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ISOMETRIC IMMERSIONS OF MANIFOLDS INTO SPACE FORMS WITH THE SAME CONSTANT CURVATURES

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ABSTRACT. We study a unifying method to obtain nondegenerate isometric immersions of manifolds of constant sectional curvatures c with flat normal bundles into the space forms of curvature c for $c = -1, 0, 1$.

1. Introduction

The problem of immersions of the space forms $N^n(c)$ into the space forms $N^{n+k}(c')$ is one of the most important problems in differential geometry. The first result about this was given by Hilbert [3], who proved that a complete 2-dimensional Riemannian manifold of constant negative curvature, say, the hyperbolic space form $\mathbb{H}^2$ cannot be isometrically immersed into 3-dimensional Euclidean space $\mathbb{R}^3$. If the dimension of $\mathbb{H}^n$ is bigger than 2, much has been known, even for the local immersions. A starting work on it was due to Cartan [2]. He proved that $N^n(c)$ cannot be locally, isometrically immersed in $N^{2n-2}(c+1)$, but can be into $N^{2n-1}(c+1)$. Moreover, if such an immersion exists, then the normal bundle is automatically flat. Later, many results have been obtained for this problem in [4], [5], [6], [7], [8]. In particular, Terng [8] studied immersions of $N^n(c)$ in $N^{2n}(c)$ for each case $c = -1, 0, 1$, separately, using the so-called n-dimensional systems on symmetric spaces. This work was extensively generalized to an arbitrary codimension case in [1], recently.

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In this paper, we give a simple proof for local nondegenerate immersions $N^n(c)$ into $N^{n+k}(c)$ with flat normal bundles for $k \geq n$, unifying all the cases $c = -1, 0, 1$ using only one real symmetric space related to isometry groups of $N^{n+k}(c)$. Also, we investigate in detail on the geometric meaning of the conditions of nondegeneracy and flat normal curvatures for such immersions.

2. Submanifolds in a space form

The space form $N^n(c)$ is the complete, simply connected Riemannian manifold with constant sectional curvature c. For example, when $c = 0, 1$, or -1, it is the Euclidean space $\mathbb{R}^n$, the unit sphere $\mathbb{S}^n$, or the hyperbolic space $\mathbb{H}^n$.

Suppose $X : M^n \to N^{n+k}(c)$ is an isometric immersion. Take a local orthonormal frame field $e_1, \cdots, e_{n+k}$ on $N^{n+k}(c)$, so that if when restricted to M, $e_1, \cdots, e_n$ are tangent to M. From now on, we shall use the following index convention:

$$1 \leq A, B, C \leq n+k, \quad 1 \leq i, j, k \leq n, \quad n+1 \leq \alpha, \beta, \gamma \leq n+k.$$

Let ω_A be the dual coframe to e_A, that is, $\omega_A(e_B) = \delta_{AB}$. The first fundamental form on M is given by $I = \sum_i \omega_i \otimes \omega_i$. Let ω_{AB} be the connection 1-form corresponding to the canonical connection d on $N^{n+k}(c)$,

$$de_A = \sum_B e_B \otimes \omega_{BA}.$$

This induces the Levi-Civita connection ∇ on M by

$$\nabla e_i = \sum_j e_j \otimes \omega_{ji},$$

and the structure equations on M are

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j. \tag{2.1}$$

The Gauss, Codazzi and Ricci equations are

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \sum_\alpha \omega_{i\alpha} \wedge \omega_{\alpha j} + c\, \omega_i \wedge \omega_j, \tag{2.2}$$

$$d\omega_{i\alpha} = -\sum_k \omega_{ik} \wedge \omega_{k\alpha} - \sum_\beta \omega_{i\beta} \wedge \omega_{\beta\alpha}, \tag{2.3}$$

$$d\omega_{\alpha\beta} = -\sum_i \omega_{\alpha i} \wedge \omega_{i\beta} - \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}. \tag{2.4}$$

From (2.2) and (2.4), we obtain the curvature 2-form Ω on M and the normal curvature 2-form Ω^ν as

$$\Omega_{ij} = -\sum_\alpha \omega_{i\alpha} \wedge \omega_{j\alpha} + c\,\omega_i \wedge \omega_j, \tag{2.5}$$

$$\Omega^\nu_{\alpha\beta} = -\sum_i \omega_{i\alpha} \wedge \omega_{i\beta}. \tag{2.6}$$

The shape operator A_{e_α} in the direction e_α is defined by

$$A = \sum_{j,\alpha} \omega_{j\alpha} \otimes \omega_\alpha \otimes e_j, \tag{2.7}$$

which is identified with the second fundamental form $I\!I$ under the metric isomorphism $TN^{n+k}(c) \simeq TN^{n+k}(c)$:

$$I\!I = \sum_{j,\alpha} \omega_{j\alpha} \otimes \omega_j \otimes e_\alpha. \tag{2.8}$$

Now, suppose the normal bundle νM is flat, i.e., $\Omega^\nu = 0$. Then there exists a parallel normal frame $\{e_\alpha\}$ and it is easy to see that all the shape operators commute by (2.6), and thus they are simultaneously diagonalizable.

DEFINITION 2.1. A submanifold M^n is called *nondegenerate* if dim $\{A_v |\ v \in \nu_p M\} = \dim M = n$ for any $p \in M$.

The conditions of being nondegenerate and $\Omega^\nu = 0$ give a strong geometric restriction to M. To see this, let $T_pM = E_1 \oplus \cdots \oplus E_r$ be the common eigen-decomposition for $\{A_v | v \in \nu_p M\}$. Then

$$A|_{E_i} = \lambda_i \otimes Id_{E_i}$$

for some $\lambda_1, \cdots, \lambda_r \in (\nu_p M)^*$. The curvature normals $v_1, \cdots, v_r$ in $\nu_p M$ are defined as the dual to λ_i, that is, $\lambda_i(v) = \langle v, v_i \rangle$.

LEMMA 2.2. *Suppose M^n is nondegenerate and has a flat normal bundle. Then the curvature normals $v_1, \cdots, v_r$ are linearly independent and $r = n$.*

PROOF. We claim that $A_{v_1}, \cdots, A_{v_r}$ span $\{A_v \mid v \in \nu_p M\}$. For any $w \in \nu_p M$, let $w = v + u \in \mathrm{Span}\{v_i\} \oplus \mathrm{Span}\{v_i\}^{\perp}$. Then

$$A_w|_{E_i} = \langle w, v_i \rangle Id_{E_i} = \langle v, v_i \rangle Id_{E_i} = A_v|_{E_i}.$$

Thus $A_w = A_v$. Since M is nondegenerate, $r = n$ and $A_{v_1}, \cdots, A_{v_n}$ should be a basis of $\{A_v \mid v \in \nu_p M\}$. Now it is easy to see that $v_1, \cdots, v_n$ are linearly independent. □

REMARK 2.3. From the above lemma, we can see that $\dim E_i = 1$ and thus there exist a unique orthonormal tangent frame $\{e_i\}$ which diagonalize the shape operators simultaneously, up to signs and permutations, and they are smooth. Also, it is obvious that $k \geq n$.

Now, suppose e_α are a parallel orthonormal normal frame, i.e., $\omega_{\alpha\beta} = 0$. The curvature normals v_i can be expressed as

$$v_i = \sum_{\alpha} \lambda_{i\alpha} e_\alpha, \tag{2.9}$$

where, $\lambda_{i\alpha} = \langle e_\alpha, v_i \rangle = \lambda_i(e_\alpha)$. Using this frame $\{e_i, e_\alpha\}$, we have

$$\omega_{i\alpha} = \lambda_{i\alpha} \omega_i. \tag{2.10}$$

Furthermore, suppose M has constant sectional curvature c, that is,

$$\Omega_{ij} = c\, \omega_i \wedge \omega_j. \tag{2.11}$$

Then by (2.5), (2.10) and (2.11),

$$\sum_{\alpha} \lambda_{i\alpha} \lambda_{j\alpha} = 0 \qquad \text{for} \quad i \neq j, \tag{2.12}$$

and hence the curvature normals v_i are orthogonal. From the Codazzi equations (2.3), using (2.10) and $\omega_{\alpha\beta} = 0$, we obtain

$$d\lambda_{i\alpha}(e_j) + (\lambda_{i\alpha} - \lambda_{j\alpha})\gamma_{iji} = 0 \quad \text{for } i \neq j, \tag{2.13}$$

(2.14) $$(\lambda_{i\alpha}-\lambda_{k\alpha})\gamma_{ikj}=(\lambda_{i\alpha}-\lambda_{j\alpha})\gamma_{ijk} \text{ for distinct } i,j,k,$$
where $\omega_{ij}=\sum_k \gamma_{ijk}\omega_k$.

Put $b_i=1/|v_i|$. Multiplying $\lambda_{i\alpha}$ to (2.13) and (2.14) and summing up over α, we obtain

$$|v_i|^2\gamma_{ijk}=0 \quad \text{for distinct } i,j,k, \quad \text{and} \quad \gamma_{iji}=\frac{db_i(e_j)}{b_i} \quad \text{for} \quad i\neq j.$$

Therefore,

(2.15) $$\omega_{ij}=\frac{db_i(e_j)}{b_i}\omega_i-\frac{db_j(e_i)}{b_j}\omega_j.$$

PROPOSITION 2.4. *Suppose M^n is a nondegenerate submanifold of $N^{n+k}(c)$ with constant sectional curvature c and a parallel normal frame e_α. Then there exists a coordinate system $(x_1,\cdots,x_n)$ such that $e_i=\frac{1}{b_i}\frac{\partial}{\partial x_i}$ is a principal tangent frame.*

PROOF. By (2.15), we see that $\nabla_{e_i}e_j=\frac{db_i(e_j)}{b_i}e_i$. It is a direct calculation that

$$[b_ie_i,b_je_j]=\nabla_{b_ie_i}b_je_j-\nabla_{b_je_j}b_ie_i=0. \qquad \square$$

We now conclude the local geometry of the above submanifold as follows;

THEOREM 2.5. *Let $X: M^n\longrightarrow N^{n+k}(c)$ be a nondegenerate local immersion of the Riemannian manifold M^n of constant sectional curvature c with flat normal bundle. Then, for a local parallel normal frame e_α, there exists a curvature coordinate system $(x_1,\cdots,x_n)$, a map $b=(b_1,\cdots,b_n)^t$ and a $k\times n$ matrix-valued $B_1=(b_{ij})$ such that $B_1^tB_1=Id$ and the first and second fundamental forms are given by*

$$I=\sum_i b_i^2dx_i^2, \qquad I\!I=\sum_{i,j} b_{ji}b_idx_i^2\otimes e_{n+j}.$$

PROOF. It remains only to prove the existence of B_1 and the second fundamental form given as above. Let $e_i=\frac{1}{b_i}\frac{\partial}{\partial x_i}$. Then $\omega_i=b_idx_i$.

Define $b_{ji}=\lambda_{i,n+j}b_i$. Then by (2.10),

$$\omega_{i,n+j}=\lambda_{i,n+j}\omega_i=b_{ji}dx_i.$$

Hence, the second fundamental form is given as above. The orthonormality of the columns of B_1 follows from the fact that the curvature normals $v_i=\sum_\alpha\lambda_{i\alpha}e_\alpha$ are orthogonal and $b_i=1/|v_i|$. $\square$

3. G/K Systems

G/K systems were introduced for a symmetric space G/K by Terng in [8]. We will review definitions briefly. For more details, see [8].

Let G/K be a rank n symmetric space, $\sigma : \mathcal{G} \to \mathcal{G}$ the corresponding involution on the Lie algebra $\mathcal{G}$ of G, $\mathcal{G} = \mathcal{K} + \mathcal{P}$ the Cartan decomposition, and $\mathcal{A} \subset \mathcal{P}$ a maximal abelian subalgebra. Let $a_1, \ldots, a_n$ be a basis for $\mathcal{A}$ consisting of regular elements with respect to the $Ad(K)$-action on $\mathcal{P}$. Let $\mathcal{A}^\perp$ denote the orthogonal complement of $\mathcal{A}$ in $\mathcal{G}$ with respect to the Killing form. Then G/K *system* for $v : \mathbb{R}^n \to \mathcal{P} \cap \mathcal{A}^-$ is

$$[a_i, v_{x_j}] - [a_j, v_{x_i}] = \big[[a_i, v], [a_j, v] \big], \qquad 1 \le i \ne j \le n,$$

where, $v_{x_i} = \frac{\partial v}{\partial x_i}$. This system is equivalent to the following Lax pair:

$$\Big[\frac{\partial}{\partial x_i} + \lambda a_i + [a_i, v], \frac{\partial}{\partial x_j} + \lambda a_j + [a_j, v]\Big] = 0 \quad \text{for any} \quad \lambda \in \mathbb{C}.$$

The Cauchy problem for G/K system can be solved for any generic data decaying rapidly along $(x_1, 0, \ldots, 0)$ (cf. [8]).

We can also express G/K system in terms of a connection 1-form on the trivial principal bundle $\mathbb{R}^n \times \mathcal{G}$ on $\mathbb{R}^n$. To see this, we need the following proposition, which can be proved by a direct computation.

Proposition 3.1. *Given smooth maps $A_i : \mathbb{R}^n \to \mathcal{G}$ for $1 \le i \le n$, the following statements are equivalent:*
(i) *$E_{x_i} = EA_i$ is solvable for $E : \mathbb{R}^n \to G$,*
(ii) $[\frac{\partial}{\partial x_i} + A_i, \frac{\partial}{\partial x_j} + A_j] = 0$,
(iii) $(A_j)_{x_i} - (A_i)_{x_j} + [A_i, A_j] = 0$,
(iv) *$d\theta + \theta \wedge \theta = 0$, where θ is the $\mathcal{G}$-valued 1-form $\sum_{i=1}^n A_{x_i} dx_i$.*
In this case, we call E a trivialization of θ and it satisfies $E^{-1}dE = \theta$.

Suppose E is a trivialization of θ, $E^{-1}dE = \theta$. Let $g : \mathbb{R}^n \to G$. The gauge transformation of E by g is defined as $g * E = Eg^{-1}$. This induces a new flat connection

$$(Eg^{-1})^{-1} d(Eg^{-1}) = g\theta g^{-1} - dgg^{-1}.$$

We call $g * \theta = g\theta g^{-1} - dgg^{-1}$, the gauge transformation of θ by g.

It is easy to see that v is a solution for G/K system if and only if the following one-parameter family of $\mathcal{G} \otimes \mathbb{C}$-valued connections on $\mathbb{R}^n$ is flat;

$$\theta_\lambda = \sum_{i=1}^{n} (a_i \lambda + [a_i, v]) dx_i. \tag{3.1}$$

PROPOSITION 3.2. *Let $a_1, \cdots, a_n$ be a basis of a maximal abelian subalgebra $\mathcal{A}$ in $\mathcal{P}$, and $u_i : \mathbb{R}^n \to \mathcal{K}_{\mathcal{A}}^{\perp}$ smooth maps for $1 \le i \le n$, where $\mathcal{K}_{\mathcal{A}}$ is the centralizer of $\mathcal{A}$ in $\mathcal{K}$. If*

$$\theta_\lambda = \sum_{i=1}^{n} (a_i \lambda + u_i) dx_i \tag{3.2}$$

is a flat connection 1-form on $\mathbb{R}^n$ for all $\lambda \in \mathbb{C}$, then there exists a unique map $v : \mathbb{R}^n \to \mathcal{P} \cap \mathcal{A}^{\perp}$ such that $u_i = [a_i, v]$.

PROOF. We may assume that $a_1, \cdots, a_n$ are regular by changing a basis and coordinates. Since θ_λ is flat for all λ,

$$[a_i, u_j] = [a_j, u_i]. \tag{3.3}$$

Because $a_1, \cdots, a_n$ are regular, $\mathrm{ad}(a_j)$ maps $\mathcal{P} \cap \mathcal{A}^{\perp}$ isomorphically to $\mathcal{K}_{\mathcal{A}}^{\perp}$. Hence there exists a unique $v_j \in \mathcal{P} \cap \mathcal{A}^{\perp}$ such that $u_j = \mathrm{ad}(a_j)(v_j)$ for $1 \le j \le n$. Then (3.3) implies that

$$\mathrm{ad}(a_i)\mathrm{ad}(a_j)(v_j) = \mathrm{ad}(a_j)\mathrm{ad}(a_i)(v_i).$$

Since $[a_i, a_j] = 0$, $\mathrm{ad}(a_i)\mathrm{ad}(a_j) = \mathrm{ad}(a_j)\mathrm{ad}(a_i)$. But $\mathrm{ad}(a_i)$ is injective on $\mathcal{P} \cap \mathcal{A}^{\perp}$ so that $v_i = v_j$, which will be denoted by v. □

Thus the existence of the solution v of the system is equivalent to the flatness of the connection θ_λ of the form (3.2).

To explain the geometry of a submanifold M^n in $N^{n+k}(c)$, we need to know the isometry group of $N^{n+k}(c)$. First, we identify $\mathbb{R}^{n+k}$ with $\mathbb{R}^{n+k} \times \{1\} \subset \mathbb{R}^{n+k+1}$ by $X \leftrightarrow (X, 1)$. Then all the space form $N^{n+k}(c)$ can be regarded as subsets of the vector space $\mathbb{R}^{n+k+1}$. It is well-known that the isometry groups of $\mathbb{R}^{n+k}$, $\mathbb{S}^{n+k}$ and $\mathbb{H}^{n+k}$ (corresponding to $0, 1$ and -1, respectively) are

$$\left\{ \begin{pmatrix} A & \xi \\ 0 & 1 \end{pmatrix} \mid A \in O(n+k),\ \xi^t \in \mathbb{R}^{n+k} \right\},$$

$$O(n+k+1) = \left\{ A \in GL(n+k+1, \mathbb{R}) \mid A^t A = I \right\}$$

and

$$O(n+k, 1) = \left\{ A \in GL(n+k+1, \mathbb{R}) \mid A^t J A = J \right\},$$

where $J = \mathrm{diag}(1, \cdots, 1, -1)$.

LEMMA 3.3. *The Lie algebras of the isometry groups of $N^{n+k}(c)$ can be expressed as the Lie algebra*

$$\mathcal{G}_c = \left\{ \begin{pmatrix} Y & \xi \\ -c\xi^t & 0 \end{pmatrix} \mid Y \in \mathrm{so}(n+k),\ \xi^t \in \mathbb{R}^{n+k} \right\}.$$

PROOF. The Lie algebras of the isometry groups of $N^{n+k}(c)$ are of the form

$$\left\{ \begin{pmatrix} Y & \xi \\ 0 & 0 \end{pmatrix} \mid Y \in \mathrm{so}(n+k),\ \xi^t \in \mathbb{R}^{n+k} \right\}, \ \mathrm{so}(n+k+1) \text{ and } \mathrm{so}(n+k, 1),$$

respectively. It is trivial that all of these are $\mathcal{G}_c$ for $c = 0, 1$ or -1, respectively. □

Now, for $\mathcal{G}_c$, define an involution

$$\sigma(X) = \begin{pmatrix} I_k & 0 & 0 \\ 0 & -I_n & 0 \\ 0 & 0 & -1 \end{pmatrix} X \begin{pmatrix} I_k & 0 & 0 \\ 0 & -I_n & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ X \in \mathcal{G}_c.$$

Here, the matrix is partitioned into 3×3-blocks with sizes $(k, n, 1)$, and I_n is the $n \times n$ identity matrix. Then the Cartan decomposition becomes $\mathcal{G}_c = \mathcal{K} + \mathcal{P}$, where

$$\mathcal{K} = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & a_1 \\ 0 & -ca_1^t & 0 \end{pmatrix} \mid A \in \mathrm{so}(k),\ B \in \mathrm{so}(n),\ a_1^t \in \mathbb{R}^n \right\},$$

$$\mathcal{P} = \left\{ \begin{pmatrix} 0 & C & a_2 \\ -C^t & 0 & 0 \\ -ca_2^t & 0 & 0 \end{pmatrix} \;\middle|\; C \text{ is a } k \times n \text{ matrix}, \; a_2^t \in \mathbb{R}^k \right\}.$$

Put

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & -D & 0 \\ D & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; D = (d_{ij}), \; d_{ij} = 0 \text{ for } i \neq j \right\},$$

then it is an abelian subalgebra in $\mathcal{P}$ with a basis

$$a_i = \begin{pmatrix} 0 & -D_i & 0 \\ D_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad 1 \leq i \leq n,$$

where D_i has 1 as the (i,i)-entry and zero elsewhere. Let $(\mathcal{G}_c)_{\mathcal{A}}$ denote the centralizer of $\mathcal{A}$ in $\mathcal{G}_c$. Then $\mathcal{P} \cap (\mathcal{G}_c)_{\mathcal{A}}^{\perp}$ is the space of elements of the form

$$v = \begin{pmatrix} 0 & 0 & F & b \\ 0 & 0 & G & 0 \\ -F^t & -G^t & 0 & 0 \\ -cb^t & 0 & 0 & 0 \end{pmatrix},$$

where v is partitioned into 4×4-blocks with sizes $(n, n-k, n, 1)$ and $F = (f_{ij}) \in \mathrm{gl}(n)$ with $f_{ii} = 0$.

Since v is completely determined by (F, G, b), we will say that (F, G, b) is a solution of this system instead of v being a solution.

Put $\delta = \mathrm{diag}(dx_1, \cdots, dx_n)$. Then the connection 1-form θ_λ in (3.1) becomes

$$\theta_\lambda = \begin{pmatrix} \delta F^t - F\delta & \delta G^t & -\lambda\delta & 0 \\ -G\delta & 0 & 0 & 0 \\ \lambda\delta & 0 & \delta F - F^t\delta & \delta b \\ 0 & 0 & -cb^t\delta & 0 \end{pmatrix}. \tag{3.4}$$

It is obvious that

PROPOSITION 3.4. *(F, G, b) is a solution of G/K-system associated to $\mathcal{G}_c$ if and only if θ_λ in (3.4) is flat for any λ.*

4. Main Theorems

THEOREM 4.1. *A nondegenerate local immersion X of the Riemannian manifold M^n into $N^{n+k}(c)$ of constant sectional curvature c with a flat normal bundle as in Theorem 2.5 gives rise to a solution (F, G, b) of the system associated to $\mathcal{G}_c$.*

In fact, they are related by

$$F = \left(\frac{(b_i)_{x_j}}{b_j}\right), \quad \omega = \delta F - F^t\delta, \quad \text{and} \quad B_1^t dB_1 = \delta F^t - F\delta.$$

PROOF. Choose a parallel normal frame $\{e_\alpha\}$ and a tangent frame $\{e_i\}$ as in Theorem 2.5 so that $\omega_i = b_i dx_i$. Put $b = (b_1, \cdots, b_n)^t$. Then from the structure equations, Gauss, Codazzi and Ricci equations,

$$\tilde{\theta}_1 = \begin{pmatrix} 0 & -B_1\delta & 0 \\ \delta B_1^t & \omega & \delta b \\ 0 & -cb^t\delta & 0 \end{pmatrix}$$

is flat. It is an easy computation that

$$\tilde{\theta}_\lambda = \begin{pmatrix} 0 & -\lambda B_1\delta & 0 \\ \lambda\delta B_1^t & \omega & \delta b \\ 0 & -cb^t\delta & 0 \end{pmatrix} \tag{4.1}$$

is also flat for any λ.

Let $F = (f_{ij}) \in \mathrm{gl}(n)$, where $f_{ij} = \frac{(b_i)_{x_j}}{b_j}$ for $i \neq j$ and $f_{ii} = 0$. Since the connection 1-form $\omega = (\omega_{ij})$ on M satisfies

$$\omega_{ij} = \frac{(b_i)_{x_j}}{b_j} dx_i - \frac{(b_j)_{x_i}}{b_i} dx_j \quad \text{for} \quad i \neq j$$

by (2.15), we obtain

$$\omega = \delta F - F^t\delta. \tag{4.2}$$

On the other hand, from the flatness of $\tilde{\theta}_\lambda$,

$$dB_1 \wedge \delta = -B_1\delta \wedge \omega = B_1(\delta F^t - F\delta) \wedge \delta$$

and thus

$$dB_1 = B_1(\delta F^t - F\delta) + C\delta \tag{4.3}$$

for some $k \times n$ matrix C. Extend B_1 to $B = (B_1, B_2) \in O(k)$. Then from (4.3),

$$B_2^t dB_1 = B_2^t C\delta. \tag{4.4}$$

Since $B^{-1}dB$ is flat and

$$B^{-1}dB = \begin{pmatrix} B_1^t dB_1 & B_1^t dB_2 \\ B_2^t dB_1 & B_2^t dB_2 \end{pmatrix},$$

we have

$$dB_2^t \wedge dB_2 + B_2^t dB_1 \wedge B_1^t dB_2 + B_2^t dB_2 \wedge B_2^t dB_2 = 0. \tag{4.5}$$

By (4.4),

$$B_1^t dB_2 = (dB_2^t B_1)^t = -(B_2^t dB_1)^t = -\delta C^t B_2.$$

So it follows from (4.5) that $B_2^t dB_2$ is flat, and hence $h^{-1}dh = B_2^t dB_2$ for some $h \in O(k-n)$. Thus if we take a gauge transformation on $\tilde{\theta}_\lambda$ by

$$g = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} B^t & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then the resulting flat connection 1-form θ_λ is

$$\theta_\lambda = g * \tilde{\theta}_\lambda = \begin{pmatrix} B_1^t dB_1 & -\delta C^t B_2 h^t & -\lambda\delta & 0 \\ hB_2^t C\delta & 0 & 0 & 0 \\ \lambda\delta & 0 & \omega & b \\ 0 & 0 & -cb^t & 0 \end{pmatrix}.$$

Set $G = -hB_2^t C$. From (4.3), we have $B_1^t dB_1 - (\delta F^t - F\delta) = Y\delta$, where $Y = B_1^t C$. Since the left-hand side is skew-symmetric, so is $Y\delta$. But $Y\delta = -\delta Y^t$ implies that $Y = 0$. It follows that the flat connection $g * \tilde{\theta}_\lambda$ is of the form θ_λ defined by (3.4). Therefore (F, G, b) is a solution of the system associated to $\mathcal{G}_c$. □

Conversely, we have

THEOREM 4.2. *A solution (F, G, b) of the system associated to $\mathcal{G}_c$ gives rise to a nondegenerate local immersion X_c of M^n of constant sectional curvature c with flat normal bundle into $N^{n+k}(c)$, which has a parallel normal frame $\{e_\alpha\}$ and a coordinate system $(x_1, \cdots, x_n)$ such that the first and second fundamental forms are given by*

$$I = \sum_i b_i^2 dx_i^2, \qquad I\!I = \sum_{i,j} b_{ji} b_i dx_i^2 \otimes e_{n+j}.$$

PROOF. We have a flat connection θ_λ as in (3.4) obtained from (F, G, b).

Since $\eta = \begin{pmatrix} \delta F^t - F\delta & \delta G^t \\ -G\delta & 0 \end{pmatrix}$ is flat, $B^t dB = \eta$ for some $B \in O(k)$. Taking a gauge transformation $h = \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}$ on θ_λ gives $h * \theta_\lambda = \tilde{\theta}_\lambda$, where $\tilde{\theta}_\lambda$ is of the form (4.1). Now, let E be a trivialization of $\tilde{\theta}_1$, that is, $dE = E\tilde{\theta}_1$. Denote by e_α, e_i, X_c the columns of E. Then from

$$d(e_\alpha, e_i, X_c) = (e_\alpha, e_i, X_c)\, \tilde{\theta}_1,$$

we obtain

$$dX_c = \sum b_i dx_i \otimes e_i, \quad de_{n+j} = \sum_i b_{ji} dx_i \otimes e_i,$$

and thus e_α are a parallel normal frame and $I\!I$ is given as above. Therefore, X_c gives a desired immersion. □

We can conclude that there is a correspondence between the nondegenerate isometric immersions of n-dimensional manifolds of constant sectional curvatures c with flat normal bundles into $N^{n+k}(c)$ and the solutions of systems associated to the Lie algebra $\mathcal{G}_c$ for $k \geq n$.

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