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STRONG CONVERGENCE THEOREMS FOR LOCALLY PSEUDO-CONTRACTIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty bounded open subset of X, and T a continuous mapping from the closure of C into X which is locally pseudo-contractive mapping on C. We show that if the closed unit ball of X has the fixed point property for nonexpansive self-mappings and T satisfies the following condition: there exists $z \in C$ such that $\|z - T(z)\| < \|x - T(x)\|$ for all x on the boundary of C, then the trajectory $t \longmapsto z_t \in C$, $t \in [0, 1)$ defined by the equation $z_t = tT(z_t) + (1-t)z$ is continuous and strongly converges to a fixed point of T as $t \to 1^-$.

1. Introduction

Let X and X^* be a real Banach space with norm $\|\cdot\|$ and the dual space, respectively, and C a subset of X. A mapping $T : C \to X$ is said to be *pseudo-contractive* ([3], [19]) if for each u, $v \in X$ and $\lambda > 1$

$$(1) \qquad (\lambda - 1)\|u - v\| \leq \|(\lambda I - T)(u) - (\lambda I - T)(v)\|.$$

Following Kato [14], we can describe an equivalent formulation to this definition. A mapping T from C to X is pseudo-contractive if and only if for every u, $v \in C$ there exists $j \in J(u - v)$ such that

$$(2) \qquad \langle T(u) - T(v), j\rangle \leq \|u - v\|^2,$$

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where $J : X \to 2^{X^*}$ is the (normalized) *duality mapping* which is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2, \ \|x\| = \|x^*\|\}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If (2) holds *locally*, that is, if for each $x \in C$ has a neighborhood U such that the restriction of T to U is pseudo-contractive.

The pseudo-contractive mappings which are closely more general than the nonexpansive mappings (mappings $T : C \to X$ for which $\|T(x) - T(y)\| \le \|x - y\|$, $x, y \in C$) are characterized by the fact that a mapping $T : C \to X$ is pseudo-conractive if and only if the mapping $A = I - T$ is accretive on C ([3], [14]). Recent interest in mapping theory for accretive operators, particularly as it relate to existence theorems for nonlinear differential equations, has prompted a corresponding interest in fixed point theory for pseudo-contractive mappings (e.g., [3], [5], [7], [14] $\sim$ [16], [18]).

The purpose of this paper is to continue the discussion concerning the strong convergence of the path $\{z_t\}$, $0 \le t < 1$ defined by the equation $z_t = tT(z_t) + (1 - t)z$ for $x \in C$. Actually, we prove for a locally pseudo-contractive mapping as well as a locally nonexpansive mapping, that the strong $\lim_{t \to 1^-} z_t$ exists and is a fixed point of T under a certain boundary conditions in a reflexive Banach space with a uniformly Gâteaux differentiable norm. Our results are extensions of results obtained by Bruck et al. [4] and Morales [21] to more general types of spaces. However, our proofs reply mainly on that of given in [4] and [21]. Further, we obtain the strong convergence results in reflexive Banach spaces with a weakly sequentially continuous duality mapping. The study of this type of problems begun over thirty years ago can be found in [2], [12], [13], [17], [23], [27], [29], [31], and [32] among others.

2. Preliminaries

Throughout this paper, we assume X being a Banach space and denote the set of all nonnegative real numbers by $\mathbb{R}^+$. When $\{x_n\}$ is a sequence in X, then $x_n \to x$ (resp. $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp. weak, weak*) convergence of the sequence $\{x_n\}$ to x.

Let $U = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. The norm of X is said to be *Gâteaux differentiable* (and X is said to be *smooth*) if

$$(3) \qquad \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x, y in U. It is said to be *uniformly Gâteaux differentiable* if each $y \in U$, this limit is attained uniformly for $x \in U$. The norm is said to be *Fréchet differentiable* if for each $x \in U$, the limit is obtained uniformly for $y \in U$. Finally, the norm is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit in (3) is attained uniformly for $(x, y) \in U \times U$. Since the dual X^* of X is uniformly convex if and only if the norm of X is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. But there are reflexive Banach spaces with a uniformly Gâteaux differentiable norm that are not even isomorphic to a uniformly smooth Banach space. To see this, consider X to be the direct sum $l^2(l^{p_n})$, the class of all those sequences $x = \{x_n\}$ with $x_n \in l^{p_n}$ and $\|x\| = (\sum_{n<\infty} \|x_n\|^2)^{\frac{1}{2}}$ ([6]). Now if $1 < p_n < \infty$ for all $n \in \mathbb{N}$, where either $\limsup_{n\to\infty} p_n = \infty$ or $\liminf_{n\to\infty} p_n = 1$, then X is a reflexive Banach space ([6]) with a uniformly Gâteaux differentiable norm ([33]), but is not uniformly smooth ([6]) (cf. [26]).

The (normalized) duality mapping J is single valued if and only if X is smooth. It is also well-known that if X has a uniformly Gâteaux differentiable norm, then J is uniformly continuous on bounded subsets of X from the strong topology of X to the weak-star topology of X^*. This is explicitly proved in Lemma 2.2 of [28] (also [8]). Suppose that J is single valued. Then J is said to be *weakly sequentially continuous* if for each $\{x_n\} \in X$ with $x_n \rightharpoonup x$, $J(x_n) \overset{*}{\rightharpoonup} J(x)$ ([10]).

Let μ be a mean on positive integers $\mathbb{N}$, that is, a continuous linear functional on ℓ^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \leq \mu(a) \leq \sup\{a_n : n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, ...) \in \ell^\infty$. We sometimes use $\mu_n(a_n)$ in place of $\mu(a)$. A mean μ on $\mathbb{N}$ is called a *Banach limit* if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, ...) \in \ell^\infty$. Using the Hahn-Banach theorem, we can prove the existence of a Banach limit. We know that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, ...) \in \ell^\infty$. Let $\{x_n\}$ be a bounded sequence in E. Then we can define the real valued continuous convex function ϕ on E by

$$\phi(z) = \mu_n \|x_n - z\|^2$$

for each $z \in E$.

The following lemma which was given in [11, 13, 30] is, in fact, a variant of Lemma 1.2 in [26].

LEMMA 1. *Let C be a nonempty closed convex subset of a Banach space X with a uniformly Gâteaux differentiable norm and let $\{x_n\}$ be a bounded sequence in X. Let μ be a Banach limit and $u \in C$. Then*

$$\mu_n \|x_n - u\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2$$

if and only if

$$\mu_n (x - u, J(x_n - u)) \leq 0$$

for all $x \in C$, where J is the duality mapping.

Recall that a closed convex subset C of X is said to have the *fixed point property* for nonexpansive self-mappings if every nonexpansive mapping $T : C \to C$ has a fixed point, i.e. there is a point $p \in C$ such that $T(p) = p$. It is well-known that every bounded closed convex subset of a uniformly smooth Banach space has the fixed point property for nonexpansive self-mapping (cf. [9, p. 45]). We also observe that spaces which enjoy the fixed point property for nonexpansive self-mappings are not necessarily spaces with a uniformly Gâteaux differentiable norm. The converse of this fact appears to be unknown as well.

Finally, to fix the notation, we will denote the closure and boundary of C by $\overline{C}$ and ∂C respectively, and for u, $v \in X$ we use $\mathrm{seg}[u, v]$ to denote the segment $\{tu + (1-t)v : t \in [0,1]\}$. We will also use $B(x; r)$ and $\overline{B}(x; r)$ to stand for the open ball $\{z \in X : \|x - z\| < r\}$ and the closed ball $\{z \in X : \|x - z\| \leq r\}$ respectively. We denote the distance between the sets A and B by $\mathrm{dist}(A, B)$, i.e.,

$$\mathrm{dist}(A, B) = \inf\{\|a - b\| : a \in A, \ b \in B\}.$$

3. Main results

We begin with the following lemma which was given in [4].

LEMMA 2. *Let X be a Banach space, C a bounded open subset of X, and $T : \overline{C} \to X$ a continuous mapping which is locally nonexpansive on C. Suppose that for each $z \in C$, there exists a continuous path $t \longmapsto z_t$, $0 \le t < 1$ satisfying*

$$z_t = tT(z_t) + (1-t)z \in C$$

and

$$\operatorname{dist}(\{z_t\}, \partial C) > 0.$$

Let $z \in C$, $d = \operatorname{dist}(\{z_t\}, \partial C)$, and $w \in C$ with $\|w - z\| < d$. If $\{w_t\}$ is the path corresponding to w, then $\|w_t - z_t\| \le \|w - z\|$ for all $0 \le t < 1$.

As in [4], we prepare the following more general proposition for our main results by using Lemma 1 and 2.

PROPOSITION 1. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm for which the closed unit ball has the fixed point property for nonexpansive self-mappings. Let C be a bounded open subset of X and let $T : \overline{C} \to X$ be a continuous mapping which is locally nonexpansive on C. Suppose that for each $z \in C$, there exists a continuous path $t \longmapsto z_t$, $0 \le t < 1$ satisfying*

$$z_t = tT(z_t) + (1-t)z \in C$$

and

$$\operatorname{dist}(\{z_t\}, \partial C) > 0,$$

and that each component of C contains a fixed point of T. Then for each $z \in C$, the strong $\lim_{t \to 1^-} z_t$ exists and is a fixed point of T.

PROOF. Since each component of C contains a fixed point of T, the set

$$E = \{z \in C : \text{the strong } \lim_{t \to 1^-} z_t \text{ exists}\}$$

is nonempty. So it suffices to show that E is both open and closed in C.

To see that E is closed in C, let $\{z^n\} \subset E$ and $z^n \to z \in C$. We can choose $n_0 \in \mathbb{N}$ such that $\|z^n - z\| < \operatorname{dist}(\{z_t\}, \partial C)$ for $n \geq n_0$, and by Lemma 2, $\|z_t^n - z_t\| \leq \|z^n - z\|$ for all $0 \leq t < 1$. Therefore

$$\begin{aligned} \|z_t - z_s\| &\leq \|z_t - z_t^n\| + \|z_t^n - z_s^n\| + \|z_s^n - z_s\| \\ &\leq 2\|z^n - z\| + \|z_t^n - z_s^n\| \end{aligned}$$

for all $n \geq n_0$. Consequently, $\{z_t\}$ is a Cauchy sequence and $z \in E$.

Now we show that E is open in C by using Lemma 1 and a variant patterned after [4] and [27]. Let $z \in E$ and $w = \lim_{t\to 1^-} z_t$. Then w is a fixed point of T. Let $d = \operatorname{dist}(\{z_t\}, \partial C)$, and suppose $y \in B(z, \frac{d}{3}) \cap C$. Let $\frac{2d}{3} < d_1 < d$. Then the closed ball $\overline{B}(w; d_1) \subset C$ is invariant under T. Let $t_n \to 1^-$ and $x_n = y_{t_n}$. Since

$$\begin{aligned} \|x_n - w\| &\leq \|y_{t_n} - z_{t_n}\| + \|z_{t_n} - w\| \\ &\leq \|z - y\| + \|z_{t_n} - w\| \\ &< \frac{d}{3} + \|z_{t_n} - w\|, \end{aligned}$$

$x_n \in \overline{B}(w; d_1)$ for all n sufficiently large. We also have $x_n - T(x_n) \to 0$ as $n \to \infty$. We now define $\phi : X \to \mathbb{R}^+$ by $\phi(x) = \mu_n \|x_n - x\|^2$. Since ϕ is continuous and convex and X is reflexive, ϕ attains its infimum over $\overline{B}(w; d_1)$ (cf. [1, p. 79]). Let

$$K = \{u \in \overline{B}(w; d_1) : \phi(u) = \inf\{\phi(x) : x \in \overline{B}(w; d_1)\}\}.$$

Then it is clearly that K is nonempty closed bounded and convex. If $u \in K$, then

$$\begin{aligned} \phi(T(u)) &= \mu_n \|x_n - T(u)\|^2 \\ &= \mu_n \|T(x_n) - T(u)\|^2 \\ &\leq \mu_n \|x_n - u\|^2 \\ &= \phi(u), \end{aligned}$$

so that K is invariant under T. Therefore, due to the assumption, T has a fixed point in K. Denote such a fixed point v. Since T is nonexpansive on $\overline{B}(w; d_1)$ and v is a fixed point of T, we have

$$\begin{aligned} \langle \frac{1}{t_n} x_n - (\frac{1}{t_n} - 1)y - v, J(v - x_n) \rangle &= \langle T(x_n) - T(v), J(v - x_n) \rangle \\ &\geq -\|T(x_n) - T(v)\| \|J(v - x_n)\| \\ &\geq -\|v - x_n\|^2 \\ &= \langle x_n - v, J(v - x_n) \rangle \end{aligned}$$

and hence $\langle (\frac{1}{t_n} - 1)(x_n - y), J(v - x_n)\rangle \geq 0$, where J is the duality mapping. So, we obtain

$$\langle x_n - y, J(x_n - v)\rangle \leq 0$$

for all sufficiently large n, and

$$\mu_n \langle x_n - y, J(x_n - v)\rangle \leq 0. \tag{4}$$

On the other hand, without loss of generality, we assume that $y - v \notin \overline{B}(w; d_1)$. But, since $v \in \overline{B}(w; d_1)$, we can choose $\tau > 0$ such that $v + t(y - v) \in \overline{B}(w; d_1)$ for all $t \in (0, \tau]$. Then, since it follows from Lemma 1 that

$$\mu_n \langle x - v, J(x_n - v)\rangle \leq 0$$

for all $x \in \overline{B}(w; d_1)$, we have

$$\mu_n \langle y - v, J(x_n - v)\rangle \leq 0. \tag{5}$$

Combining (4) and (5), we obtain

$$\mu_n \|x_n - v\|^2 = \mu_n \langle x_n - v, J(x_n - v)\rangle \leq 0.$$

Therefore, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to v. To complete the proof, let $\{x_{m_k}\}$ be another subsequence of $\{y_t : t \in [0, 1)\}$ such that $x_{m_k} = x_{t_{m_k}}$, $t_{m_k} \to 1^-$ as $k \to \infty$, and $x_{m_k} \to u$, where $T(u) = u$. Then (4) and (5) implies that

$$\langle v - y, J(v - u)\rangle \leq 0$$

and

$$\langle u - y, J(u - v)\rangle \leq 0.$$

Hence $v = u$ and the strong $\lim_{t \to 1^-} y_t$ exists. Therefore E is open in C. This completes the proof. □

REMARK 1. Proposition 1 is an improvement of Proposition 1 of [4].

The following lemma is essentially [21, Theorem 1]. For the sake of completion, we include its proof.

LEMMA 3. *Let X be a Banach space for which the closed unit ball has the fixed point property for nonexpansive self-mapping. Let C be a bounded open subset of X and $T : \overline{C} \to X$ a continuous mapping which is locally pseudo-contractive on C. Suppose that there exists $z \in C$ such that*

$$\|z - T(z)\| < \|x - T(x)\| \qquad \text{for all } x \in \partial C. \tag{6}$$

Then T has a fixed point in C.

PROOF. Let $A = I - T$. Then A is continuous on $\overline{C}$ and locally accretive on C, which satisfies (6) on ∂C. Therefore by Theorem 1 in [21], A has a zero in C. Thus T has a fixed point in C. □

We also need the following Lemma due to Morales [21] for the proof of main results.

LEMMA 4 (Lemma 1 of [21]). *Let X be a Banach space, C a nonempty bounded open subset of X, and $T : \overline{C} \to X$ a continuous mapping which is locally nonexpansive on C. Suppose that there exists $z \in C$ so that*

$$\|z - T(z)\| < \|x - T(x)\| \qquad \text{for all } x \in \partial C.$$

Then

(i) *there exists a unique continuous path $t \longmapsto z_t \in C$, $0 \le t < 1$, such that $z_t = tT(z_t) + (1-t)z$;*
(ii) *the function $h : [0,1) \to \mathbb{R}$ defined by $h(t) = \|z_t - T(z_t)\|$ is nonincreasing;*
(iii) *if $\|u - T(u)\| < \rho$ for $u \in C$, then $B(u;\rho) \subset C$, where $\rho = \frac{\|z-T(z)\|}{3}$;*
(iv) *the set $\{T(z_t) : 0 \le t < 1\}$ is bounded provided that $\{z_t : 0 \le t < 1\}$ is a bounded set.*

Now, we study the strong convergence of $\{z_t\}$ in a reflexive Banach space with a uniformly Gâteaux differentiable norm. Our first result will be for locally nonexpansive mappings.

THEOREM 1. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty bounded open subset of X, and $T : \overline{C} \to X$ a continuous mapping which is locally nonexpansive on C. Suppose that the closed unit ball of X has the fixed point property for nonexpansive self-mapping. Suppose that there exists $z \in C$ such that*

$$\|z - T(z)\| < \|x - T(x)\| \quad \text{for all } x \in \partial C.$$

Then there exists a unique path $t \longmapsto z_t \in C$, $0 \le t < 1$, satisfying

$$z_t = tT(z_t) + (1-t)z,$$

where the strong $\lim_{t\to 1^-} z_t$ exists and this limit is a fixed point of T.

PROOF. We make use of the idea in [21]. We first observe that due to Lemma 3 we may derive that the component of C which contains z has a fixed point of T. Since

$$z_t - T(z_t) = (1-t)(z - T(z_t)),$$

it follows from Lemma 4 (iv) that $\|z_t - T(z_t)\| \to 0$ as $t \to 1^-$. This implies that there exists $\alpha \in (0,1)$ so that $\|z_t - T(z_t)\| < \rho$ (ρ as in Lemma 4 (iii)) for all $t \in (\alpha, 1)$. Then by Lemma 4 (iii), we have

$$\operatorname{dist}(\{z_t\}, \partial C) > 0.$$

Hence the result follows from Proposition 1. □

We are ready to establish the main result for a more general class of mappings as a consequence of the above result.

THEOREM 2. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty bounded open subset of X, and $T : \overline{C} \to X$ a continuous mapping which is locally pseudo-contractive on C. Suppose that the closed unit ball of X has the fixed point property for nonexpansive self-mapping. Suppose that there exists $z \in C$ such that*

$$\|z - T(z)\| < \|x - T(x)\| \quad \text{for all } x \in \partial C.$$

Then there exists a unique path $t \longmapsto z_t \in C$, $0 \le t < 1$, satisfying

$$z_t = tT(z_t) + (1-t)z, \tag{7}$$

where the strong $\lim_{t\to 1^-} z_t$ exists and this limit is a fixed point of T.

PROOF. We also follow the arguments developed in [21]. Let $F(x) = 2x - Tx$. Then by Proposition 2 of [24], F^{-1} exists on $F(C)$, and by the assumption on T, F^{-1} is also locally nonexpansive on $F(C)$. Now since it is not known whether F is one-to-one on ∂C, we will redefine the domain of T to assure that F is also invertible on the boundary of its domain. Due to Theorem 2 of [19], we may choose $w \in C$ such that

$$\|w - T(w)\| < \|z - T(z)\|.$$

We now define the set

$$C_0 = \{x \in C : \|x - T(x)\| < \|z - T(z)\|\}.$$

Then $\partial C_0 \subset C$, and

$$\|w - T(w)\| < \|x - T(x)\| \quad \text{for } x \in \partial C_0.$$

This means that the path $t \longmapsto w_t$ exists and is uniquely defined on $(0, 1)$ (Lemma 3 of [19]). Also by Assertion 1 of [15], we know that $\text{seg}[w, F(w)] \subset F(C_0)$. Therefore F^{-1} is nonexpansive on $\text{seg}[w, F(w)]$, and

$$\|w - F^{-1}(w)\| \leq \|w - F(w)\| < \|x - T(x)\| \quad \text{for } x \in \partial C_0.$$

Since $\partial F(C_0) = F(\partial C_0)$, for each $y \in \partial F(C_0)$ there exists $x \in \partial C_0$ so that $y = F(x)$ and

$$\|w - F^{-1}(w)\| < \|y - F^{-1}(y)\|.$$

Consequently, by Theorem 1, there exists a unique path $t \longmapsto u_t \in F(C_0)$, $t \in [0, 1)$, satisfying the equation

$$u_t = tF^{-1}(u_t) + (1 - t)w,$$

where the strong $\lim_{t \to 1^-} u_t$ exists and is a fixed point of F^{-1}. By the uniqueness of the path $t \longmapsto w_t$, $F^{-1}(u_t) = w_s$, where $s = \frac{1}{2-t}$, and hence the strong $\lim_{t \to 1^-} w_t$ exists. Finally, to prove that the path described by (7) strongly converges to a fixed point of T, we can follow the arguments of the proof of Theorem 1 of [24]. □

As the immediate consequences of Theorem 1 and 2, we have following:

COROLLARY 1 ([21, Theorem 3]). *Let X^* be a uniformly convex Banach space, C a nonempty bounded open subset of X, and $T : \overline{C} \to X$ a continuous mapping which is locally nonexpansvive on C Suppose that there exists $z \in C$ such that*

$$\|z - T(z)\| < \|x - T(x)\| \quad \text{for all } x \in \partial C.$$

Then there exists a unique path $t \longmapsto z_t \in C$, $0 \le t < 1$, satisfying

$$z_t = tT(z_t) + (1 - t)z,$$

where the strong $\lim_{t\to 1^-} z_t$ exists and this limit is a fixed point of T.

COROLLARY 2 ([21, Theorem 4]). *Let X^* be a uniformly convex Banach space, C a nonempty bounded open subset of X, and $T : \overline{C} \to X$ a continuous mapping which is locally pseudo-contractive on C. Suppose that there exists $z \in C$ such that*

$$\|z - T(z)\| < \|x - T(x)\| \quad \text{for all } x \in \partial C.$$

Then there exists a unique path $t \longmapsto z_t \in C$, $0 \le t < 1$, satisfying

$$z_t = tTz_t + (1 - t)z,$$

where the strong $\lim_{t\to 1^-} z_t$ exists and this limit is a fixed point of T.

REMARK 2. Theorem 1 is also an extension of Theorem 1 of [4] and Theorem 2 is an improvement of Theorem 1 of [20].

We next obtain convergences of the path $\{z_t\}$ for spaces with a weakly sequentially continuous duality mapping.

THEOREM 3. *Let X be a reflexive Banach space with a weakly sequentially continuous duality mapping, C a nonempty bounded open subset of X, and $T : \overline{C} \to X$ a continuous mapping which is locally nonexpansive on C. Suppose that there exists $z \in C$ such that*

$$\|z - T(z)\| < \|x - T(x)\| \quad \text{for all } x \in \partial C.$$

Then there exists a unique path $t \longmapsto z_t \in C$, $0 \le t < 1$, satisfying

$$z_t = tT(z_t) + (1 - t)z,$$

where the strong $\lim_{t\to 1^-} z_t$ exists and this limit is a fixed point of T.

PROOF. We first observe that due to Lemma 1 of [10], the duality mapping J is single-valued. Now let $x_n = z_{t_n}$ for $t_n \to 1^-$ as $n \to \infty$ and let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v$. Since $z_t - T(z_t) = (1-t)(z - T(z_t))$, we also have from Lemma 4 (iv) that

$$\|z_t - T(z_t)\| \to 0 \quad \text{as } t \to 1^-.$$

So, we have

$$\begin{aligned}\limsup_{k\to\infty} \|x_{n_k} - T(v)\| &\le \limsup_{k\to\infty} \{\|x_{n_k} - T(x_{n_k})\| + \|T(x_{n_k}) - T(v)\|\} \\ &\le \limsup_{k\to\infty} \|x_{n_k} - v\|.\end{aligned}$$

If $T(v) \neq v$, from Theorem 1 in [10], we have

$$\begin{aligned}\limsup_{k\to\infty} \|x_{n_k} - v\| &< \limsup_{k\to\infty} \|x_{n_k} - T(v)\| \\ &\le \limsup_{k\to\infty} \|x_{n_k} - v\|.\end{aligned}$$

This is a contradiction. Hence v is a fixed point of T. Since T is nonexpansive, by the same argument in proof of Proposition 1, we have

$$\langle z - x_{n_k}, J(x_{n_k} - v\rangle \ge 0.$$

Thus it follows that

$$\begin{aligned}\|x_{n_k} - v\|^2 &\le \langle x_{n_k} - v, J(x_{n_k} - v)\rangle + \langle z - x_{n_k}, J(x_{n_k} - v)\rangle \\ &= \langle z - v, J(x_{n_k} - v)\rangle.\end{aligned}$$

Since $x_{n_k} \rightharpoonup v$ and J is weakly sequentially continuous, we have $x_{n_k} \to v$. We now follow the last argument given in the proof of Proposition 1, to conclude that $\{z_t\}$ converges strongly to v in C. This completes the proof. □

We also have the following result as a consequence of Theorem 3.

THEOREM 4. *Let X be a reflexive Banach space with a weakly sequentially continuous duality mapping, C a nonempty bounded open subset of X, and $T : \overline{C} \to X$ a continuous mapping which is locally pseudo-contractive on C. Suppose that there exists $z \in C$ such that*

$$\|z - T(z)\| < \|x - T(x)\| \quad \text{for all } x \in \partial C.$$

Then there exists a unique path $t \longmapsto z_t \in C$, $0 \leq t < 1$, satisfying

$$z_t = tT(z_t) + (1 - t)z,$$

where the strong $\lim_{t \to 1^-} z_t$ exists and this limit is a fixed point of T.

PROOF. The result follows from the proof of Theorem 2. □

REMARK 3. Theorem 4 is a local version of Theorem 3 in [23].

REMARK 4. In the case that $T : \overline{C} \to X$ is a closed mapping which is continuous and locally pseudo-contractive as well as locally nonexpansive on C in the above results, we can also obtain the same results using the methods developed in [22] and [25].

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