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q-IDEALLY DIFFERENTIAL AND i-IDEALLY DIFFERENTIAL COMMUTATIVE ARTINIAN RINGS

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ABSTRACT. We characterize the commutative Artinian rings R every proper quotient ring (respectively every proper ideal) in which is invariant with respect to all derivations.

0. A mapping $d : R \to R$ is called a derivation of an associative ring R if

$$d(a + b) = d(a) + d(b) \text{ and } d(ab) = d(a)b + ad(b)$$

for all elements a, b of R. If every (left) ideal of R is invariant with respect to all derivations of R, then R is called (left) ideally differential. The ideally differential rings were studied in [1] and [9]. The class of ideally differential rings contains the class of differentially trivial rings, i.e., rings having no nontrivial derivations. The differentially trivial rings R with the additive group R^+ of finite (Prüfer) rank were characterized in [2]. An associative ring R in which all proper quotient rings (respectively all proper (left) ideals) are (left) ideally differential is said to be a (left) q-ideally differential (respectively a (left) i-ideally differential) ring.

In this paper we characterize the q-ideally differential (and respectively the i-ideally differential) commutative Artinian rings.

Throughout the paper p is a prime, $\mathcal{J}(R)$ will always denote the Jacobson radical of a ring R, $\mathrm{char}(R)$ the characteristic of R and $C[x]$ the commutative ring of polynomials in x over a field C.

We will also use some other terminology from [3] and [6].

1. First we characterize the q-ideally differential commutative Artinian rings.

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As defined in I. S. Cohen [5], a v-ring V is a commutative unramified complete regular local rank one domain of characteristic zero with a residue field of characteristic p.

EXAMPLE 1.1. If $R \cong C[x]/(x)^p$, where C is a field (or a v-ring) of characteristic $m > 0$.

If $m = 2^n$ $(n \geq 1)$ and $p = 2$, then R is not differentially trivial. Put $s = 2^{n-1}$. It is clear that there is nontrivial element z such that $z^2 = 0$ and $R = C + Cz$ is a group direct sum. Thus for every element $r \in R$ there are unique elements $a, b \in C$ such that $r = a + bz$. Let e be the identity element of C. Then a map $d : R \to R$ given by the rule

$$d(r) = d(a + bz) = b(z + se), \;\; r \in R$$

determines a nontrivial derivation of R. Since $d(z) \notin zR$, R is not an ideally differential ring and, furthermore, R is i-ideally differential.

If $m = p$ is a prime, then R also is not differentially trivial. In fact, $R = C + jC + \cdots + j^{p-1}C$ for some element j of the nilpotency index p and a map $\delta : R \to R$ given by the rule $\delta(c_0 + jc_1 + \cdots + j^{p-1}c_{p-1}) = c_1 + 2jc_2 + \cdots + (p-1)j^{p-2}c_{p-1}$, where $c_0, c_1, \ldots, c_{p-1} \in C$, determines a nontrivial derivation δ of R such that $\delta(j) \notin jR$.

LEMMA 1.2. *Let R be a commutative local Artinian ring such that char(R)=char($R/\mathcal{J}(R)$). If R is a q-ideally differential ring, then one of the following statements is satisfied:*

(i) *R is a field;*

(ii) *$R \cong C[x]/(x)^m$, where C is a field of a prime characteristic p, $2 \leq m \leq p$;*

(iii) *$R \cong C[x]/(x)^m$ $(m \geq 2)$, where C is a field of characteristic zero.*

Proof. By Theorem 9 of [5] (see also [10, Chapter VIII, Theorem 27]) R contains a subfield C such that $R = \mathcal{J}(R) + C$ is a group direct sum. If R is a differentially trivial ring, then by Lemma 2.5 of [2] R is a field. Therefore we assume that R is not a field and so it has a nontrivial derivation (see Lemma 2.2 of [2]).

Suppose that $\mathcal{J}(R)$ is not a minimal ideal of R. By B we denote $R/\mathcal{J}(R)^2$ and by D the quotient ring $(C + \mathcal{J}(R)^2)/\mathcal{J}(R)^2$. Then $B = \mathcal{J}(B) + D$ and

$$\mathcal{J}(B) = b_1 D + \cdots + b_n D$$

for some nontrivial elements $b_1, \dots, b_n$ $(n \geq 1)$. Thus for every element b of B there are unique elements $x_1, \dots, x_n, u \in D$ such that

$$b = \sum_{i=1}^{n} b_i x_i + u. \tag{1}$$

Then a map $\delta : B \to B$ given by the rule $\delta(b) = b_1 x_1,\ b \in B$, with b_1 and x_1 as in (1), determines a nontrivial derivation of B.

Assume that $n \geq 2$. Since $\delta(b_1 + b_2) = b_1$, we conclude that $b_1 \in (b_1 + b_2)B$, a contradiction. Therefore $n = 1$. By Theorem 1 of [8], the Jacobson radical $\mathcal{J}(R)$ is a principal ideal and consequently $\mathcal{J}(R) = zR$ for some its nontrivial element z. Hence

$$R = z^{m-1}C + \cdots + zC + C,$$

where m is the nilpotency index of $\mathcal{J}(R)$. This gives that

$$R \cong C[x]/(x)^m \quad (m \geq 2).$$

If char(C) is a prime p, then, in view of Example 1.1, this means that $2 \leq m \leq p$. Therefore we assume that char(C)=0. Let d be a nontrivial derivation of R. Then

$$d(z) = a_0 z^l + \cdots + a_{l-1} z + a_l$$

for some elements $a_0, \dots, a_l \in C$ and some integer $l \leq m - 1$, and

$$0 = d(z^m) = m z^{m-1} d(z) = m a_l z^{m-1}$$

yields that $a_l = 0$. Hence $d(z) \in zR$. Let I be any proper nontrivial ideal of R. Then $I = f(z)R$, where

$$f(z) = b_0 z^k + \cdots + b_{k-i} z^i$$

with $b_0 \neq 0,\ b_{k-i} \neq 0,\ 1 \leq i \leq k \leq m - 1$. Since it is easy to see that $z^i \in I$, we have $I = z^i R$, and therefore R is an ideally differential ring. The lemma is proved. $\square$

LEMMA 1.3. *Let R be a commutative local Artinian ring of prime power characteristic p^n $(n \geq 2)$. If R is a q-ideally differential ring, then $R \cong C$ or $R \cong C[x]/(x)^m$, where $C \cong V/p^nV$, V is a v-ring and $2 \leq m \leq p$.*

Proof. As a consequence of Theorem 11 from [5, p. 79], R contains a subring C such that $R = C + \mathcal{J}(R)$ and $C \cong V/p^nV$ for some v-ring V. Since $\mathcal{J}(C) = pC$, in view of Proposition 9.2 of [3] it easily to seen that C is an ideally differential ring. By our hypothesis the quotient ring R/pR is a q-ideally differential ring of characteristic p and thus, by Lemma 1.2, R/pR is a field or $R/pR = D + D\alpha + \cdots + D\alpha^{m-1}$ for some its element α of the nilpotency index m and some subfield D of characteristic p, where $2 \leq m < p$. This gives that $\mathcal{J}(R) = pR$ or $\mathcal{J}(R) = pR + aC + \cdots + a^{m-1}C$, where a is an inverse image of α. Hence $R = C + \mathcal{J}(R) = C + pC + p^2R = \ldots = C + pC + p^2C + \ldots + p^{n-1}C = C$ or $R = C + \mathcal{J}(R) = C[a] \cong C[x]/(x)^m$ with $2 \leq m \leq p$. The lemma is proved. □

THEOREM 1.4. *Let R be a commutative Artinian ring. Then R is a q-ideally differential ring if and only if it is of one of the following types:*

(i) *R is a field;*

(ii) *$R \cong V/p^nV$ $(n \geq 2)$, where V is a v-ring;*

(iii) *$R \cong C[x]/(x)^m$ $(m \geq 2)$, where C is a field of characteristic zero;*

(iv) *$R \cong C[x]/(x)^m$, where $2 \leq m \leq p$, C is a field of a prime characteristic p or $C \cong V/p^nV$ $(n \geq 2)$, where V is a v-ring;*

(v) *$R = R_1 \times \cdots \times R_s$ $(s \geq 2)$ is a ring direct product of $R_1, \ldots, R_s$, and each R_i is either a ring of type* (i), *or* (ii), *or* (iii), *or* (iv) *with $2 \leq m < p$.*

Proof. $(\Leftarrow)$. It is clear that the rings R of types (i), (ii), (iii) and (iv) are q-ideally differential. The results of [7] and [4] yield that every quotient ring of R of type (v) is distributive and consequently R is a q-ideally differential ring.

$(\Rightarrow)$. By Theorem 8.7 of [6] it follows that

$$R = R_1 \times \cdots \times R_s \ (s \geq 1)$$

is a ring direct product of q-ideally differential local Artinian rings R_1, $\ldots$, R_s. If $s \geq 2$, then Example 1.1, Lemmas 1.2 and 1.3 yield that each R_i is either of type (i), or (ii), or (iii). If $s = 1$, then by using Lemmas 1.2 and 1.3 we complete the proof. □

2. In this part we obtain the characterization of i-ideally differential commutative Artinian rings.

LEMMA 2.1. *Let R be a commutative local Artinian ring such that char(R)=char($R/\mathcal{J}(R)$). If R is an i-ideally differential ring, then one of the following statements is satisfied:*

(i) *R is a field;*

(ii) *$R \cong C[x]/(x)^n$, where C is a field and $n = 2$ or 3.*

Proof. As in the proof of Lemma 1.2, $R = \mathcal{J}(R) + C$ for some subfield C. Suppose that $\mathcal{J}(R)$ is nontrivial. Let n be the nilpotency index of $\mathcal{J}(R)$. Then

$$\mathcal{J}(R)^{n-1} = j_1C + \cdots + j_mC$$

for some nontrivial elements $j_1, \dots, j_m \in \mathcal{J}(R)$. If $m \geq 2$, then an identical argument to that in the proof of Lemma 1.2 gives a contradiction. Hence $m = 1$.

Let A be any ideal of R such that $A^2 = \{0\}$. If i is any nontrivial element of A, then iC is an ideal of A. Since A is ideally differential, $A = iC$. It is clear that $A = \mathcal{J}(R)^{n-1}$. Let $k = [\frac{n}{2}] + 1$, where $[\frac{n}{2}]$ is the maximal integer not exceeding $\frac{n}{2}$. Then $(\mathcal{J}(R)^k)^2 = \{0\}$ and so $\mathcal{J}(R)^k \leq \mathcal{J}(R)^{n-1}$. This yields that $\mathcal{J}(R)^{k+1} = \{0\}$. Therefore $[\frac{n}{2}] + 2 \geq n$ and $n = 2$ or 3. Hence $R \cong C[x]/(x)^n$, where $n = 2$ or 3. The lemma is proved. □

LEMMA 2.2. *Let R be a commutative local Artinian ring such that char(R) $\neq$ char($R/\mathcal{J}(R)$). If R is an i-ideally differential ring, then $R \cong C/p^mC$ $(m \geq 2)$, where C is a v-ring.*

Proof. By Theorem 11 of [5, p.79], there is a subring V such that $R = V + \mathcal{J}(R)$, where $V \cong C/p^mC$ for some integer $m \geq 2$ and C is a v-ring. Then m is the nilpotency index of $\mathcal{J}(R)$.

1) Suppose that $V \neq R$. Since the annihilator $\mathrm{Ann}(\mathcal{J}(R))$ of $\mathcal{J}(R)$ in R is a finitely generated ideal of R, we obtain that $\mathrm{Ann}(\mathcal{J}(R))$ is a finite-dimensional K-algebra, where $K \cong R/\mathcal{J}(R)$. Therefore there exists a K-basis $\{x_1, \dots, x_s\}$ of $\mathrm{Ann}(\mathcal{J}(R))$. If $s \geq 2$, then a map $\delta : \mathrm{Ann}(\mathcal{J}(R)) \to \mathrm{Ann}(\mathcal{J}(R))$ given by the rule

$$\delta\Big(\sum_{i=1}^{s} u_i x_i\Big) = u_2x_1 + u_1x_2$$

determines a nontrivial derivation of $\operatorname{Ann}(\mathcal{J}(R))$. If $Kx_1 \cap Kx_2$ contains a nontrivial element w, then $w = c_1x_1 = c_2x_2$ for some $c_2, c_2 \in K$, and we obtain a contradiction with minimality of the system $\{x_1, \dots, x_s\}$. Hence $Kx_1 \cap Kx_2 = \{0\}$ and so $\delta(x_1R) \nsubseteq x_1R$, a contradiction. This yields $s = 1$ and thus $\operatorname{Ann}(\mathcal{J}(R)) = Kx$ for some its nontrivial element x. It is clear that $\mathcal{J}(R)^{m-1}$ and $p^{m-1}\mathcal{J}(R)$ are the K-subalgebras of $\operatorname{Ann}(\mathcal{J}(R))$ and consequently

$$p^{m-1}\mathcal{J}(R) = \operatorname{Ann}(\mathcal{J}(R)) = \mathcal{J}(R)^{m-1}. \tag{2}$$

Moreover, $\operatorname{Ann}(\mathcal{J}(R))$ is a minimal ideal of R.

2) Let $\overline{R} = R/\operatorname{Ann}(\mathcal{J}(R))$. Then $\operatorname{Ann}(\mathcal{J}(\overline{R}))$ is a finite-dimensional K-algebra and so there exists a K-basis $\{\overline{y}_1, \dots, \overline{y}_l\}$ of $\operatorname{Ann}(\mathcal{J}(\overline{R}))$. Let y_j be an inverse image of $\overline{y}_j$ in R $(j = 1, \dots, l)$.

a_2) If $pR \leq y_jR$ for all j $(1 \leq j \leq l)$, then, in view of condition

$$y_jR \cdot \mathcal{J}(R) \leq \operatorname{Ann}(\mathcal{J}(R)),$$

we obtain that $pR \cdot \mathcal{J}(R) \leq \operatorname{Ann}(\mathcal{J}(R))$ and consequently $p^2\mathcal{J}(R) = \{0\}$. Now, (2) yields that $m = 2$. Hence $\mathcal{J}(R) = \operatorname{Ann}(\mathcal{J}(R))$ is a principal ideal, a contradiction.

b_2) Suppose, for example, that $pR \nsubseteq y_1R$ and without loss of generality $py_2 \notin y_1R$. A map $D : \overline{R} \to \overline{R}$ given by the rule

$$D\Big(\sum_{k=1}^{l} u_k\overline{y}_k\Big) = u_2\overline{y}_1 + u_1\overline{y}_2$$

determines a nontrivial derivation D of $\overline{R}$. Then a map $d : R \to R$ such that

$$d(r) = pD(r + \operatorname{Ann}(\mathcal{J}(R)), \quad r \in R$$

is a derivation of R with $d(y_1) \notin y_1R$, a contradiction. Hence $\operatorname{Ann}(\mathcal{J}(\overline{R}))$ is a principal ideal. The same argument, as in the line 1), shows that

$$p^{m-2}\mathcal{J}(\overline{R}) = \operatorname{Ann}(\mathcal{J}(\overline{R})) = \mathcal{J}(\overline{R})^{m-2}.$$

3) Let $\overline{\overline{R}} = \overline{R}/\operatorname{Ann}(\mathcal{J}(\overline{R}))$. As above, there exists a K-basis $\{\overline{\overline{z}}_1, \dots, \overline{\overline{z}}_n\}$ of $\operatorname{Ann}(\mathcal{J}(\overline{\overline{R}}))$. Let $\overline{z}_g$ be an inverse image of $\overline{\overline{z}}_g$ in $\overline{R}$ $(g = 1, \dots, n)$.

a_3) If $p\overline{R} \leq \overline{z}_g\overline{R}$ for all g $(1 \leq g \leq n)$, then, as above, we can prove that $p^2\mathcal{J}(\overline{R}) = \{\overline{0}\}$ and so $\mathcal{J}(R)^3 = \{0\}$. Since $\mathcal{J}(R/\mathcal{J}(R)^2)$ is a principal ideal, we obtain that $\mathcal{J}(R)$ is also principal by Theorem 1 of [8], a contradiction.

b_3) Now, assume, for example, that $p\overline{\overline{R}} \not\subseteq \overline{z}_1\overline{R}$ and without restricting of generality $p\overline{z}_2 \notin \overline{z}_1\overline{R}$. As in b_2), we can prove that $\mathrm{Ann}(\mathcal{J}(\overline{\overline{R}}))$ is a principal ideal and

$$p^{m-3}\mathcal{J}(\overline{\overline{R}}) = \mathrm{Ann}(\mathcal{J}(\overline{\overline{R}})) = \mathcal{J}(\overline{\overline{R}})^{m-3}.$$

Continuing this argument after finitely many steps we obtain that $\mathcal{J}(R)/\mathcal{J}(R)^2$ is a principal ideal, a final contradiction. The lemma is proved.□

THEOREM 2.3. *Let R be a commutative Artinian ring. Then R is an i-ideally differential ring if and only if it is of one of the following types:*

(i) *R is a field;*

(ii) *$R \cong V/p^nV$ $(m \geq 2)$, where V is a v-ring;*

(iii) *$R \cong C[x]/(x)^n$, where C is a field and $n = 2$ or 3;*

(iv) *$R = R_1 \times \cdots \times R_n$ $(n \geq 2)$ is a ring direct product of $R_1, \ldots, R_n$, and each R_i is a ring of type* (i), *or* (ii), *or* (iii), *where $n \neq \mathrm{char}(C)$.*

Proof. $(\Rightarrow)$ it follows from Theorem 8.7 of [6], Lemmas 2.1, 2.2 and Example 1.1.

$(\Leftarrow)$ is obvious. The theorem is proved. □

Open question. Characterize the left Artinian rings in which every proper quotient ring (respectively every proper ideal) is left ideally differential.

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