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ON THE LARGE DEVIATION PROPERTY OF RANDOM MEASURES ON THE d-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. We give a formulation of the large deviation property for rescalings of random measures on the d-dimensional Euclidean space R^d. The approach is global in the sense that the objects are Radon measures on R^d and the dual objects are the continuous functions with compact support. This is applied to the cluster random measures with Poisson centers, a large class of random measures that includes the Poisson processes.

1. Introduction and preliminaries

Cramèr presented the first large deviation property at a probability symposium in 1937. Since 1937, this property has undergone an extensive development and this original work was extended in various directions. There have been many developments in the large deviation property over the last two decades. For more details about the large deviation property, the reader is referred to Ellis (1985), Deuschel and Stroock (1989).

Let B^d denote the collection of Borel subsets of d-dimensional Euclidean space R^d. The space $M(R^d)$ (denoted by M, later)of all non-negative measures defined on (R^d, B^d) and finite on bounded sets (i.e., Radon measures) will be equipped with the smallest σ-algebra $\mathcal{M}$ containing basic sets of the form $\{\mu \in M : \mu(B) \leq k\}$ for a bounded Borel set $B \in B^d$ and $0 < k < \infty$.

A random measure X is a measurable mapping from a fixed probability space (Ω, Σ, P) into $(M, \mathcal{M})$. The induced measure $P_X = P \circ X^{-1}$ on $(M, \mathcal{M})$ is the distribution of X. If X is a random measure and

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$B \subseteq R^d$ is a Borel subset, then we let $X(B)$ be the random amounts of mass the measure X gives the set B.

For $X \in M$ and a real measurable function f on R^d, we also define the integral $X(f)$ by

$$X(f) = \int_{R^d} f dX = \int_{R^d} f(x) X(dx)$$

if the integral on the right exists.

Now, we define

$$X_r(A) = X(rA) \quad \text{for a bounded Borel subset } A \text{ of } R^d$$

and

$$X_r(f) = X(f_r), \quad \text{where } f_r(x) = f(x/r).$$

We denote the collection of Radon measures on R^d by M, which is endowed with the weak topology. Let $M[0,L]^d$ (denoted by $\bar{M}$, later) be the restriction of measures in M to $[0,L]^d$. In other words, $M(R^d)$ maps onto $M([0,L]^d)$ by restriction. Let X be a random measure in M and let X^L(denoted by $\bar{X}$, later) be a random measure in $\bar{M}$ obtained by restricting X to $[0,L]^d$. We denote $\bar{X}_r = (X_r)^L$.

For the random measure $\bar{X}_r$, if $E[\bar{X}_r[0,L]^d] = r^d\delta$ and $\delta = E[\bar{X}[0,L]^d]$ is the intensity of X, the ergodic theorem implies that $\bar{X}_r/r^d \to \delta|\cdot|$, where $|\cdot|$ denotes the Lebesgue measure on R^d.

In this paper, we consider the large deviation property for rescalings of random measures on R^d. We are interested in estimates of deviations of $\bar{X}_r/r^d$ from $\delta|\cdot|$.

The dual objects to random measures are continuous functions with compact support. Functional approaches to the large deviation property using test functions as dual objects to random measures are developed. Also, this property is applied to some important classes of models, i.e., Poisson point process and Poisson center cluster random measure.

Let $\bar{M}$ be the set of finite measures on $[0,L]^d$ with the weak topology.

K_c is the set of nonnegative and continuous functions defined on $[0,L]^d$.

DEFINITION 1.1. We say that a function $I : \bar{M} \to [0,\infty]$ is a rate function if

(a) $I(\cdot)$ is lower semicontinuous on $\bar{M}$, and
(b) $I(\cdot)$ has compact level sets, i.e., for each real number $l < \infty$ the level sets $K_l = \{\mu \in \bar{M} | I(\mu) \le l\}$ are compact in M.

Now we define the cumulant generating functional of $\bar{X}_r$ as follows: $\Phi_{\bar{X}_r}(f) = \frac{1}{r^d} \cdot \log E[e^{\bar{X}_r(f)}]$ for each $f \in K_c$.

We shall assume here that

(a) Each function $\Phi_{\bar{X}_r}(f)$ is finite for each $f \in K_c$ and

(b) $\Phi(f) = \lim_{r \to \infty} \Phi_{\bar{X}_r}(f)$ exists and is also finite for $f \in K_c$.

The Cramer-Fenchel transform $I(\mu)$ is defined as the convex conjugate of $\Phi(f)$ by

$$I(\mu) = \sup_{f \in K_c} \{\mu(f) - \Phi(f)\}.$$

DEFINITION 1.2. The sequence $\left\{ \frac{\bar{X}_r}{r^d} : r \in R^+ \right\}$ is said to satisfy a large deviation property with a rate function $I(\cdot)$ if the following holds:

(a) $I(\cdot)$ is a rate function.

(b) (Large deviation upper bound): for all closed subsets $F \in \bar{M}$,

$$\limsup_{r \to \infty} \frac{1}{r^d} \cdot \log P\left(\frac{\bar{X}_r}{r^d} \in F \right) \leq - \inf_{\mu \in F} I(\mu).$$

(c) (Large deviation lower bound): for all open subsets $G \in \bar{M}$,

$$\liminf_{r \to \infty} \frac{1}{r^d} \cdot \log P\left(\frac{\bar{X}_r}{r^d} \in G \right) \geq - \inf_{\mu \in G} I(\mu).$$

2. Large deviation upper bound and lower bound

DEFINITION 2.1. Following Deuschel and Stroock [4], we sat that a family of distribution $\{P_r : r > 0\}$ satisfies large deviation tightness if, for each $a < \infty$, there exists a compact set K_a such that

$$\limsup_{r \to \infty} \frac{1}{r^d} \cdot \log P_r({K_a}^c) \leq -a.$$

Let us recall the notation $\bar{X}_r$ and then let P_r be the distribution of $\frac{\bar{X}_r}{r^d}$. A weak compactness argument shows the following.

LEMMA 2.2. *If $\frac{\bar{X}_r}{r^d}[0, L]^d$ satisfies large deviation tightness as random variables, then P_r also satisfies large deviation tightness.*

In order to show that the large deviation upper bound holds, i.e. for all closed subsets F in $\bar{M}$,

$$\limsup_{r\to\infty} \frac{1}{r^d} \cdot \log P\left(\frac{\bar{X}_r}{r^d} \in F\right) \le - \inf_{\mu\in F} I(\mu),$$

it is consequence of Theorem 4.5.3 of Dembo and Zeitouni [3] by applying the fact that the large deviation upper bound holds for any compact subset K in $\bar{M}$ and large deviation tightness.

Now we obtain the large deviation lower bound estimate as follows. For each open subset G in $\bar{M}$,

$$\liminf_{r\to\infty} \frac{1}{r^d} \cdot \log P\left(\frac{\bar{X}_r}{r^d} \in G\right) \ge - \inf_{\mu\in G} I(\mu).$$

We need two assumptions.

Assumption A : If $\{f_n\}$ is a sequence of uniformly bounded measurable functions on R^d, which converges μ-almost everywhere to a bounded measurable function f, then $\Phi(f_n) \to \Phi(f)$, i.e.,Φ is continuous with respect to f.

Assumption B : The large deviation lower bound holds for in the sense of the finite dimensional distribution.

Let us consider a random vector $\underline{x} = (\mu(A_1), \cdots, \mu(A_n))$ for disjoint and bounded Borel subsets $A_1, ..., A_n$ of R^d. The moment generating function of the random vector $\underline{x}$ is defined by

$$M_n(\underline{t}) = E[e^{<\underline{t}\cdot\underline{x}>}] = E\Big[\exp\Big(\sum_{i=1}^{n} t_i\mu(A_i)\Big)\Big] \quad \text{for all } \underline{t} = (t_1, \cdots, t_n).$$

We define $\Phi(\underline{t}) = \lim_{n\to\infty} \frac{1}{n} \cdot \log M_n(\underline{t})$. In general, we define a rate function $I(\underline{x})$ for a random vector by $I(\underline{x}) = \sup_{\underline{t}\in R^n} \{< \underline{t}, \underline{x} > -\Phi(\underline{t})\}$.

LEMMA 2.3. *Under the* Assumption A, *the following property holds:*

$$\sup_{f\in K_c} \{\mu(f) - \Phi(f)\} = \sup_{n, A_1, \cdots, A_n} \sup_{\underline{t}\in R^n} \left\{ \sum_{i=1}^{n} t_i\mu(A_i) - \Phi(\underline{t}) \right\},$$

where f is approximated as a simple function by disjoint subsets A_1, $\cdots$, A_n.

PROOF. Given $f \in K_c$. We can find uniformly bounded simple functions $f_n = \sum_{i=1}^{n} t_i 1_{A_i}$ such that $f_n \to f$. It is clear that $\mu(f_n) \to \mu(f)$

and $\Phi(f_n) \to \Phi(f)$ by Assumption A. Then we derive that

$$\mu(f) - \Phi(f) \leq \sup_{n,A_1,\dots,A_n} \sup_{\underline{t}\in R^n} \{\mu(f_n) - \Phi(f_n)\}.$$

Taking sup over all $f \in K_c$, then we get the following inequality

$$\sup_{f\in K_c} \{\mu(f) - \Phi(f)\} \leq \sup_{n,A_1,\dots,A_n} \sup_{\underline{t}\in R^n} \left\{ \sum_{i=1}^{n} t_i \mu(A_i) - \Phi(\underline{t}) \right\}.$$

On the other hand, given $g = \sum_{i=1}^{n} t_i 1_{A_i}$ on $[0, L]^d$, we can find uniformly bounded continuous functions g_m on $[0, L]^d$ such that $g_m \to g$ a.e.. By Bounded Convergence Theorem and Assumption A, we get $\mu(g_m) \to \mu(g)$ and $\Phi(g_m) \to \Phi(g)$. It is clear that $\mu(g_m) - \Phi(g_m) \leq \sup_{f\in K_c} \{\mu(f) - \Phi(f)\}$. Taking sup over all g_m, that is, over all $n, A_1, \cdots, A_n$ and $t_1, t_2, \cdots, t_n$, then we have the reverse inequality as follows

$$\sup_{f\in K_c} \{\mu(f) - \Phi(f)\} \geq \sup_{n,A_1,\dots,A_n} \sup_{\underline{t}\in R^n} \left\{ \sum_{i=1}^{n} t_i \mu(A_i) - \Phi(\underline{t}) \right\}.$$

So, the above equality is proved.

THEOREM 2.4. *We assume the properties A and B. Then the large deviation lower bound holds, i.e., for any open subset G in $\bar{M}$,*

$$\liminf_{r\to\infty} \frac{1}{r^d} \cdot \log P\left(\frac{\bar{X}_r}{r^d} \in G \right) \geq - \inf_{\mu\in G} I(\mu).$$

PROOF. Our proof follows the lines of Dawson and Gärtner [2] using Lemma 2.3. Let G be an open subset in $\bar{M}$. Then G is a union of open sets from Θ, since Θ forms a base for the topology generated by weak convergence, where

$$\begin{aligned}
\Theta = \{B, \text{open in } \bar{M} : B = B(\nu, F_1, \cdots, F_n, \epsilon)\}, \text{ with} \\
B = \{\mu \in \bar{M} : \mu(F_i) < \nu(F_i) + \epsilon, i = 1, \cdots, n \text{ and} \\
|\mu[0, L]^d - \nu[0, L]^d| < \epsilon\}
\end{aligned}$$

for Borel sets $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq [0, L]^d$.

Now let us take as follows:

$$E_0 = [0, L]^d / F_n, \tag{2.1}$$

$$E_1 = F_1, \tag{2.2}$$

$$E_k = F_k / F_{k-1}, \quad k = 2, \cdots, n. \tag{2.3}$$

Then there exists an open set $U \subseteq [0, L]^{n+1}$ depending on $\nu(F_1), \cdots,$ $\nu(F_n)$, $\nu[0, L]^d$ and ϵ.

$$\begin{aligned} B &= \{\mu \in \bar{M} : (\mu(E_0), \cdots, \mu(E_n)) \in U\} \\ &= C(E_0, \cdots, E_n, U) \text{ (we denote this by } C), \end{aligned}$$

where all $E_i's$ are disjoint. So we have the relation such that

$$G = \bigcup_{B \in \Theta, B \subseteq G} B = \bigcup_{C \in \Theta, C \subseteq G} C, \tag{2.4}$$

where C is obtained from B by letting each F_i disjoint as (2.1), (2.2) and (2.3). Thus,

$$\begin{aligned} P\left(\frac{\bar{X}_r}{r^d} \in B\right) &= P\left(\frac{\bar{X}_r}{r^d}(F_i) < \nu(F_i) + \epsilon \text{ for } i = 1, \cdots, n \right. \\ &\qquad \left. \text{and } \nu([0, L]^d) - \epsilon < \frac{\bar{X}_r([0, L]^d)}{r^d} < \nu([0, L]^d) + \epsilon\right) \\ &= P^{f.d.}(C), \end{aligned}$$

where $P^{f.d.}$ denotes a probability of a finite dimension and C depends on $F_1, \cdots, F_n$ and $[0, L]^d$ which are obtained by making each F_i disjoint as (2.1), (2.2) and (2.3). Since $I(G) = \inf_{\nu \in G} I(\nu)$ by definition, using the relation (2.4) we have

$$\begin{aligned} -I(G) &= -\inf_{\nu \in G} I(\nu) = \sup_{\nu \in G} -I(\nu) \\ &= \sup_{B \in \Theta, B \subseteq G} -I(B) = \sup_{C \in \Theta, C \subseteq G} -I(C). \end{aligned}$$

Moreover if $\nu \in G$, there exists $C \in \Theta$ (obtained from B) depending upon a finite dimension which contains ν such that for each $\epsilon > 0$, $-I(G) \leq -I(B) + \epsilon$. Therefore,

$$\begin{aligned}
\liminf_{r\to\infty}\frac{1}{r^d}\cdot\log P\left(\frac{\bar X_r}{r^d}\in G\right) &\geq \liminf_{r\to\infty}\frac{1}{r^d}\cdot\log P\left(\frac{\bar X_r}{r^d}\in B\right)\\
&= \liminf_{r\to\infty}\frac{1}{r^d}\cdot\log P^{f.d.}(C)\\
&\geq -\inf_{\underline{t}\in C} I(\underline{t}) \text{ by Assumption B}\\
&\geq -\inf_{\mu\in B} I(\mu) \text{ by Lemma 2.3}\\
&= -I(B)\geq -I(G)-\epsilon.
\end{aligned}$$

Since ϵ is arbitrary, letting $\epsilon\longrightarrow 0$, we obtain that

$$\liminf_{r\to\infty}\frac{1}{r^d}\cdot\log P\left(\frac{\bar X_r}{r^d}\in G\right)\geq -\inf_{\mu\in G} I(\mu).$$

3. Application of large deviation property

In this section we consider Poisson center cluster random measures. These are general classes of cluster models which include the Poisson, Neyman-Scott cluster processes and self-exciting processes. For more details the reader may consult Kallenberg [10] and Karr [11].

DEFINITION 3.1. Let X be a random measure and let A be a bounded Borel set in R^d. If a random measure X has independent increments and $X(A)$ is a Poisson random variable with parameter $\alpha|A|$ where $|A|$ denotes the Lebesgue measure, then we say that X is a Poisson point process with intensity $\alpha > 0$.

EXAMPLE 3.2. (Poisson point process) Let X be a Poisson point process with intensity $\alpha = 1$ by rescaling without loss of generality. We define an appropriate compact subset K_a to make $P_r({K_a}^c)$ small enough by Chebyshev's inequality for any real number $a < \infty$ such that

$$P_r({K_a}^c) = P\left(\frac{\bar X_r}{r^d}[0,L]^d > 2a\right)\leq \frac{e^{L^d r^d(e-1)}}{e^{2ar^d}} = e^{-r^d\{2a-L^d(e-1)\}}.$$

K_a is compact in the weak topology by the Banach-Alaoglu theorem. Now, we have that

$$\limsup_{r\to\infty}\frac{1}{r^d}\cdot\log P_r({K_a}^c)\leq -a$$

for such real number a. This means that P_r satisfies large deviation tightness. Thus it is consequence of Theorem 4.5.3 of Dembo and Zeitouni [3] that P_r satisfies the large deviation upper bound.

In order to show that the large deviation lower bound holds, recall that the moment generating functional of a Poisson random measure X is

$$E[e^{X(f)}] = \exp\left(\alpha \int_{[0,L]^d} [e^{f(x)} - 1]dx\right) \text{ for each } f \in K_c.$$

Then we get that $\Phi(f) = \alpha \int_{[0,L]^d} [e^{f(x)} - 1]dx$ for each $f \in K_c$.

Now let us check the condition of the Assumption A. For the functions f_n, f in K_c with $f_n \to f$ a.e.,

$$\begin{aligned} \lim_{n\to\infty} \Phi(f_n) &= \lim_{n\to\infty} \alpha \int_{[0,L]^d} (e^{f_n(x)} - 1)dx \\ &= \alpha \int_{[0,L]^d} \lim_{n\to\infty} [e^{f_n(x)} - 1]dx \text{ by L.D.C.T.} \\ &= \alpha \int_{[0,L]^d} [e^{f(x)} - 1]dx = \Phi(f). \end{aligned}$$

Since $\Phi(f_n) \to \Phi(f)$ as $f_n \to f$ a.e., the Assumption A is satisfied for a Poisson random measure X. Now, for the Assumption B, we have already proved such property in details in [9]. Thus Theorem 2.4. shows that a large deviation lower bound holds.

From now on, we follow Burton and Dehling (1990) for terminology.

DEFINITION 3.3. Let U be a stationary Poisson process on R^d with intensity $\alpha > 0$. V is a random measure so that $E[V(R^d)] = \zeta < \infty$. Let x_i be the random occurrences of U and let V_i be independent identically distributed(i.i.d.) copies of V that are independent of U. The resulting cluster process X is said to be a Poisson center cluster random measure, which is defined by superimposing i.i.d. copies of V centered at the occurrences of U.

In other words, if A is a bounded Borel subset of R^d, then X is defined by

$$X(A) = \sum_i V_i(A - x_i).$$

The moment generating function of $V(R^d)$ is $M_{V(R^d)}(t) = E[e^{tV(R^d)}]$ for each $t \in R$. In addition, we assume that $V(R^d)$ has a finite moment generating function $M_{V(R^d)}(t)$ for each $t \in R$.

Note that $E[X(A)] = \alpha\zeta|A|$.

EXAMPLE 3.4. (Poisson center cluster random measure) Let X be a Poisson center cluster random measure. For a bounded Borel subset A of R^d, we have the moment generating function of $X_r(A)$ from Campbell's formula,

$$M_{X_r(A)}(t) = \exp\left\{\alpha \cdot \int_{\mathrm{R^d}} \mathrm{E[e^{tV(rA-x)} - 1]dx}\right\}, \ \mathrm{t \in R}.$$

From $M_{X_r(A)}(t)$, we also get the following inequality, so

$$M_{X_r(A)}(t) \leq \exp\left\{r^d\alpha|A|[M_{V(R^d)}(t) - 1]\right\}.$$

Now, let us take an appropriate subset K_a for any real number $a < \infty$,

$$K_a = \left\{\frac{\bar{X}_r}{r^d} \in \bar{M} : \frac{\bar{X}_r}{r^d}[0, L]^d \leq 2a\right\}.$$

Then K_a is compact in the weak topology by the Banach-Alaoglu theorem. Thus we make $P_r({K_a}^c)$ small enough by Chebyshev's inequality such that

$$\begin{aligned} P_r({K_a}^c) &= P\left(\frac{\bar{X}_r}{r^d}[0, L]^d > 2a\right) \\ &\leq \frac{e^{r^d\alpha L^d[M_{v(R^d)}(1)-1]}}{e^{2ar^d}} = e^{-r^d\{2a-\alpha L^d[M_{V(R^d)}(1)-1]\}}. \end{aligned}$$

Now, we have that $\limsup_{r\to\infty}\frac{1}{r^d} \cdot \log P_r({K_a}^c) \leq -a$ for such real number a. This means that P_r satisfies large deviation tightness. Thus it is consequence of Theorem 4.5.3 of Dembo and Zeitouni [3] that P_r satisfies the large deviation upper bound.

In order to show that the large deviation lower bound holds, recall the moment generating functional of a Poisson center cluster random measure X is

$$E\left[e^{X(f)}\right] = \exp\left(\alpha \int_{\mathrm{[0,L]^d}} \mathrm{[M_{V(R^d)}(T_x f) - 1]dx}\right) \text{ for each } \mathrm{f \in K_c}.$$

Then we get that $\Phi(f) = \alpha \int_{[0,L]^d}[M_{V(R^d)}(f) - 1]dx$ for each $f \in K_c$. Here T_x is the translation operator, $(T_x f)(y) = f(x+y)$ and $M_{V(R^d)}(f) = E[e^{V(R^d)f(x)}]$.

Now let us check the condition of the Assumption A. For the functions f_n, f in K_c with $f_n \to f$ a.e.,

$$\begin{aligned}\lim_{n\to\infty} \Phi(f_n) &= \lim_{n\to\infty} \alpha \int_{[0,L]^d} [M_{V(R^d)}(f_n) - 1]dx \\ &= \alpha \int_{[0,L]^d} \lim_{n\to\infty} E[e^{V(R^d)f_n(x)} - 1]dx \text{ by L.D.C.T.} \\ &= \alpha \int_{[0,L]^d} E[e^{V(R^d)f(x)} - 1]dx = \Phi(f).\end{aligned}$$

Since $\Phi(f_n) \to \Phi(f)$ as $f_n \to f$ a.e., the Assumption A is satisfied for a Poisson center cluster random measure X. Now, for the Assumption B, we have already proved such property in details in [9]. Thus Theorem 2.4. shows that a large deviation lower bound holds.

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