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A CENTRAL LIMIT THEOREM FOR THE STATIONARY MULTIVARIATE LINEAR PROCESS GENERATED BY ASSOCIATED RANDOM VECTORS

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Abstract. A central limit theorem is obtained for a stationary multivariate linear process of the form $\mathbb{X}_t = \sum_{u=0}^{\infty} A_u \mathbb{Z}_{t-u}$, where $\{\mathbb{Z}_t\}$ is a sequence of strictly stationary m-dimensional associated random vectors with $E\mathbb{Z}_t = \mathbb{O}$ and $E\|\mathbb{Z}_t\|^2 < \infty$ and $\{A_u\}$ is a sequence of coefficient matrices with $\sum_{u=0}^{\infty} \|A_u\| < \infty$ and $\sum_{u=0}^{\infty} A_u \neq O_{m\times m}$.

1. Introduction and main result

A finite sequence $\{Y_1, \cdots, Y_m\}$ of random variables is said to be associated if for the coordinatewise increasing functions $f, g : \mathbb{R}^m \to \mathbb{R}$

$$\mathrm{Cov}(f(Y_1, \cdots, Y_m), g(Y_1, \cdots, Y_m)) \geq 0, \tag{1}$$

where the covariance is defined. An infinite family of random variables is associated if every finite subfamily is associated. This prevalent positive dependence notion was first defined by Esary, Proschan and Walkup (1967).

Let $\{\mathbb{X}_t,\ t = 0, \pm 1, \cdots\}$ be an m-variate linear process of the form

$$\mathbb{X}_t = \sum_{u=0}^{\infty} A_u \mathbb{Z}_{t-u} \tag{2}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$, where $\{\mathbb{Z}_t\}$ is a sequence of stationary m-variate associated random vectors (see Definition 2.1 below) with $E\mathbb{Z}_t = \mathbb{O}$, $E\|\mathbb{Z}_t\|^2 < \infty$ and positive definite covariance matrix

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$\Gamma : m \times m$. Throughout this paper we shall assume that

$$\sum_{u=0}^{\infty} \|A_u\| < \infty \text{ and } \sum_{u=0}^{\infty} A_u \neq \mathbb{O}_{m\times m}, \tag{3}$$

where for any $m \times m$, $m \geq 1$, matrix $A = (a_{ij})$, $\|A\| = \sum_{i=1}^{m}\sum_{j=1}^{m} |a_{ij}|$ and $\mathbb{O}_{m\times m}$ denotes the $m \times m$ zero matrix. Let

$$T = \left(\sum_{j=0}^{\infty} A_j\right) \Gamma \left(\sum_{j=0}^{\infty} A_j\right)', \tag{4}$$

where the prime denotes transpose, and the matrix $\Gamma = [\sigma_{kj}]$ with

$$\sigma_{kj} = E(Z_{1k}Z_{1j}) + \sum_{t=2}^{\infty} \left(E(Z_{1k}Z_{tj}) + E(Z_{1j}Z_{tk})\right). \tag{5}$$

Further, let $\mathbb{S}_n = \sum_{t=1}^{n} \mathbb{X}_t, (n \geq 1)(\mathbb{S}_0 = \mathbb{O})$.

Fakhre-Zakeri and Lee (1993) proved a central limit theorem for multivariate linear processes generated by independent multivariate random vectors and Fakhre-Zakeri and Lee (2000) also derived a functional central limit theorem for multivariate linear processes generated by multivariate random vectors with martingale difference sequence.

In this paper we prove a central limit theorem for an m-variate linear process generated by stationary m-variate associated random vectors.

THEOREM 1.1. *Let $\{\mathbb{Z}_t, t = 0, \pm 1, \cdots\}$ be a strictly stationary associated sequence of m-dimensional random vectors with $E(\mathbb{Z}_t) = \mathbb{O}$, $E\|\mathbb{Z}_t\|^2 < \infty$ and positive definite covariance matrix Γ as in (5). Let $\{\mathbb{X}_t\}$ be an m-variate linear process defined as in (2). Assume that*

$$E\|\mathbb{Z}_1\|^2 + 2\sum_{t=2}^{\infty}\sum_{i=1}^{m} \mathrm{Cov}(Z_{1i}, Z_{ti}) = \sigma^2 < \infty. \tag{6}$$

Then, the multivariate linear process $\{\mathbb{X}_t\}$ fulfills the central limit theorem, that is,

$$n^{-\frac{1}{2}}\mathbb{S}_n \xrightarrow{\mathcal{D}} N(\mathbb{O}, T), \tag{7}$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution and $N(\mathbb{O}, T)$ indicates a normal distribution with mean zero vector and covariance matrix T defined in (4).

2. Proofs

DEFINITION 2.1 (Burton, et. al. 1986). A finite family $\{Z_1, \cdots, Z_n\}$ of m-variate random vectors is said to be associated if for all coordinatewise increasing functions $f, g : R^{mn} \longrightarrow R$ $\mathrm{Cov}(f(Z_1, \cdots, Z_n), g(Z_1, \cdots, Z_n)) \geq 0$ where the covariance is defined. A infinite family of m-variate random vectors is associated if every finite subfamily is associated.

Note that Newman (1980) has proved the central limit theorem for associated random variables (See Theorem 2 of [6]). Thus by means of the simple device due to Cramer Wold the following result holds.

LEMMA 2.1. *Let $\{\mathbb{Z}_t\}$ be a sequence of stationary associated m-variate random vectors with $E(Z_t) = \mathbb{O}$ and $E\|\mathbb{Z}_t\|^2 < \infty$. If (6) holds then*

$$n^{-\frac{1}{2}} \sum_{t=1}^{n} \mathbb{Z}_t \xrightarrow{\mathcal{D}} N(\mathbb{O}, \Gamma),$$

where $\Gamma = [\sigma_{kj}]$ is defined as in (5); that is, $\{\mathbb{Z}_t\}$ satisfies the central limit theorem.

LEMMA 2.2. *Let $\{\mathbb{Z}_t\}$ be a sequence of stationary associated random vectors with $E(\mathbb{Z}_t) = \mathbb{O}$, $E\|\mathbb{Z}_t\|^2 < \infty$. Let $\tilde{\mathbb{X}}_t = (\sum_{j=0}^{\infty} A_j)\mathbb{Z}_t$ and $\tilde{\mathbb{S}}_k = \sum_{t=1}^{k} \tilde{\mathbb{X}}_t$. Assume that (6) holds. Then*

$$n^{-\frac{1}{2}} \max_{1 \leq k \leq n} \|\tilde{\mathbb{S}}_k - \mathbb{S}_k\| = o_p(1). \tag{8}$$

PROOF. See Appendix. □

PROOF OF THEOREM 1.1 As in Lemma 2.2, set $\tilde{\mathbb{X}}_t = (\sum_{j=0}^{\infty} A_j)\mathbb{Z}_t$ and $\tilde{\mathbb{S}}_n = \sum_{t=1}^{n} \tilde{\mathbb{X}}_t$. First note that

$$\begin{aligned} & E\|\tilde{\mathbb{X}}_1\|^2 + 2\sum_{t=2}^{\infty}\sum_{i=1}^{m} E(\tilde{X}_{1i}\tilde{X}_{ti}) \\ &= \Big(\sum_{j=1}^{\infty} A_j\Big)^2 \Big(E\|\mathbb{Z}_1\|^2 + 2\sum_{t=2}^{\infty}\sum_{i=1}^{m} E(Z_{1i}Z_{ti})\Big) \\ &= \Big(\sum_{j=1}^{\infty} A_j\Big)^2 \sigma^2. \end{aligned} \tag{9}$$

Since $\tilde{\mathbb{X}}_t$ is associated by Lemma 2.1 $\{\tilde{\mathbb{X}}_t\}$ satisfies the central limit theorem, that is,

$$n^{-\frac{1}{2}}\tilde{\mathbb{S}}_n \xrightarrow{\mathcal{D}} N(\mathbb{O}, T), \tag{10}$$

where T is defined as in (4). According to Lemma 2.2 we also have

$$n^{-\frac{1}{2}}|\tilde{\mathbb{S}}_n - \mathbb{S}_n| = o_p(1). \tag{11}$$

Hence from (10) and (11) the desired conclusion follows by Theorem 4.1 of Billingsley (1968). □

Appendix

To prove Lemma 2.2 we use the ideas in the proof of Lemma 3 of [5] and apply Newman and Wrights' inequality instead of Doob's maximal inequality.

PROOF OF LEMMA 2.2. First observe that

$$\begin{aligned}\tilde{S}_k &= \sum_{t=1}^{k}\left(\sum_{j=0}^{k-t} A_j\right)\mathbb{Z}_t + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} A_j\right)\mathbb{Z}_t \\ &= \sum_{t=1}^{k}\left(\sum_{j=0}^{t-1} A_j\mathbb{Z}_{t-j}\right) + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} A_j\right)\mathbb{Z}_t\end{aligned}$$

and thus,

$$\begin{aligned}\tilde{S}_k - S_k &= -\sum_{t=1}^{k}\sum_{j=t}^{\infty} A_j\mathbb{Z}_{t-j} + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} A_j\right)\mathbb{Z}_t \\ &= I + II \text{ (say)}.\end{aligned}$$

To prove

$$n^{-\frac{1}{2}} \max_{1<k<n} \|I\| = o_p(1), \tag{A.1}$$

consider that

$$
\begin{aligned}
& n^{-1} E \max_{1\le k\le n} \left\| \sum_{t=1}^{k} \sum_{j=t}^{\infty} A_j \, \mathbb{Z}_{t-j} \right\|^2 \\
= & \; n^{-1} \, E \max_{1\le k\le n} \left\| \sum_{j=1}^{\infty} \sum_{t=1}^{j\wedge k} A_j \, \mathbb{Z}_{t-j} \right\|^2 \\
\le & \; n^{-1} \left(\sum_{j=1}^{\infty} \|A_j\| \left\{ E \max_{1\le k\le n} \left\| \sum_{t=1}^{j\wedge k} \mathbb{Z}_{t-j} \right\|^2 \right\}^{\frac{1}{2}} \right)^2 \\
\le & \; \sigma^2 \left[\sum_{j=1}^{\infty} \|A_j\| \left(\frac{j\wedge k}{n} \right)^{\frac{1}{2}} \right]^2 ,
\end{aligned}
$$

where we have used Newman and Wrights' maximal inequality for associated sequence (See Theorem 2. of [7]) and (6). The first inequality above is obtained by Minkowski's inequality and by the dominated convergence theorem the last term above tends to zero as $n \longrightarrow \infty$. Thus (A.1) is proved by Markov inequality.

Next, we show that

$$
(A.2) \qquad n^{-\frac{1}{2}} \max_{1\le k\le n} \|II\| \;=\; o_p(1).
$$

Write

$$
II = II_1 + II_2,
$$

where

$$
II_1 = A_1\mathbb{Z}_k + A_2(\mathbb{Z}_k + \mathbb{Z}_{k-1}) + \cdots + A_k(\mathbb{Z}_k + \cdots + \mathbb{Z}_1)
$$

and

$$
II_2 = (A_{k+1} + A_{k+2} + \cdots)\ (\mathbb{Z}_k + \cdots + \mathbb{Z}_1).
$$

Let p_n be sequence of positive integers such that

$$
(A.3) \qquad p_n \to \infty \text{ and } p_n/n \to 0 \text{ as } n \to \infty.
$$

Note that

$$\begin{aligned} n^{-\frac{1}{2}} \max_{1\le k\le n} \|II_2\| &\le \sum_{i=0}^{\infty} \|A_i\|\, n^{-\frac{1}{2}} \max_{1\le k\le p_n} \|\mathbb{Z}_1+\cdots+\mathbb{Z}_k\| \\ &\quad +\Big(\sum_{i>p_n} \|A_i\|\Big)\, n^{-\frac{1}{2}} \max_{1\le k\le n} \|\mathbb{Z}_1+\cdots+\mathbb{Z}_k\| \\ &= O_p\left(\frac{p_n}{n}\right)+O_p\Big(\sum_{i>p_n} \|A_i\|\Big) \\ &= o_p(1), \end{aligned}$$

by Newman and Wrights' maximal inequality, (3) and (A.3). It remains to prove that

$$Y_n \;:= n^{-\frac{1}{2}} \max_{1\le k\le n} \|II_1\| \;=\; o_p(1).$$

To this end, define for each $l \ge 1$

$$II_{1,l} = \; B_1\mathbb{Z}_k + B_2(\mathbb{Z}_k+\mathbb{Z}_{k-1}) + \cdots + B_k(\mathbb{Z}_k+\cdots+\mathbb{Z}_1),$$

where

$$B_k = \begin{cases} A_k, & k \le l \\ O_{m\times m}, & k > l. \end{cases}$$

Let

$$Y_{n,l} = n^{-\frac{1}{2}} \max_{1\le k\le n} \|II_{1,l}\|.$$

Clearly, for each $l \ge 1$,

$$(A.4) \qquad\qquad Y_{n,l} = o_p(1).$$

On the other hand,

$$
\begin{aligned}
n(Y_{n,l} - Y_n)^2 &\le \max_{1\le k\le n} \Big\| \sum_{i=1}^{k} (A_i - B_i)\ (\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}) \Big\|^2 \\
&= \max_{1<k\le n} \Big(\sum_{i=l+1}^{k} \|A_i\|\ \|\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}\| \Big)^2 \\
&\le \Big(\sum_{i>l} \|A_i\| \Big)^2 \max_{l<k\le n}\ \max_{l\le i\le k} \|\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}\|^2 \\
&\le \Big(\sum_{i>l} \|A_i\| \Big)^2 \max_{l\le k\le n}\ \max_{l<i\le k} (\|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| \\
&\qquad + \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_{k-i}\|)^2 \\
&\le \Big(\sum_{i>l} \|A_i\| \Big)^2 (2 \max_{l<k\le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\|^2 \\
&\qquad + 2 \max_{l<k\le n}\ \max_{l<i\le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_{k-i}\|^2) \\
&\le 4\Big(\sum_{i>l} \|A_i\| \Big)^2 \max_{l\le i\le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_i\|^2.
\end{aligned}
$$

From this result (3) and (6), for any $\delta > 0$,

$$
\begin{aligned}
&\lim_{l\to\infty} \limsup_{n\to\infty}\ P(|Y_{n,l} - Y_n|^2 > \delta) \\
&\le \lim_{l\to\infty} \limsup_{n\to\infty}\ 4\delta^{-2}\Big(\sum_{i>l} \|A_i\| \Big)^2\ n^{-1}E \\
&\quad \max_{1\le i\le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_i\|^2 \\
&\le 4\delta^{-2}\sigma^2 \lim_{l\to\infty} \Big(\sum_{i>l} \|A_i\| \Big)^2 = 0.
\end{aligned}
\tag{A.5}
$$

In view of (A.4) and (A.5), it follows from Theorem 4.2 of Billingsley (1968, p.25) that $Y_n = o_p(1)$. This completes the proof of Lemma 2.2. □

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