{\Omega_k^h}). \tag{2.3}
$$

In particular, one may be interested in Gauss-Legendre(G-L) quadrature rules. Let L_q denote the cross-products of q-point G-L rules along the $\widehat{x}_1$ and $\widehat{x}_2$ axes on $\widehat{\Omega} = \widehat{I} \times \widehat{I}$, given by

$$
L_q(\widehat{g}) = \sum_{i=1}^{q} \sum_{j=1}^{q} \widehat{w}_i^q\, \widehat{w}_j^q\, \widehat{g}(\widehat{x}_{ij}^q) \quad \text{for all} \quad \widehat{g} \in L_2(\widehat{\Omega}),
$$

where $\widehat{x}_{ij}^q = (\widehat{x}_i^q, \widehat{x}_j^q) \in \widehat{\Omega} = \widehat{I} \times \widehat{I}$ with the weights $\widehat{w}_i^q$ and $\widehat{w}_j^q$. We consider a family $\{L_q\}_{q \geq l(p)}$ of G-L quadrature rules with respect to $U_p(\widehat{\Omega})$ such that $l(p) = p + 1$. Then, $\{L_q\}_{q \geq l(p)}$ satisfy the properties $(K1) - (K4)$. In fact, when $q \geq p+1$ $L_q(\widehat{g})$ is exact for all $\widehat{g} \in U_{d(q)}(\widehat{\Omega})$ with $d(q) \geq 2p + 1 > 0$, so that (K2) and (K3) hold with $C_1 = C_2 = 1$.

Here, one may employ the numerical quadrature rules scheme for computing the integrals in the discrete variational form (1.17). Especially, since the model problem (1.6) is a non-constant coefficients elliptic problem the numerical quadrature rules $I_m \in G_p$ can be used for calculating the integrals in the stiffness matrix. Thus, we denote by DF the 2×2 Jacobian matrix of $F : R^2 \to R^2$, and define two discrete inner products

$$
(u, v)_{m,\Omega_k^h} = I_{m,\Omega_k^h}\big((uv)_{\Omega_k^h}\big) = I_m(\widehat{J_k^h}\,\widehat{(uv)_{\Omega_k^h}}) \text{ on } \Omega_k^h \in \mathcal{J}^h, \tag{2.4}
$$

$$
(u, v)_{m,\Omega} = \sum_{\Omega_k^h \in \mathcal{J}^h} (u, v)_{m.\Omega_k^h} \text{ on } \Omega. \tag{2.5}
$$

Then, under the assumption that all integrations in the load vector of (1.17) are performed exactly, using the quadrature rules $I_m \in G_p$ for computing the integrals in the stiffness matrix of (1.17) we obtain the following actual problem of (1.17): Find $\widetilde{u}_p^h \in S_{p,0}^h(\Omega)$ such that

$$B_{m,\Omega}(\widetilde{u}_p^h, v_p^h) = (f, v_p^h)_\Omega \text{ for all } v_p^h \in S_{p,0}^h(\Omega), \tag{2.6}$$

where

$$\begin{aligned} &B_{m,\Omega}(\widetilde{u}_p^h, v_p^h)\\ &= I_{m,\Omega}(a\nabla\widetilde{u}_p^h \cdot \nabla v_p^h)\\ &= \sum_{\Omega_k^h \in \mathcal{J}^h} I_{m,\Omega_k^h}\left(a_{\Omega_k^h}\nabla(\widetilde{u}_p^h)_{\Omega_k^h} \cdot \nabla(v_p^h)_{\Omega_k^h}\right)\\ &= \sum_{\Omega_k^h \in \mathcal{J}^h} I_m\Big(\widehat{J_k^h}\widehat{a_{\Omega_k^h}}\Big[\big(\widehat{DT_k^{h^{-1}}}\big)^t\big(\nabla\widehat{(\widetilde{u}_p^h)_{\Omega_k^h}}\big)\Big]^t\Big[\big(\widehat{DT_k^{h^{-1}}}\big)^t\big(\nabla\widehat{(v_p^h)_{\Omega_k^h}}\big)\Big]\Big). \end{aligned}$$

Here, if we let $\left(DT_k^{h^{-1}}\right)\left(DT_k^{h^{-1}}\right)^t = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}$, then $\widehat{(a_{ij})_{\Omega_k^h}} = \widehat{J_k^h}\widehat{(b_{ij})_{\Omega_k^h}}$ are the entries of the matrix $\widehat{J_k^h}\big(\widehat{DT_k^{h^{-1}}}\big)\big(\widehat{DT_k^{h^{-1}}}\big)^t$. For the simplicity of notation, if the restrictions $\widehat{a_{\Omega_k^h}}$, $\widehat{(a_{ij})_{\Omega_k^h}}$, $\widehat{(\widetilde{u}_p^h)_{\Omega_k^h}}$ and $\widehat{(v_p^h)_{\Omega_k^h}}$ are simply denoted by $\widehat{a}$, $\widehat{a_{ij}}$, $\widehat{\widetilde{u}_p^h}$ and $\widehat{v_p^h}$ respectively, then we have

$$\begin{aligned} &B_{m,\Omega}(\widetilde{u}_p^h, v_p^h)\\ &= \sum_{\Omega_k^h \in \mathcal{J}^h} I_m\Big(\widehat{J_k^h}\widehat{a_{\Omega_k^h}}\big(\nabla\widehat{(\widetilde{u}_p^h)_{\Omega_k^h}}\big)^t\big(\widehat{DT_k^{h^{-1}}}\big)\big(\widehat{DT_k^{h^{-1}}}\big)^t\big(\nabla\widehat{(v_p^h)_{\Omega_k^h}}\big)\Big)\\ &= \sum_{\Omega_k^h \in \mathcal{J}^h} I_m\left(\widehat{a}\begin{pmatrix} \frac{\partial\widehat{\widetilde{u}_p^h}}{\partial\widehat{x}_1} \\ \frac{\partial\widehat{\widetilde{u}_p^h}}{\partial\widehat{x}_2}\end{pmatrix}^t \begin{pmatrix} \widehat{a_{11}} & \widehat{a_{12}} \\ \widehat{a_{21}} & \widehat{a_{22}}\end{pmatrix}\begin{pmatrix} \frac{\partial\widehat{v_p^h}}{\partial\widehat{x}_1} \\ \frac{\partial\widehat{v_p^h}}{\partial\widehat{x}_2}\end{pmatrix}\right) \qquad (2.7)\\ &= \sum_{\Omega_k^h \in \mathcal{J}^h}\sum_{i,j=1}^{2}\left(\widehat{a}\widehat{a_{ij}}\frac{\partial\widehat{\widetilde{u}_p^h}}{\partial\widehat{x}_i}\frac{\partial\widehat{v_p^h}}{\partial\widehat{x}_j}\right)_{m,\widehat{\Omega}}\\ &= \sum_{\Omega_k^h \in \mathcal{J}^h}\sum_{i,j=1}^{2}\left(\widehat{a}\widehat{a_{ij}}\frac{\partial\widehat{\widetilde{u}_p^h}}{\partial\widehat{x}_i}, \frac{\partial\widehat{v_p^h}}{\partial\widehat{x}_j}\right)_{m,\widehat{\Omega}}. \end{aligned}$$

Now, we will derive the L_2-norm error bound for the numerical solution in (2.6). To estimate the error $\|u - \widetilde{u}_p^h\|_{0,\Omega}$ we start with the following lemma.

LEMMA 2.1. *Let u be the exact solution of* (1.6) *and u_p^h the hp-version solution of* (1.17). *Then, for an approximate solution $\widetilde{u}_p^h$ of u_p^h which satisfies* (2.6) *we have*

$$\|u - \widetilde{u}_p^h\|_{0,\Omega} \tag{2.8}$$
$$\leq C \sup_{w \in H^0(\Omega)} \inf_{v_p^h \in S_{p,0}^h(\Omega)} \frac{1}{\|w\|_{0,\Omega}} \{\|u - \widetilde{u}_p^h\|_{1,\Omega} \|s_w - v_p^h\|_{1,\Omega}$$
$$+ |B(\widetilde{u}_p^h, v_p^h) - B_{m,\Omega}(\widetilde{u}_p^h, v_p^h)|\},$$

where for each $w \in H^0(\Omega)$, $s_w \in H_0^1(\Omega)$ denotes the solution of variational problem:

$$B(s_w, v) = (w, v)_\Omega \quad \text{for all} \quad v \in H_0^1(\Omega). \tag{2.9}$$

Proof. $\|u - \widetilde{u}_p^h\|_{0,\Omega}$ can be characterized as

$$\|u - \widetilde{u}_p^h\|_{0,\Omega} = \sup_{w \in H^0(\Omega)} \frac{|(w, u - \widetilde{u}_p^h)_\Omega|}{\|w\|_{0,\Omega}}. \tag{2.10}$$

Since $u - \widetilde{u}_p^h \in H_0^1(\Omega)$ we have from (2.9) that

$$B(s_w, u - \widetilde{u}_p^h) = (w, u - \widetilde{u}_p^h)_\Omega. \tag{2.11}$$

Hence, for each $v_p^h \in S_{p,0}^h(\Omega)$

$$(w, u - \widetilde{u}_p^h)_\Omega = B(s_w, u - \widetilde{u}_p^h)$$
$$= B(u - \widetilde{u}_p^h, s_w) - B(u - \widetilde{u}_p^h, v_p^h) + B(u - \widetilde{u}_p^h, v_p^h) \tag{2.12}$$
$$= B(u - \widetilde{u}_p^h, s_w - v_p^h) + B(u, v_p^h) - B(\widetilde{u}_p^h, v_p^h).$$

Further, since

$$B(u, v_p^h) = (f, v_p^h)_\Omega = B_{m,\Omega}(\widetilde{u}_p^h, v_p^h) \tag{2.13}$$

it follows that

$$|(w, u - \widetilde{u}_p^h)_\Omega|$$
$$\leq |B(u - \widetilde{u}_p^h, s_w - v_p^h)| + |B(\widetilde{u}_p^h, v_p^h) - B_{m,\Omega}(\widetilde{u}_p^h, v_p^h)| \tag{2.14}$$
$$\leq C \inf_{v_p^h \in S_{p,0}^h(\Omega)} \{\|u - \widetilde{u}_p^h\|_{1,\Omega} \|s_w - v_p^h\|_{1,\Omega}$$
$$+ |B(\widetilde{u}_p^h, v_p^h) - B_{m,\Omega}(\widetilde{u}_p^h, v_p^h)|\}.$$

The lemma follows from (2.10) and (2.14). □

The error $\|u - \widetilde{u}_p^h\|_{0,\Omega}$ will depend upon the smoothness of the exact solution u, the coefficient a and $\widehat{a_{ij}}$. In this connection, we give some results.

LEMMA 2.2. *Let $\widehat{u_p}$, $\widehat{w_p} \in U_p(\widehat{\Omega})$ and $\widehat{g} \in L_\infty(\widehat{\Omega})$. Then, for all $\widehat{v_q^1}, \widehat{v_q^2} \in U_q(\widehat{\Omega})$, $\widehat{f_r} \in U_r(\widehat{\Omega})$ with $0 < q \le p$ and $r = d(m) - p - q > 0$ we have*

$$
\begin{aligned}
&|\,(\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}\,| \\
&\le C\{\,\|\widehat{g_r}\|_{0,\infty,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^1}\|_{0,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^2}\|_{0,\widehat{\Omega}} \\
&\quad + \|\widehat{g} - \widehat{g_r}\|_{0,\infty,\widehat{\Omega}} \|\widehat{u_p}\|_{0,\widehat{\Omega}}\ \|\widehat{u_p}\|_{0,\widehat{\Omega}}\},
\end{aligned} \tag{2.15}
$$

where C is independent of p, q and m.

Proof. For any $\widehat{g_r} \in U_r(\widehat{\Omega})$ we have

$$
\begin{aligned}
&|\,(\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}\,| \\
&\le |\,(\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}}\,| + |\,(\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}\,| \\
&\quad + |\,(\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}} - (\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}\,|.
\end{aligned} \tag{2.16}
$$

Thank to (K4),

$$
\begin{aligned}
&(\widehat{g_r}\widehat{v_q^1}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{v_q^1}, \widehat{u_p})_{m,\widehat{\Omega}} = 0 \quad \text{for any} \quad \widehat{v_q^1} \in U_q(\widehat{\Omega}) \quad \text{and} \\
&(\widehat{g_r}\widehat{u_p}, \widehat{v_q^2})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{v_q^2})_{m,\widehat{\Omega}} = 0 \quad \text{for any} \quad \widehat{v_q^2} \in U_q(\widehat{\Omega}).
\end{aligned} \tag{2.17}
$$

Hence,

$$
\begin{aligned}
&|\,(\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}\,| \\
&\le |\,(\widehat{g_r}\widehat{u_p}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}}\,| \\
&\quad + |\,(\widehat{g_r}\widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}}\,|.
\end{aligned} \tag{2.18}
$$

By the Schwarz inequality we obtain

$$
\begin{aligned}
&|\,(\widehat{g_r}\widehat{u_p}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}}\,| \\
&\le (\widehat{g_r}(\widehat{u_p} - \widehat{v_q^1}), \widehat{g_r}(\widehat{u_p} - \widehat{v_q^1}))_{\widehat{\Omega}}^{\frac{1}{2}} (\widehat{u_p} - \widehat{v_q^2}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}}^{\frac{1}{2}} \\
&\le C\,\|\widehat{g_r}\|_{0,\infty,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^1}\|_{0,\widehat{\Omega}}\ \|\widehat{u_p} - \widehat{v_q^2}\|_{0,\widehat{\Omega}}.
\end{aligned} \tag{2.19}
$$

Also, from (K2) we have

$$
\begin{aligned}
&|\,(\widehat{g_r}\widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}}\,| \\
&\le (\widehat{g_r}(\widehat{u_p} - \widehat{v_q^1}), \widehat{g_r}(\widehat{u_p} - \widehat{v_q^1}))_{m,\widehat{\Omega}}^{\frac{1}{2}} (\widehat{u_p} - \widehat{v_q^2}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}}^{\frac{1}{2}} \\
&\le C\|\widehat{g_r}\|_{0,\infty,\widehat{\Omega}} (\widehat{u_p} - \widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^1})_{m,\widehat{\Omega}}^{\frac{1}{2}} (\widehat{u_p} - \widehat{v_q^2}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}}^{\frac{1}{2}} \\
&\le C\|\widehat{g_r}\|_{0,\infty,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^1}\|_{0.\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^2}\|_{0.\widehat{\Omega}}.
\end{aligned} \tag{2.20}
$$

Hence, combining (2.19) and (2.20) we estimate

$$
\begin{aligned}
&|\,(\widehat{g}_r\widehat{u_p},\widehat{u_p})_{\widehat{\Omega}} - (\widehat{g}_r\widehat{u_p},\widehat{u_p})_{m,\widehat{\Omega}}\,| \\
\le\ & C\,\|\widehat{g}_r\|_{0,\infty,\widehat{\Omega}}\|\widehat{u_p}-\widehat{v_q^1}\|_{0,\widehat{\Omega}}\|\widehat{u_p}-\widehat{v_q^2}\|_{0,\widehat{\Omega}}.
\end{aligned}
\tag{2.21}
$$

Similarly, since $\widehat{g}\in L_\infty(\widehat{\Omega})$ we obtain

$$
\begin{aligned}
&\|(\widehat{g}\,\widehat{u_p},\widehat{u_p})_{\widehat{\Omega}} - (\widehat{g}_r\widehat{u_p},\widehat{u_p})_{\widehat{\Omega}}\| \\
&\le ((\widehat{g}-\widehat{g}_r)\widehat{u_p},(\widehat{g}-\widehat{g}_r)\widehat{u_p})^{\frac{1}{2}}_{\widehat{\Omega}}\,(\widehat{u_p},\widehat{u_p})^{\frac{1}{2}}_{\widehat{\Omega}} \\
&\le C\,\|\widehat{g}-\widehat{g}_r\|_{0,\infty,\widehat{\Omega}}\|\widehat{u_p}\|_{0,\widehat{\Omega}}\|\widehat{u_p}\|_{0,\widehat{\Omega}}
\end{aligned}
\tag{2.22}
$$

and

$$
\begin{aligned}
&|(\widehat{g}_r\widehat{u_p},\widehat{u_p})_{m,\widehat{\Omega}} - (\widehat{g}\,\widehat{u_p},\widehat{u_p})_{m,\widehat{\Omega}}| \\
&\le ((\widehat{g}_r-f)\widehat{u_p},(\widehat{g}_r-\widehat{g})\widehat{u_p})^{\frac{1}{2}}_{m,\widehat{\Omega}}(\widehat{u_p},\widehat{u_p})^{\frac{1}{2}}_{m,\widehat{\Omega}} \\
&\le C\,\|\widehat{g}_r-\widehat{g}\|_{0,\infty,\widehat{\Omega}}(\widehat{u_p},\widehat{u_p})^{\frac{1}{2}}_{m,\widehat{\Omega}}(\widehat{u_p},\widehat{u_p})^{\frac{1}{2}}_{m,\widehat{\Omega}} \\
&\le C\,\|\widehat{g}_r-\widehat{g}\|_{0,\infty,\widehat{\Omega}}\|\widehat{u_p}\|_{0,\widehat{\Omega}}\|\widehat{u_p}\|_{0,\widehat{\Omega}}.
\end{aligned}
\tag{2.23}
$$

The lemma follows from (2.21), (2.22), (2.23), and (2.16). □

Further, we let

$$
M_{p,q} = \max_{\Omega_k^h\in\mathcal{J}^h}\max_{i,j}\|\widehat{a}\widehat{a_{ij}}\|_{p,q,\widehat{\Omega}},
\tag{2.24}
$$

where the subscript q will be omitted when $q=2$.

Then, we obtain the following lemma which gives an estimate for the last term of the right side in (2.8).

LEMMA 2.3. *Suppose that $u\in H^\sigma(\Omega), a\in H^\alpha(\Omega)$ and $\widehat{a_{ij}}\in H^\rho(\widehat{\Omega})$ for $i,j=1,2,$ such that $\lambda=\min(\alpha,\rho)\ge 2$. For any $w\in H^0(\Omega)$ and an approximation $\widetilde{u}_p^h$ which satisfies (2.6) let $v_p^h=\Pi_p^1 s_w\in S_{p,0}^h(\Omega)$ with respect to (2.9), then we have*

$$
\begin{aligned}
&\frac{|B(\widetilde{u}_p^h,v_p^h)-B_{m,\Omega}(\widetilde{u}_p^h,v_p^h)|}{\|w\|_{0,\Omega}} \\
&\le C\big[\{r^{-(\lambda-1)}(q^{-1}+1)hM_\lambda+q^{-1}hM_{0,\infty}\}\|u-\widetilde{u}_p^h\|_{1,\Omega} \\
&\quad+\{r^{-(\lambda-1)}(p^{-1}+q^{-\sigma}+1)h^\sigma M_\lambda+q^{-\sigma}h^\sigma M_{0,\infty}\}\|u\|_{\sigma,\Omega}\big],
\end{aligned}
\tag{2.25}
$$

where q is a positive integer such that $0<q\le p$ and $r=d(m)-p-q>0$.

Proof. For any $w \in H^0(\Omega)$ we have

$$\begin{aligned} &\left| B(\widetilde{u}_p^h, v_p^h) - B_{m,\Omega}(\widetilde{u}_p^h, v_p^h) \right| \\ &\leq C \sum_{\Omega_k^h \in \mathcal{J}^h} \max_{i,j} \left| \left(\widehat{a}\,\widehat{a_{ij}} \frac{\partial \widehat{\widetilde{u}_p^h}}{\partial \widehat{x}_i}, \frac{\partial \widehat{v_p^h}}{\partial \widehat{x}_j} \right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a_{ij}} \frac{\partial \widehat{\widetilde{u}_p^h}}{\partial \widehat{x}_i}, \frac{\partial \widehat{v_p^h}}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} \right|. \end{aligned} \tag{2.26}$$

For each $\widehat{a_{ij}}$ $i,j = 1,2$ and $\Omega_k^h \in \mathcal{J}^h$ we let q be any integer such that $0 < q \leq p$ and $r = d(m) - p - q > 0$. Then since $\widehat{a}\,\widehat{a_{ij}} \in L_\infty(\widehat{\Omega})$, due to Lemma 2.2 with $\widehat{v}_q^1 = \frac{\partial \Pi_q^1 \widehat{u}}{\partial \widehat{x}_i}$, $\widehat{v}_q^2 = \frac{\partial \Pi_q^1 \widehat{v_p^h}}{\partial \widehat{x}_j} \in U_q(\widehat{\Omega})$ and $\widehat{g_r} = \Pi_r^2(\widehat{a}\,\widehat{a_{ij}})$, we have

$$\begin{aligned} &\left| \left(\widehat{a}\,\widehat{a_{ij}} \frac{\partial \widehat{\widetilde{u}_p^h}}{\partial \widehat{x}_i}, \frac{\partial \widehat{v_p^h}}{\partial \widehat{x}_j} \right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a_{ij}} \frac{\partial \widehat{\widetilde{u}_p^h}}{\partial \widehat{x}_i}, \frac{\partial \widehat{v_p^h}}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} \right| \\ &\leq C \Bigg\{ \|\Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}} \left\| \frac{\partial \widehat{\widetilde{u}_p^h}}{\partial \widehat{x}_i} - \frac{\partial \Pi_q^1 \widehat{u}}{\partial \widehat{x}_i} \right\|_{0,\widehat{\Omega}} \left\| \frac{\partial \widehat{v_p^h}}{\partial \widehat{x}_j} - \frac{\partial \Pi_q^1 \widehat{v_p^h}}{\partial \widehat{x}_j} \right\|_{0,\widehat{\Omega}} \\ &\qquad + \|\widehat{a}\,\widehat{a_{ij}} - \Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}} \left\| \frac{\partial \widehat{\widetilde{u}_p^h}}{\partial \widehat{x}_i} \right\|_{0,\widehat{\Omega}} \left\| \frac{\partial \widehat{v_p^h}}{\partial \widehat{x}_j} \right\|_{0,\widehat{\Omega}} \Bigg\}. \end{aligned} \tag{2.27}$$

Since $\widehat{a}\,\widehat{a_{ij}} \in H^\lambda(\widehat{\Omega})$ with $\lambda = \min(\alpha, \rho) \geq 2$ we obtain from Lemma 1.2 and (1.14) that

$$\begin{aligned} &\|\widehat{a}\,\widehat{a_{ij}} - \Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}} \left\| \frac{\partial \widehat{\widetilde{u}_p^h}}{\partial \widehat{x}_i} \right\|_{0,\widehat{\Omega}} \left\| \frac{\partial \widehat{v_p^h}}{\partial \widehat{x}_j} \right\|_{0,\widehat{\Omega}} \\ &\leq C\, r^{-(\lambda-1)} \|\widehat{a}\,\widehat{a_{ij}}\|_{\lambda,\widehat{\Omega}} (\|\widehat{u} - \widehat{\widetilde{u}_p^h}\|_{1,\widehat{\Omega}} + \|\widehat{u}\|_{1,\widehat{\Omega}}) \|\widehat{v_p^h}\|_{1,\widehat{\Omega}} \\ &\leq C\, r^{-(\lambda-1)} \|\widehat{a}\,\widehat{a_{ij}}\|_{\lambda,\widehat{\Omega}} (\|\widehat{u} - \widehat{\widetilde{u}_p^h}\|_{1,\widehat{\Omega}} \\ &\qquad + \|\widehat{u}\|_{1,\widehat{\Omega}}) (\|\widehat{s_w}\|_{1,\widehat{\Omega}} + \|\widehat{s_w} - \Pi_p^1 \widehat{s_w}\|_{1,\widehat{\Omega}}) \\ &\leq C\, r^{-(\lambda-1)} \|\widehat{a}\,\widehat{a_{ij}}\|_{\lambda,\widehat{\Omega}} (\|\widehat{u} - \widehat{\widetilde{u}_p^h}\|_{1,\widehat{\Omega}} + \|\widehat{u}\|_{1,\widehat{\Omega}}) \\ &\qquad \times (\|\widehat{s_w}\|_{2,\widehat{\Omega}} + p^{-1} \|\widehat{s_w}\|_{2,\widehat{\Omega}}) \\ &\leq C\, r^{-(\lambda-1)} h (1 + p^{-1}) \\ &\qquad \times M_\lambda (\|u - \widetilde{u}_p^h\|_{1,\Omega_k^h} + h^{(\sigma-1)} \|u\|_{\sigma,\Omega_k^h}) \|s_w\|_{2,\Omega_k^h}. \end{aligned} \tag{2.28}$$

Further, it follows from Lemma 1.1, Lemma 1.2 and (1.14) that

$$
\begin{aligned}
&\|\Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}} \left\| \frac{\partial \widehat{\widetilde{u}_p^h}}{\partial \widehat{x}_i} - \frac{\partial \Pi_q^1 \widehat{u}}{\partial \widehat{x}_i} \right\|_{0,\widehat{\Omega}} \left\| \frac{\partial \widehat{v_p^h}}{\partial \widehat{x}_j} - \frac{\partial \Pi_q^1 \widehat{v_p^h}}{\partial \widehat{x}_j} \right\|_{0,\widehat{\Omega}} \\
&\le C \left\{ \|\widehat{a}\,\widehat{a_{ij}} - \Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}} \right. \\
&\quad \left. + \|\widehat{a}\,\widehat{a_{ij}}\|_{0,\infty,\widehat{\Omega}} \right\} \|\widehat{\widetilde{u}_p^h} - \Pi_q^1 \widehat{u}\|_{1,\widehat{\Omega}} \|\widehat{v_p^h} - \Pi_q^1 \widehat{v_p^h}\|_{1,\widehat{\Omega}} \\
&\le C \left\{ \|\widehat{a}\,\widehat{a_{ij}} - \Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}} + M_{0,\infty} \right\} \\
&\quad \times \left\{ \|\widehat{u} - \widehat{\widetilde{u}_p^h}\|_{1,\widehat{\Omega}} + \|\widehat{u} - \Pi_q^1 \widehat{u}\|_{1,\widehat{\Omega}} \right\} \|\widehat{v_p^h} - \Pi_q^1 \widehat{v_p^h}\|_{1,\widehat{\Omega}} \\
&\le C\, q^{-1} \left\{ r^{-(\lambda-1)} \|\widehat{a}\,\widehat{a_{ij}}\|_{\lambda,\widehat{\Omega}} + M_{0,\infty} \right\} \\
&\quad \times \left\{ \|\widehat{u} - \widehat{\widetilde{u}_p^h}\|_{1,\widehat{\Omega}} + q^{-(\sigma-1)} \|\widehat{u}\|_{\sigma,\widehat{\Omega}} \right\} \|\widehat{v_p^h}\|_{2,\widehat{\Omega}} \\
&\le C\, q^{-1} h \left\{ r^{-(\lambda-1)} M_\lambda + M_{0,\infty} \right\} \\
&\quad \times \left\{ \|u - \widetilde{u}_p^h\|_{1,\Omega_k^h} + q^{-(\sigma-1)} h^{(\sigma-1)} \|u\|_{\sigma,\Omega_k^h} \right\} \|v_p^h\|_{2,\Omega_k^h},
\end{aligned}
\tag{2.29}
$$

where C is independent of p and q.

Thus, Since $\|v_p^h\|_{2,\Omega_k^h} \le C\|\Pi_p^2 s_w\|_{2,\Omega_k^h} \le C\|s_w\|_{2,\Omega_k^h}$, substituting (2.28) and (2.29) in (2.27) we have

$$
\begin{aligned}
&|\, B(\widetilde{u}_p^h, v_p^h) - B_{m,\Omega}(\widetilde{u}_p^h, v_p^h)\, | \\
&\le C \sum_{\Omega_k^h \in \mathcal{J}^h} \max_{i,j} \left| \left(\widehat{a}\,\widehat{a_{ij}} \frac{\partial \widehat{\widetilde{u}_p^h}}{\partial \widehat{x}_i}, \frac{\partial \widehat{v_p^h}}{\partial \widehat{x}_j} \right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a_{ij}} \frac{\partial \widehat{\widetilde{u}_p^h}}{\partial \widehat{x}_i}, \frac{\partial \widehat{v_p^h}}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} \right| \\
&\le C \sum_{\Omega_k^h \in \mathcal{J}^h} \Big\{ q^{-1} h (r^{-(\lambda-1)} M_\lambda + M_{0,\infty}) \\
&\qquad \times (\|u - \widetilde{u}_p^h\|_{1,\Omega_k^h} + q^{-(\sigma-1)} h^{(\sigma-1)} \|u\|_{\sigma,\Omega_k^h}) \\
&\qquad + r^{-(\lambda-1)} h (1 + p^{-1}) \\
&\qquad \times M_\lambda (\|u - \widetilde{u}_p^h\|_{1,\Omega_k^h} + h^{(\sigma-1)} \|u\|_{\sigma,\Omega_k^h}) \Big\} \\
&\qquad \times \|s_w\|_{2,\Omega_k^h} \\
&\le C \Big\{ q^{-1} h (r^{-(\lambda-1)} M_\lambda + M_{0,\infty}) \\
&\quad \times (\|u - \widetilde{u}_p^h\|_{1,\Omega} + q^{-(\sigma-1)} h^{(\sigma-1)} \|u\|_{\sigma,\Omega}) \\
&\quad + r^{-(\lambda-1)} h (1 + p^{-1}) \\
&\quad \times M_\lambda (\|u - \widetilde{u}_p^h\|_{1,\Omega} + h^{(\sigma-1)} \|u\|_{\sigma,\Omega}) \Big\} \|s_w\|_{2,\Omega}
\end{aligned}
\tag{2.30}
$$

$$\leq C\big[\{r^{-(\lambda-1)}(q^{-1}+1)hM_\lambda + q^{-1}hM_{0,\infty}\}\|u-\widetilde{u}_p^h\|_{1,\Omega} + \{r^{-(\lambda-1)}(p^{-1}+q^{-\sigma}+1)h^\sigma M_\lambda + q^{-\sigma}h^\sigma M_{0,\infty}\} \times \|u\|_{\sigma,\Omega}\big]\|s_w\|_{2,\Omega}.$$

In addition, since Ω is convex, due to the smoothness of a and $\widehat{a_{ij}}$ given by (1.8), (1.10) and (1.11), it follows from the regularity of the variational problem (2.9) that

$$\|s_w\|_{2,\Omega} \leq C\|w\|_{0,\Omega}. \tag{2.31}$$

This completes the proof. □

By a direct application of Lemma 2.3 to Lemma 2.1 we obtain the following result which gives an asymptotic $L_2(\Omega)$-norm estimate for the rate of convergence of the $hp-$version with numerical integration.

THEOREM 2.4. *For any $I_m \in G_p$, let $u \in H_0^\sigma(\Omega)$ be the exact solution of (1.6) and $\widetilde{u}_p^h \in S_{p,0}^h(\Omega)$ an approximate solution of u_p^h which satisfies (2.6). Suppose that $a \in H^\alpha(\Omega)$ and $\widehat{a_{ij}} \in H^\rho(\widehat{\Omega})$ for $i,j=1,2$, such that $\lambda = \min(\alpha,\rho) \geq 2$. Then, for any integer q such that $0 < q \leq p$ and $r = d(m)-p-q > 0$ we have*

$$\begin{aligned}&\|u-\widetilde{u}_p^h\|_{0,\Omega}\\ &\leq C\big[\{r^{-(\lambda-1)}(q^{-1}+1)hM_\lambda + q^{-1}hM_{0,\infty}\}\|u-\widetilde{u}_p^h\|_{1,\Omega}\\ &\quad+\{r^{-(\lambda-1)}(p^{-1}+q^{-\sigma}+1)h^\sigma M_\lambda + q^{-\sigma}h^\sigma M_{0,\infty}\}\|u\|_{\sigma,\Omega}\big],\end{aligned} \tag{2.32}$$

where C is independent of p and q.

Proof. For each $w \in H^0(\Omega)$ it follows from Lemma 1.1 and (1.14) that

$$\begin{aligned}\|s_w - v_p^h\|_{1,\Omega} &\leq C\|\widehat{s_w} - \Pi_p^2\widehat{s_w}\|_{1,\widehat{\Omega}}\\ &\leq Cp^{-1}\|\widehat{s_w}\|_{2,\widehat{\Omega}} \leq Cp^{-1}h\|s_w\|_{2,\Omega_k^h}.\end{aligned} \tag{2.33}$$

Since $\|s_w - v_p^h\|_{1,\Omega}^2 = \sum_{\Omega_k^h\in\mathcal{J}^h}\|s_w-v_p^h\|_{1,\Omega_k^h}^2$ we have from (2.31) that

$$\begin{aligned}&\inf_{v_p^h\in S_{p,0}^h(\Omega)}\|u-\widetilde{u}_p^h\|_{1,\Omega}\|s_w-v_p^h\|_{1,\Omega}\\ &\leq Cp^{-1}h\|u-\widetilde{u}_p^h\|_{1,\Omega}\|s_w\|_{2,\Omega}\\ &\leq Cp^{-1}h\|u-\widetilde{u}_p^h\|_{1,\Omega}\|w\|_{0,\Omega}.\end{aligned} \tag{2.34}$$

Thus, by a direct application of Lemma 2.3 and (2.34) to Lemma 2.1 we see that the first term of the right side in (2.8) is dominated by the term $q^{-1}hM_{0,\infty}\|u-\widetilde{u}_p^h\|_{1,\Omega}$ in (2.25). This completes the proof. □

In [6, Theorem 4.8] it has been shown that

$$\|u-u_p^h\|_{1,\Omega} \le C\,p^{-(\sigma-1)}h^{(\sigma-1)}\|u\|_{\sigma,\Omega}. \tag{2.35}$$

Thus, if $d(m)$ is large enough with $q=p$, then the rate of convergence for $\|u-\widetilde{u}_p^h\|_{1,\Omega}$ is asymptotically $O(p^{-(\sigma-1)}h^{(\sigma-1)})$, which coincides with that of $\|u-u_p^h\|_{1,\Omega}$. It follows that the $L_2(\Omega)$ error $\|u-\widetilde{u}_p^h\|_{0,\Omega}$ in (2.32) is asymptotically $O(p^{-\sigma}h^{\sigma})$ under nearly exact integrations. This implies that the L_2 error $\|u-\widetilde{u}_p^h\|_{0,\Omega}$ has nearly $O(p^{-1}h)$ improvement over the H^1 error $\|u-\widetilde{u}_p^h\|_{1,\Omega}$. Further, we see the following fact.

In the case where α and ρ are large enough, the terms containing the factor $r^{-(\lambda-1)}$ in (2.32) may be dominated by the other terms, so that the rate of convergence for $\|u-\widetilde{u}_p^h\|_{0,\Omega}$ is asymptotically $O(p^{-\sigma}h^{\sigma})$. Consequently, if a and $\widehat{a_{ij}}$ are sufficiently smooth, then we have no need of overintegration. Even when $d(m)\approx 2p+1$ with $q=p$ we obtain the optimal rate of convergence $O(p^{-\sigma}h^{\sigma})$.

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