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POLYTOPES OF MINIMAL NULL DESIGNS

SOOJIN CHO

ABSTRACT. Null designs form a vector space and there are only finite number of minimal null designs(up to scalar multiple), hence it is natural to look at the convex polytopes of minimal null designs. For example, when $t = 0$, $k = 1$, the convex polytope of minimal null designs is the polytope of roots of type A_n. In this article, we look at the convex polytopes of minimal null designs and find many general properties on the vertices, edges, dimension, and some structural properties that might help to understand the structure of polytopes for big n, t through the structure of smaller n, t.

1. Introduction

For a given positive integer n, we let B_n be the Boolean algebra, i.e. the lattice of all subsets of a finite set $[n] = \{1, 2, \ldots, n\}$. For $\omega = \sum_{S \in B_n} \alpha_S S$, where $\alpha_S \in \mathbb{R}$, supp(ω) is defined to be the subset $\{S \mid \alpha_S \neq 0\}$ of B_n. For each $i = 0, 1, \ldots, n$, let us define the *fibers* X_i of B_n as $X_i = \{S \in B_n \mid |S| = i\}$. Then, for given integers $0 \leq t < k \leq n$, the vector space of *null* (t, k, n)*-designs* $N(t, k, n)$ is defined as follows:

$$N(t,k,n) = \{\omega = \sum_{S \in X_k} \alpha_S S \mid \alpha_S \in \mathbb{R}, \sum_{S \supset T} \alpha_S = 0 \text{ for any } T \in X_t\}.$$

We call a nonzero null (t, k, n)-design *minimal* if the size of its support is the minimal possible number.

Null designs have been considered in many different aspects [4, 6, 7]. Null designs are useful to understand designs or to construct new designs from a known one. They also deserve research as pure combinatorial object. Especially, people have been interested in the minimum of the

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support size of nonzero null designs and the characterization of minimal null designs. Minimal null designs were used to construct explicit bases of the space of null designs.

In [6], Frankl and Pach proved that the size of the support of nonzero null (t, k, n)-design is at least 2^{t+1}. Let us identify a subset $\{a_1, a_2, \ldots, a_l\}$ of $[n]$ with the monomial $x_{a_1}x_{a_2}\cdots x_{a_l}$ of l distinct indeterminates. Then

$$\omega_0 = (x_1 - x_2)(x_3 - x_4)\cdots(x_{2t+1} - x_{2t+2})x_{2t+3}\cdots x_{k+t+1}$$

is a null (t, k, n)-design whose support size is 2^{t+1}. In [5], Liebler and Zimmerman proved that for $k = t + 1$, ω_0 and its images under the natural permutation group action are the only null t-designs which have the support size 2^{t+1} up to scalar multiple. For general t and k, it was proved that the null t-designs of support size 2^{t+1} are of the form $(x_1-x_2)(x_3-x_4)\cdots(x_{2t+1}-x_{2t+2})x_{2t+3}\cdots x_{k+t+1}$ up to scalar multiple, if $n = k + t + 1$ [4].

Since null (t, k, n)-designs are linear combinations of k-subsets, with a chosen ordering on the set of k-subsets, they can be thought as vectors in $\mathbb{R}^{\binom{n}{k}}$. In $\mathbb{R}^{\binom{n}{k}}$, if we only consider the minimal null designs with coefficients $0, 1, -1$, then the vectors of minimal null designs are all on a sphere. Hence, it is natural to look at the convex polytopes of (vector representation of) minimal null designs. For example, when $t = 0$, $k = 1$, the convex polytope of minimal null designs is the polytope of roots of type A_n. Moreover, convex polytopes of minimal null designs are examples of Young orbit polytopes that are served as domains for many optimization problems.

In this article, we consider the convex hull of vectors of minimal null $(t, t+1, n)$-designs, and call it $P_{t,n}$. Note that we only consider the case $k = t + 1$. So, $P_{t,n}$ is a convex polytope in $\mathbb{R}^{\binom{n}{t+1}}$.

The special case $t = 0$ has been considered in [5] as a convex polytope of all roots of type A_n, since when $t = 0$ minimal null designs are of the form $x_i - x_j$ and these are exactly the roots of type A_n ([8]). If $t > 0$ then the set of minimal null designs does not form a root system of any type, we however believe that $P_{t,n}$ can be constructed from $P_{0,n}$'s with some basic operations. Hence, it is worthwhile to investigate the general properties of $P_{t,n}$.

Also, note that $P_{t,n}$ is called as the *Young orbit polytope* corresponding to the partition $\lambda = (n - t - 1, t + 1)$, which was introduced as a framework to many combinatorial optimization problems [10].

In Section 2, we state basics on convex polytopes, summarize known results on minimal null designs and polytopes $P_{0,n}$. In Section 3, we investigate general properties of $P_{t,n}$, about vertices, edges, dimensions, and some more structural properties. We believe that the results we prove will become a base to understand the full structure of $P_{t,n}$. In the final section we conclude with some remarks.

2. Preliminaries

In this section we give very basic definitions on convex polytopes, give the formal definition of convex polytopes $P_{t,n}$, and state the known results on the minimal null designs and the structure of $P_{0,n}$.

We refer to [1, 12] for detailed information on convex polytopes, while we give some basic definitions related to convex polytopes. For two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, we let $(\mathbf{u}, \mathbf{v})$ denote the usual inner product in $\mathbb{R}^d$. A *(convex) polytope* is the convex hull $\mathrm{Conv}(K) = \{\sum_{i=1}^{l} \lambda_i \mathbf{u}_i : \sum_i \lambda_i = 1, \lambda_i \geq 0\}$ of a finite set $K = \{\mathbf{u}_1, \ldots, \mathbf{u}_l\}$ in $\mathbb{R}^d$ for some d. The *dimension* of a polytope $\mathrm{Conv}(K)$ is the dimension of its *affine hull* $\{\sum_{i=1}^{l} \lambda_i \mathbf{u}_i : \sum_i \lambda_i = 1\}$, i.e. the size of the largest affinely independent subset of K subtracted by 1. A *face* of a polytope $P \in \mathbb{R}^d$ is any set of the form $F = P \cap \{\mathbf{x} \in \mathbb{R}^d : (\mathbf{c}, \mathbf{x}) = c_0\}$, when $(\mathbf{c}, \mathbf{x}) \leq c_0$ for all $\mathbf{x} \in P$. The *dimension* of a face is the dimension of its affine hull. P itself is a face with $(\mathbf{0}, \mathbf{x}) \leq 0$, and $\varnothing$ is a face given by $(\mathbf{0}, \mathbf{x}) \leq -1$. We call these faces *trivial.* A 0-dimensional face of a polytope is called a *vertex* and a 1-dimensional face is called an *edge.* A face F of a polytope P is called a *facet* if the dimension of F is one less than the dimension of P. Observe that faces are characterized as the subsets of P, whose elements maximize a given linear functional. In the definition, we can replace $\mathbf{c}$, c_0 by $-\mathbf{c}$ and $-c_0$ respectively, and change the inequality at the same time. Hence, the faces of a polytope is also characterized as subsets *minimizing* a linear functional.

We state the theorems in [6, 9] on the minimal null designs for future references.

PROPOSITION 1. *The minimum of the support size of nonzero null $(t, t+1, n)$-designs is 2^{t+1}.*

PROPOSITION 2. *If* $\omega \in N(t, t+1, n)$ *and* $|\text{supp}(\omega)| = 2^{t+1}$, *then* ω *is a multiple of* $(x_{i_1} - x_{i_2})(x_{i_3} - x_{i_4}) \cdots (x_{i_{2t+1}} - x_{i_{2t+2}})$ *for a* $(2t+2)$*-subset* $\{i_1, \dots, i_{2t+2}\}$ *of* $[n]$.

For $t < n$, fix a linear order on the set of $(t+1)$-subsets of $[n]$, then each minimal null $(t, t+1, n)$-design can be represented as a vector in $\mathbb{R}^{\binom{n}{t+1}}$ with exactly 2^{t+1} nonzero entries. We normalize each minimal null design vector by multiplying a *positive* scalar so that each nonzero entry of the vector is either $+1$ or -1. From now on, in this paper, by *minimal null* $(t, t+1, n)$*-vectors* (or just *minimal null vectors* if there is no confusion) we mean the normalized (vector representation of) minimal null designs i.e. the vector of minimal null $(t, t+1, n)$-designs whose nonzero entries are ± 1. We now can give a formal definition of the polytope $P_{t,n}$:

$$P_{t,n} = \text{Conv}(\{\mathbf{v} \mid \mathbf{v} \text{ is a minimal null } (t, t+1, n)\text{-vector}\}) \subset \mathbb{R}^{\binom{n}{t+1}}.$$

EXAMPLE 1. We consider the case $n = 4$ and $t = 1$. By Proposition 2, minimal null designs are multiples of $(x_{i_1} - x_{i_2})(x_{i_3} - x_{i_4})$ where $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$. Therefore, minimal null designs are multiples of one of the followings:

$$\begin{aligned}
\omega_1 &= (x_1 - x_2)(x_3 - x_4) = \{1,3\} + \{2,4\} - \{1,4\} - \{2,3\}, \\
\omega_2 &= (x_1 - x_3)(x_2 - x_4) = \{1,2\} + \{3,4\} - \{1,4\} - \{2,3\}, \\
\omega_3 &= (x_1 - x_4)(x_2 - x_3) = \{1,2\} + \{3,4\} - \{1,3\} - \{2,4\}.
\end{aligned}$$

We give a linear order on the set of 2-subsets of [4] in the following way:

$$\{1,2\} < \{1,3\} < \{1,4\} < \{2,3\} < \{2,4\} < \{3,4\}.$$

Then, the vector representation of ω_1 in $\mathbb{R}^6$ is $\mathbf{v}_1 = (0, 1, -1, -1, 1, 0)$, the vector representation of ω_2 is $\mathbf{v}_2 = (1, 0, -1, -1, 0, 1)$, and the vector representation of ω_3 is $\mathbf{v}_3 = (1, -1, 0, 0, -1, 1)$. Note that the number of nonzero entries of each vector are $2^2 = 4$.

Let ω be a given minimal null design. Then for some nonzero constant $c \in \mathbb{R}$ and $i \in \{1, 2, 3\}$, we have $\omega = c\omega_i$, and its normalized vector is $\mathbf{v}_i$ if $c > 0$, and $-\mathbf{v}_i$ if $c < 0$. Hence we have 6 minimal null vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, -\mathbf{v}_1, -\mathbf{v}_2, -\mathbf{v}_3$, and

$$P_{1,4} = \text{Conv}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, -\mathbf{v}_1, -\mathbf{v}_2, -\mathbf{v}_3\}) \subset \mathbb{R}^6.$$

The followings are theorems on $P_{0,n}$ proved in [5].

PROPOSITION 3. *The dimension of $P_{0,n}$ is $n-1$.*

Let ϵ_i be the ith elementary vector in $\mathbb{R}^n$. The following proposition characterizes all the faces of $P_{0,n}$ in a very natural way.

PROPOSITION 4. *Every m-dimensional ($m = 0, \ldots, n-2$) face of $P_{0,n}$ is given by the convex hull of the vectors in $\{\epsilon_i - \epsilon_j : i \in I, j \in J\}$ where I, J are disjoint non-empty subsets of $[n]$ such that $|I| + |J| = m + 2$.*

The following is an immediate corollary of Proposition 4.

COROLLARY 5. *There is a one to one correspondence between the set of non-trivial faces of $P_{0,n}$ and the set of ordered partitions of subsets of $[n]$ with two blocks, where the dimension of the face corresponding to (I, J) is $|I| + |J| - 2$.*

Propositions above show the explicit characterization of all faces of $P_{0,n}$. When $t > 0$, we however do not have a characterization of the faces of $P_{t,n}$. In the next section we investigate the properties of $P_{t,n}$ for general t, and try to find a way of characterizations of the faces of $P_{t,n}$ through the known results on $P_{0,n}$.

3. Polytopes $P_{t,n}$

In this section, we investigate some general properties of $P_{t,n}$.

LEMMA 6. *The number of minimal null $(t, t+1, n)$-vectors is*

$$\frac{n!}{2^t\ (t+1)!\ (n-2t-2)!}.$$

PROOF. This is immediate from Proposition 2 that the minimal null $(t, t+1, n)$-designs are of the form $(x_{i_1} - x_{i_2}) \cdots (x_{i_{2t+1}} - x_{i_{2t+2}})$. □

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_N\}$ be the set of minimal null $(t, t+1, n)$-vectors, where $N = \frac{n!}{2^t\ (t+1)!\ (n-2t-2)!}$. Remember that for vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^{\binom{n}{t+1}}$, $(\mathbf{u}, \mathbf{v})$ denote the usual inner product in $\mathbb{R}^{\binom{n}{t+1}}$. Note that $\text{supp}(\mathbf{v}_i) = (\mathbf{v}_i, \mathbf{v}_i) = 2^{t+1}$ for $i = 1, \ldots, N$, since $\mathbf{v}_i$ has exactly 2^{t+1} nonzero entries and the nonzero entries are ± 1.

LEMMA 7. *$(\mathbf{v}_i, \mathbf{v}_j) \le 2^t$ if $i \neq j$.*

PROOF. Suppose that $(\mathbf{v}_i, \mathbf{v}_j) > 2^t$, then $(\mathbf{v}_i - \mathbf{v}_j, \mathbf{v}_i - \mathbf{v}_j) = (\mathbf{v}_i, \mathbf{v}_i) + (\mathbf{v}_j, \mathbf{v}_j) - 2(\mathbf{v}_i, \mathbf{v}_j) < 2^{t+1} + 2^{t+1} - 2 \cdot 2^t = 2^{t+1}$. Since $N(t, t+1, n)$ is a vector space, $\mathbf{v}_i - \mathbf{v}_j \neq \mathbf{0}$ is (vector representation of) another null design with support size less than 2^{t+1}, and we have a contradiction to Proposition 1. Hence, $(\mathbf{v}_i, \mathbf{v}_j) \le 2^t$. □

LEMMA 8. *$\mathbf{v}_i - \mathbf{v}_j$ is another minimal null vector if and only if $(\mathbf{v}_i, \mathbf{v}_j) = 2^t$.*

PROOF. First of all, note that if $\mathrm{supp}(\mathbf{v}_i) \cap \mathrm{supp}(\mathbf{v}_j) \neq \varnothing$, then the product of coefficients of $S \in \mathrm{supp}(\mathbf{v}_i) \cap \mathrm{supp}(\mathbf{v}_j)$ in $\mathbf{v}_i$ and $\mathbf{v}_j$ is independent of the choice of S because of Proposition 2 and it is either $+1$ or -1. Hence by Lemma 7, if $i \neq j$ then $|\mathrm{supp}(\mathbf{v}_i) \cap \mathrm{supp}(\mathbf{v}_j)| \le 2^t$.

If $\mathbf{v}_i - \mathbf{v}_j$ is another minimal null vector then

$$(\mathbf{v}_i - \mathbf{v}_j, \mathbf{v}_i - \mathbf{v}_j) = 2 \cdot 2^{t+1} - 2 \cdot (\mathbf{v}_i, \mathbf{v}_j),$$

which should be 2^{t+1}. Therefore, we have $(\mathbf{v}_i, \mathbf{v}_j) = 2^t$.

For the converse, let us assume that $(\mathbf{v}_i, \mathbf{v}_j) = 2^t$. Then from the above calculation, $(\mathbf{v}_i - \mathbf{v}_j, \mathbf{v}_i - \mathbf{v}_j) = 2^{t+1}$. Since

$$|\mathrm{supp}(\mathbf{v}_i) - (\mathrm{supp}(\mathbf{v}_i) \cap \mathrm{supp}(\mathbf{v}_j))| \ge 2^{t+1} - 2^t = 2^t$$

and

$$|\mathrm{supp}(\mathbf{v}_j) - (\mathrm{supp}(\mathbf{v}_i) \cap \mathrm{supp}(\mathbf{v}_j))| \ge 2^t,$$

$\mathbf{v}_i - \mathbf{v}_j$ has at least $2 \cdot 2^t$ many ± 1's. Therefore we conclude that $\mathbf{v}_i - \mathbf{v}_j$ has exactly 2^{t+1} many nonzero entries which are ± 1, and it is a minimal null vector. □

The following theorem characterizes the vertices(0-dimensional faces) of $P_{t,n}$.

THEOREM 9. *In $P_{t,n}$, each $\mathbf{v}_i$ is a vertex. In other words, the set of vertices of $P_{t,n}$ is exactly the set of minimal null vectors.*

PROOF. For each $i = 1, \ldots, N$, let us define a linear functional $f : \mathbb{R}^{\binom{n}{t+1}} \to \mathbb{R}$ as $f(\mathbf{y}) = (\mathbf{v}_i, \mathbf{y})$. Then, $f(\mathbf{v}_i) = 2^{t+1}$ while $f(\mathbf{v}_j) < 2^{t+1}$ for $j \neq i$ by Lemma 7. Hence $\{\mathbf{v}_i\}$ is the subset of $P_{t,n}$ maximizing f when f is restricted to $P_{t,n}$. □

The following theorem gives information on the edges(1-dimensional faces) of $P_{t,n}$.

THEOREM 10. *If $\mathbf{v}_i - \mathbf{v}_j$, $i \neq j$, is another minimal null vector $\mathbf{v}_l$, then the line segment between $\mathbf{v}_i$ and $\mathbf{v}_j$ is an edge of $P_{t,n}$.*

PROOF. Let us define a linear functional $f : \mathbb{R}^{\binom{n}{t+1}} \to \mathbb{R}$ as $f(\mathbf{y}) = ((\mathbf{v}_i + \mathbf{v}_j), \mathbf{y})$. Then,

$$\begin{aligned} f(a\mathbf{v}_i + b\mathbf{v}_j) &= a(\mathbf{v}_i, \mathbf{v}_i) + b(\mathbf{v}_j, \mathbf{v}_j) + (a+b)(\mathbf{v}_i, \mathbf{v}_j) \\ &= (a+b)2^{t+1} + (a+b)2^t = 2^{t+1} + 2^t, \end{aligned}$$

for $a, b \geq 0$, $a + b = 1$, by Lemma 8. However, if $k \neq i, j$, then $f(\mathbf{v}_k) = (\mathbf{v}_k, \mathbf{v}_i) + (\mathbf{v}_k, \mathbf{v}_j) \leq 2^t + 2^t$. Hence the line segment between $\mathbf{v}_i$ and $\mathbf{v}_j$ is the subset of $P_{t,n}$ maximizing f on $P_{t,n}$. □

THEOREM 11. *The dimension of $P_{t,n}$ is $\binom{n}{t+1} - \binom{n}{t}$.*

PROOF. The algebraic dimension of the vector space of null designs is $\binom{n}{t+1} - \binom{n}{t}$ and there is a well known basis consisting of minimal null designs [4]. Hence, $P_{t,n} \subseteq \mathbb{R}^{\binom{n}{t+1}-\binom{n}{t}}$ and we only need to prove that there exists $\binom{n}{t+1} - \binom{n}{t} + 1$ subset of $\{\mathbf{v}_1, \ldots, \mathbf{v}_N\}$, which is affinely independent. We let $M = \binom{n}{t+1} - \binom{n}{t}$ and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_M$ be a linear basis of the vector space of null designs, and let $\mathbf{v}_{M+1} = -\mathbf{v}_1$.

Consider the sum $\sum_{i=1}^{M+1} \lambda_i \mathbf{v}_i = 0$ where $\sum_{i=1}^{M+1} \lambda_i = 0$. Rewriting this, we have $(\lambda_1 - \lambda_{M+1})\mathbf{v}_1 + \sum_{i=2}^{M} \lambda_i \mathbf{v}_i = 0$. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_M\}$ is a linearly independent set, $\lambda_1 = \lambda_{M+1}$ and $\lambda_i = 0$ for $i = 2, \ldots, M$. Now the condition $\sum_{i=1}^{M+1} \lambda_i = 0$ forces $\lambda_1 = \lambda_{M+1} = 0$. Hence $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_M, \mathbf{v}_{M+1}\}$ is an affinely independent set. □

The following theorem states that $P_{t,2t+2}$ is exactly the same polytope as $P_{t-1,2t+1}$. For example, if we let $t = 1$, then $P_{1,4}$ is an isomorphic polytope to $P_{0,3}$ about which we already know the complete face structure by Proposition 4.

THEOREM 12. *$P_{t,2t+2}$ is affinely isomorphic to $P_{t-1,2t+1}$ when $t \geq 1$. In other words, there is an affine map $\Phi : \mathbb{R}^{\binom{2t+2}{t+1}} \to \mathbb{R}^{\binom{2t+1}{t}}$ that is a bijection between the points of two polytopes $P_{t,2t+2}$ and $P_{t-1,2t+1}$.*

We might want to check some necessary conditions for two polytopes to be isomorphic. First of all, the number of vertices of $P_{t,2t+2}$ is equal to the number of vertices of $P_{t-1,2t+1}$:

$$\frac{(2t+2)!}{2^t\,(t+1)!\,(2t+2-2t-2)!} = \frac{(2t+1)!}{2^{t-1}\,t!}\,.$$

Moreover,

$$\begin{aligned}\dim(P_{t,\,2t+2}) &= \tbinom{2t+2}{t+1} - \tbinom{2t+2}{t} \\ &= \tbinom{2t+1}{t} - \tbinom{2t+1}{t-1} \quad = \dim(P_{t-1,\,2t+1}).\end{aligned}$$

For the proof of Theorem 12, we need to introduce some definitions and some known results on null designs. We refer to [11, 3] for more detailed information.

Let $(\lambda_1, \ldots, \lambda_l)$ be a partition of n, then a *standard tableau of shape* λ is a λ-tableau whose columns and rows form increasing sequences.

For null $(t, t+1, n)$-designs, we only need tableaux of shape $(n-t-1, t+1)$. Let T be a tableau of shape $(n-t-1, t+1)$ given as follows:

$$T = \begin{matrix} i_1 & i_3 & \cdots & i_{2t+1} & i_{2t+3} & \cdots i_n \\ i_2 & i_4 & \cdots & i_{2t+2} & & \end{matrix}$$

Then, the corresponding null $(t, t+1, n)$-design is defined as

$$e(T) = (x_{i_1} - x_{i_2}) \cdots (x_{i_{2t+1}} - x_{i_{2t+2}}).$$

REMARK 1. Note that exchanging two columns in the first $t+1$ columns of T does not change the value of e, and exchanging the two numbers in the same column in the first $t+1$ columns of T change the value of e by the sign.

The following is a well known result on the representation theory of the symmetric group [11].

LEMMA 13. *The set $\{e(T) \mid T$ is a standard tableau of shape $(n-t-1, t+1)\}$ is a linear basis of the space $N(t, t+1, n)$ of null $(t, t+1, n)$-designs.*

Let us call the basis given in the above lemma *GLL basis*, where GLL stands for R. L. Graham, S. -Y. R. Li and W. -C. W. Li. The following relation gives an algorithm to write a non basis element of $N(t, t+1, n)$ as a linear combination of GLL basis elements.

Garnir relation: Assume that $\omega = (x_{i_1} - x_{i_2}) \cdots (x_{2t+1} - x_{2t+2})$ is not a GLL basis element. Then $\omega = e(T)$ with some non standard tableau

$$T = \begin{matrix} i_1 & \cdots & i_{2j+1} & i_{2j+3} & \cdots & i_{2t+1} & i_{2t+3} & \cdots & i_n \\ i_2 & \cdots & i_{2j+2} & i_{2j+4} & \cdots & i_{2t+2}. \end{matrix}$$

We, however, always can assume that T has increasing columns and the increasing second row by Remark 1. Hence we assume that T has increasing columns and the increasing second row. Then, there must be j such that $i_{2j+1} > i_{2j+3}$. Let

$$T_1 = \begin{matrix} i_1 & \cdots & i_{2j+1} & i_{2j+2} & \cdots & i_{2t+1} & i_{2t+3} & \cdots & i_n \\ i_2 & \cdots & i_{2j+3} & i_{2j+4} & \cdots & i_{2t+2} \end{matrix}$$

and

$$T_2 = \begin{matrix} i_1 & \cdots & i_{2j+3} & i_{2j+1} & \cdots & i_{2t+1} & i_{2t+3} & \cdots & i_n \\ i_2 & \cdots & i_{2j+2} & i_{2j+4} & \cdots & i_{2t+2} \end{matrix}$$

Then, $e(T) - e(T_1) - e(T_2) = 0$. □

LEMMA 14. (Garnir relation [11]) *The way to write down non GLL basis elements of $N(t, t+1, n)$ as linear sums of GLL basis elements is given by the successive use of Garnir relation.*

We are now ready to finish the proof of Theorem 12.

PROOF OF THEOREM 12. We first define a bijection ϕ between the set of vertices in $P_{t,2t+2}$ and the set of vertices in $P_{t-1,2t+1}$. Let $(x_{i_1} - x_{i_2}) \cdots (x_{i_{2t+1}} - x_{i_{2t+2}})$ be a vertex in $P_{t,2t+2}$, then note that $\{i_1, \ldots, i_{2t+2}\} = [2t+2]$, since $n = 2t+2$. We also may assume that $2t+2 \in \{i_{2t+1}, i_{2t+2}\}$. Let

$$\phi((x_{i_1} - x_{i_2}) \cdots (x_{i_{2t+1}} - x_{i_{2t+2}})) = \begin{cases} (x_{i_1} - x_{i_2}) \cdots (x_{i_{2t-1}} - x_{i_{2t}}) & \text{if} \quad i_{2t+2} = 2t+2 \\ -(x_{i_1} - x_{i_2}) \cdots (x_{i_{2t-1}} - x_{i_{2t}}) & \text{if} \quad i_{2t+1} = 2t+2. \end{cases} \tag{1}$$

Then it is obvious that ϕ has its inverse

$$\phi^{-1}((x_{i_1} - x_{i_2}) \cdots (x_{i_{2t-1}} - x_{i_{2t}})) = (x_{i_1} - x_{i_2}) \cdots (x_{i_{2t-1}} - x_{i_{2t}})(x_{i_{2t+1}} - x_{2t+2})$$

where $\{i_{2t+1}\} = [2t+1] - \{i_1, \ldots, i_{2t}\}$. Hence ϕ is a bijection between the sets of vertices. Now, it would be enough to show that ϕ is actually a restriction of a linear map $\Phi : \mathbb{R}^{\binom{2t+2}{t+1}} \longrightarrow \mathbb{R}^{\binom{2t+1}{t}}$ on the subspace $N(t, t+1, 2t+2)$ and ϕ is nonsingular.

It is easy to see that ϕ sends the GLL basis elements of $N(t, t+1, 2t+2)$ to the GLL basis elements of $N(t-1, t, 2t+1)$. Hence, we can define

Φ on the subspace $N(t, t+1, 2t+2)$ of null designs as a linear map sending GLL basis elements to GLL basis elements of $N(t-1, t, 2t+1)$ through ϕ. We also can extend the GLL basis of $N(t, t+1, 2t+2)$ to the basis of $\mathbb{R}^{\binom{2t+2}{t+1}}$, and extend Φ to the whole space. What we have left to show now is ϕ is the restriction of Φ, that is, for non basis null designs, the image by ϕ is the one given by Φ. By Lemma 14, it will be sufficient to show that the Garnir relation is preserved as we apply ϕ. Let $e(T) - e(T_1) - e(T_2) = 0$ be a Garnir relation in the subspace of null $(t, t+1, 2t+2)$-designs. Then $2t+2$ is always on the last column of T, T_1, T_2, hence by applying ϕ we only drop the last column. That is $\phi(e(T) - e(T_1) - e(T_2)) = e(T') - e(T_1') - e(T_2')$, where T' is obtained from T by dropping $2t+2$. It is obvious that $e(T') - e(T_1') - e(T_2')$ is a Garnir relation on the subspace spanned by minimal null $(t-1, t, 2t+1)$-designs. □

Theorem 12 gives a way to understand $P_{t,n}$ through the structure of polytopes of smaller t, n's for some selective values t, n. The following two observations lead us to try to construct $P_{t,n}$ using polytopes with smaller t, n's.

PROPOSITION 15.

$$\dim(P_{t+1,n+1}) = \dim(P_{t,n}) + \dim(P_{t+1,n}).$$

PROOF. This is immediate from Theorem 11. □

We let $v(t, n)$ denote the number of vertices of the polytope $P_{t,n}$, i. e. the number of null $(t, t+1, n)$-designs. Then, we have the following relation on $v(t, n)$'s.

PROPOSITION 16.

$$v(t+1, n+1) = (n-2t-2)v(t, n) + v(t+1, n).$$

PROOF. This is immediate from Lemma 6 and Theorem 9. □

4. Remarks

1. Our final goal of research on the polytopes $P_{t,n}$ is to characterize all faces of each polytope in a very natural way as we did for $P_{0,n}$ in [5]. Theorem 12, Proposition 15, and Proposition 16 give us ways to study $P_{t,n}$ through the study of smaller t, n. We believe

that there must be a natural operation between two polytopes of minimal null designs to construct another in accordance with our theorems.

2. We believe that $P_{t,n}$ can be used as a domain for optimization problems also. Although we do not deal with this matter in this paper, it would be interesting to consider $P_{t,n}$ from this point of view ([2, 10]).

References

[1] A. BrØndsted, *An Introduction to Convex Polytopes*, Graduate Texts in Mathematics, no. 90, Springer-Verlag, New York, 1983.

[2] A. I. Barvinok and A. M. Vershik, *Convex hulls of orbits of representations of finite groups and combinatorial optimization*, Funktsional. Anal. i Prilozhen. **22** (1988), 66–67.

[3] S. Cho, *Algebraic structure of null designs*, RIMS Kokyuroku **991** (1997), 180–185

[4] ———, *Minimal null designs and a density theorem of posets*, European J. Combin. **19** (1998), 433–440.

[5] ———, *Polytopes of roots of type A_n*, Bulletin of the Australian Mathematical Society **59** (1999), 391–402.

[6] P. Frankl and J. Pach, *On the number of sets in a null t-design*, European J. Combin. **4** (1983), 21–23.

[7] R. L. Graham, S. -Y. R. Li, and W. -C. W. Li, *On the structure of t-designs*, SIAM J. Alg. Disc. Math. **1**(1980), 8–14.

[8] J. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics 29, Cambridge University Press, Cambridge 1990.

[9] R. A. Liebler and K. H. Zimmermann, *Combinatorial S_n-modules as codes*, J. Algebraic Combin. **4**(1995), 47–68.

[10] S. Onn, *Geometry, complexity, and combinatorics of permutation polytopes*, J. Combin. Theory Ser. A **64** (1993), 31–49

[11] B. E. Sagan, *The symmetric group*, Wadsworth & Brooks/Cole Mathematics Series, Wadsworth, Inc., 1991.

[12] G. M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics 152, Springer-Verlag New York, 1995.

Soojin Cho
Department of Applied Mathematics
Sejong University
Seoul 143-747, Korea
E-mail: sjcho@sejong.ac.kr