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NOTES ON SOME IDENTITIES INVOLVING THE RIEMANN ZETA FUNCTION

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ABSTRACT. We first review Ramaswami's and Apostol's identities involving the Zeta function in a rather detailed manner. We then present corrected, or generalized formulas, or a different method of proof for some of them. We also give closed-form evaluation of some series involving the Riemann Zeta function by an integral representation of $\zeta(s)$ and Apostol's identities given here.

1. Introduction and Preliminaries

The Riemann Zeta function $\zeta(s)$ is defined by

$$(1.1)\quad \zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \displaystyle\left(1-2^{1-s}\right)^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0;\ s \neq 1). \end{cases}$$

The Riemann Zeta function $\zeta(s)$ can, except for a simple pole at $s = 1$ with its residue 1, be continued analytically to the whole complex s-plane by means of a familiar contour integral representation (*cf.* Whittaker and Watson [9, p. 266]) or many other known integral representations (*cf.* Erdélyi *et al.* [5, p. 33]).

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The Hurwitz (or generalized) Zeta function $\zeta(s,a)$ is defined by

$$\begin{aligned}\zeta(s,a) &:= \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \\ \big(\Re(s) > 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \mathbb{Z}_0^- &:= \mathbb{Z}^- \cup \{0\}; \\ \mathbb{Z}^- &:= \{-1,\ -2,\ -3,\ \ldots\}\big),\end{aligned} \tag{1.2}$$

which can, just as $\zeta(s)$, be continued analytically to the whole complex s-plane except for a simple pole at $s = 1$ (with its residue 1). Clearly, we have

$$\zeta(s,1) = \zeta(s) = (2^s - 1)^{-1}\, \zeta\left(s, \tfrac{1}{2}\right) \quad \text{and} \quad \zeta(s,2) = \zeta(s) - 1. \tag{1.3}$$

Ramaswami [6] presented numerous interesting recursion formulas which can be employed to get the analytic continuation of Riemann Zeta function $\zeta(s)$ over the whole s-plane. Apostol [1] also gave some interesting formulas involving the Riemann Zeta function some of which are generalizations of the above mentioned Ramaswami's identities. Here we are aiming at reviewing their results in a rather detailed manner and then presenting corrected, or generalized formulas, or a different method of proof for some of them. We also give closed-form evaluation of some series involving the Riemann Zeta function, the subject of which has an over two-century history [7], by using an integral representation of $\zeta(s)$ and Apostol's identities given here.

For these, we first recall the Bernoulli polynomials $B_n(x)$ defined by the generating function:

$$\frac{z\, e^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\, \frac{z^n}{n!} \quad (|z| < 2\pi). \tag{1.4}$$

The numbers $B_n := B_n(0)$ are called the Bernoulli numbers generated by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n\, \frac{z^n}{n!} \quad (|z| < 2\pi). \tag{1.5}$$

The Bernoulli numbers and polynomials satisfy lots of interesting and useful relations, among other things, the following are listed here:
(1.6)
$B_n(x+1) - B_n(x) = n\, x^{n-1} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},\ \ \mathbb{N} := \{1,\ 2,\ 3, \ldots\}),$

which yields

$$B_n(0) = B_n(1) \quad (n \in \mathbb{N} \setminus \{1\}); \tag{1.7}$$

$$\sum_{k=1}^{m} k^n = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1} \quad (m,\, n \in \mathbb{N}). \tag{1.8}$$

If $n \in \mathbb{N}$, denote $n = p_1^{a_1} \cdots p_k^{a_k}$ into its prime factorization. Then the Möbius function μ is defined as follows (see [2, pp. 24-25]):

$$\mu(n) := \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } a_1 = \cdots = a_k = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.9}$$

A remarkably simple formula for the divisor sum $\sum_{d|n} \mu(d)$ is given here: For $n \in \mathbb{N}$, we have

$$\sum_{d|n} \mu(d) = \left[\frac{1}{n}\right] = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{1.10}$$

where $[x]$ denotes the greatest integer $\leq x$.

The Pochhammer symbol (or the shifted factorial) $(\alpha)_n$ is defined, for any complex number α, by

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1) & \text{if } n \in \mathbb{N}, \\ 1 & \text{if } n = 0. \end{cases} \tag{1.11}$$

The generalized binomial expansion is also recalled:

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n\, z^n}{n!} \quad (|z| < 1). \tag{1.12}$$

2. Identities involving the Zeta functions

We begin with recalling three of Ramaswami's identities [6] which can be employed to get the analytic continuation of Riemann Zeta function $\zeta(s)$ over the whole s-plane:

$$\zeta(s)\left(1 - 2^{1-s}\right) = \sum_{n=1}^{\infty} \frac{(s)_n}{n!}\, \zeta(n+s)\, 2^{-n-s}, \tag{2.1}$$

$$\zeta(s)\left(1-3^{1-s}\right)=1+2\sum_{n=1}^{\infty}\frac{(s)_{2n}}{(2n)!}\,\zeta(2n+s)\,3^{-2n-s}, \tag{2.2}$$

$$\zeta(s)\left(1-2^{-s}-3^{-s}-6^{-s}\right)=1+2\sum_{n=1}^{\infty}\frac{(s)_{2n}}{(2n)!}\,\zeta(2n+s)\,6^{-2n-s}. \tag{2.3}$$

It follows from the definition (1.2) of the Hurwitz Zeta function $\zeta(s,a)$ that

$$\frac{\partial}{\partial a}\zeta(s,a)=-s\,\zeta(s+1,a), \tag{2.4}$$

$$\zeta(s,a)-\zeta(s,a+k)=\sum_{n=0}^{k-1}(a+n)^{-s}\quad (k\in\mathbb{N}), \tag{2.5}$$

which, for $k=1$, yields

$$\zeta(s,a)-\zeta(s,a+1)=a^{-s}. \tag{2.6}$$

By grouping integers modulo k, it is easy to obtain the multiplication formula for $\zeta(s,a)$:

$$\zeta(s,ka)=k^{-s}\sum_{n=0}^{k-1}\zeta\left(s,a+\frac{n}{k}\right)\quad (k\in\mathbb{N}), \tag{2.7}$$

which, upon setting $a=\frac{1}{k}$, gives

$$\zeta(s)=k^{-s}\sum_{m=1}^{k}\zeta\left(s,\frac{m}{k}\right)\quad (k\in\mathbb{N}). \tag{2.8}$$

We may use the generalized binomial expansion (1.12) to obtain the Taylor expansion of $\zeta(s,a+1)$ in the neighborhood of $a=0$ as follows:

$$\begin{aligned}\zeta(s,a+1)&=\sum_{k=1}^{\infty}k^{-s}\left(1+\frac{a}{k}\right)^{-s}=\sum_{k=1}^{\infty}k^{-s}\sum_{n=0}^{\infty}\frac{(s)_n}{n!}\left(-\frac{a}{k}\right)^{n}\\&=\sum_{n=0}^{\infty}\frac{(s)_n}{n!}(-a)^n\sum_{k=1}^{\infty}k^{-n-s}\\&=\sum_{n=0}^{\infty}\frac{(s)_n}{n!}\,\zeta(n+s)\,(-a)^n,\end{aligned}$$

the exchange of the summations being guaranteed by the absolute convergence of the involved double series. We thus have

$$\zeta(s,a+1)=\sum_{n=0}^{\infty}\frac{(s)_n}{n!}\,\zeta(n+s)\,(-a)^n \quad (|a|<1;\ \Re(s)>1), \tag{2.9}$$

which plays a key role in the work of Apostol [1] who noted (2.4) to get (2.9) and may hold for all s except $s=1$ by the analytic continuation.

By taking $a=-\frac{h}{k}$ in (2.9), $0\le h\le k-1$, and summing on h, Apostol [1] obtained

$$\sum_{h=0}^{k-1}\zeta\left(s,1-\frac{h}{k}\right)=\sum_{n=0}^{\infty}\frac{(s)_n}{n!}\,\zeta(n+s)\,k^{-n}\sum_{h=0}^{k-1}h^n,$$

which, upon employing (2.8) and (1.8), immediately leads to the Apostol's identity [1, p. 240]:

$$\zeta(s)\,\left(1-k^{1-s}\right)=\sum_{n=1}^{\infty}\frac{(s)_n\,\zeta(n+s)}{n!\,k^{n+s}}\,\frac{B_{n+1}(k)-B_{n+1}}{n+1} \quad (k\in\mathbb{N}), \tag{2.10}$$

which, for $k=2$, reduces to (2.1).

Similarly, Apostol [1] obtained the following identities:

$$\begin{aligned}&\zeta(s)\,\left(1-k^{1-s}\right)\\ &=\sum_{h=1}^{k-1}h^{-s}+\sum_{n=1}^{\infty}(-1)^n\,\frac{(s)_n\,\zeta(n+s)}{n!\,k^{n+s}}\,\frac{B_{n+1}(k)-B_{n+1}}{n+1} \quad (k\in\mathbb{N}),\end{aligned} \tag{2.11}$$

where the empty sum (as usual, in what follows) is understood to be nil;

$$\begin{aligned}&\zeta(s)\,\left(1-k^{1-s}\right)\\ &=\frac{1}{2}\sum_{h=1}^{k-1}h^{-s}+\sum_{n=1}^{\infty}\frac{(s)_{2n}\,\zeta(2n+s)}{(2n)!\,k^{2n+s}}\,\frac{B_{2n+1}(k)}{2n+1} \quad (k\in\mathbb{N});\end{aligned} \tag{2.12}$$

$$\sum_{h=1}^{k-1}h^{-s}=\sum_{n=1}^{\infty}\frac{(s)_{2n-1}\,\zeta(2n-1+s)}{(2n-1)!\,k^{2n-1+s}}\,\frac{B_{2n}(k)-B_{2n}}{n} \quad (k\in\mathbb{N}); \tag{2.13}$$

$$\begin{aligned}&\zeta(s)\sum_{d|k}\mu(d)\,d^{-s}\\ &= \phi(-s,k)+2\sum_{n=0}^{\infty}\frac{(s)_{2n}\,\zeta(2n+s)}{(2n)!\,k^{2n+s}}\,\phi(2n,k)\quad (k\in\mathbb{N}\setminus\{1,\,2\}),\end{aligned}\tag{2.14}$$

where

$$\phi(\alpha,k):=\sum_{\substack{1\le h\le k/2\\ (h,k)=1}} h^{\alpha}$$

is the sum of the αth powers of those integers not exceeding $k/2$ which are relatively prime to k, and μ denotes the Möbius function defined by (1.9). The special cases of (2.14) when $k=3$ and $k=6$ reduce to (2.2) and (2.3), respectively.

By setting $a=\pm\frac{h}{k}$ in (2.9), $1\le h\le\frac{k}{2}$, $k\ge 2$, summing on h, and combining the two resulting identities, denoting by $f(\alpha,k)$ the sum

$$f(\alpha,k):=\sum_{1\le h\le k/2} h^{\alpha},$$

we obtain

$$\begin{aligned}&\left(1-k^{-s}\right)\zeta(s)+\frac{1+(-1)^k}{2k^s}\left(2^s-1\right)\zeta(s)\\ &= f(-s,k)+2\sum_{n=0}^{\infty}\frac{(s)_{2n}\,\zeta(2n+s)}{(2n)!\,k^{2n+s}}\,f(2n,k)\quad (k\in\mathbb{N}\setminus\{1\}),\end{aligned}\tag{2.15}$$

which is a *corrected* version of Apostol's identity [1, Eq. (15)] and, when $k=2$, reduces to (2.1).

Combining (2.12) and (2.13), Apostol [1] noted that the cases of $k=2$ and $k=3$ reduce to (2.1) and (2.2), respectively. However, it may not be easy to get (2.2) from (2.12) and (2.13). In fact, setting $k=3$ in (2.12) and (2.13) with (1.6) and (1.7), we obtain

$$\zeta(s)\left(1-3^{1-s}\right)=\sum_{n=1}^{\infty}\frac{(s)_n\,\zeta(n+s)}{n!\,3^{n+s}}\left(1+2^n\right),\tag{2.16}$$

which can be obtained directly by setting $k=3$ in (2.10) with (1.6) and (1.7). The Ramaswami's identity (2.2) here has been proved as a special case of Apostol's identity (2.14) when $k=3$. So it may be interesting to compare (2.2) with (2.16), and to prove their resulting series identity directly.

We conclude this note by presenting formulas for closed-form evaluation of series involving the Zeta function, the subject of which has an over two-century history as noted in Srivastava [7] and has been an object of many mathematicians' concern since then (see, e.g. [3], [4], [8]). First recall an integral representation for $\zeta(s)$ (*cf.* [5, p. 33]):

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + 2\int_0^\infty \left(1+t^2\right)^{-\frac{s}{2}} \sin(s \arctan t) \frac{dt}{e^{2\pi t}-1}, \tag{2.17}$$

whose integral part holds for all finite values of s.

By combining (2.17) with (2.10) and (2.11), we obtain

$$\sum_{n=2}^{\infty} \frac{\zeta(n)}{k^n} \frac{B_n(k) - B_n}{n} = \log k \quad (k \in \mathbb{N}); \tag{2.18}$$

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{k^n} \frac{B_n(k) - B_n}{n} = H_{k-1} - \log k \quad (k \in \mathbb{N}), \tag{2.19}$$

where H_k denotes the harmonic numbers defined by

$$H_k := \sum_{j=1}^{k} \frac{1}{j} \ (k \in \mathbb{N}) \quad \text{and} \quad H_0 = 0. \tag{2.20}$$

Adding and subtracting two formulas (2.18) and (2.19), we get

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{k^{2n+1}} \frac{B_{2n+1}(k)}{2n+1} = \log k - \frac{1}{2} H_{k-1} \quad (k \in \mathbb{N}); \tag{2.21}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{k^{2n}} \frac{B_{2n}(k) - B_{2n}}{n} = H_{k-1} \quad (k \in \mathbb{N}). \tag{2.22}$$

Some special cases of (2.18) and (2.19) are listed here:

$$\sum_{n=2}^{\infty} \frac{\zeta(n)}{2^n} = \log 2, \tag{2.23}$$

which is recorded in [8, Eq. 3.4(44)];

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{2^n} = 1 - \log 2, \tag{2.24}$$

which is also recorded in [8, Eq. 3.4(43)];

$$\sum_{n=2}^{\infty} \frac{\zeta(n)}{3^n} \left(1 + 2^{n-1}\right) = \log 3; \tag{2.25}$$

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{3^n} \left(1 + 2^{n-1}\right) = \frac{3}{2} - \log 3. \tag{2.26}$$

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