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A STUDY ON CARLESON MEASURES WITH RESPECT TO GENERAL APPROACH REGIONS

CHOON-SERK SUH

ABSTRACT. In this paper we first introduce a space of homogeneous type X, and we consider a kind of generalized upper half-space $X \times (0, \infty)$. We are mainly concerned with some inequalities in terms of Carleson measures or in terms of certain maximal operators with respect to general approach regions in $X \times (0, \infty)$. The main tool of the proof is the Whitney decomposition.

1. Introduction

Recently enormous progress in harmonic analysis has been made. This paper will announce some problems related to harmonic analysis.

In this paper we first introduce a space of homogeneous type X, which is a more general setting than $\mathbb{R}^n$, and we also consider a kind of generalized upper half-space $X \times (0, \infty)$. Suppose that for each $x \in X$ we are given a set $\Omega_x \subset X \times (0, \infty)$. Let Ω denote the family $\{\Omega_x\}_{x \in X}$. Then we define a maximal function $A_\Omega^\infty(f)$, with respect to Ω, acting on a function f on $X \times (0, \infty)$, and an (Ω, β)-Carleson measure ν of order β, with respect to Ω, on $X \times [0, \infty)$.

The purpose of this paper is to study some inequalities in terms of an (Ω, β)-Carleson measure ν, or in terms of a maximal function $A_\Omega^\infty(f)$ in the context of the space of homogeneous type X.

Throughout this paper we shall use the letter C to denote a constant which need not be the same at each occurrence.

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2. Some preliminaries

We begin by introducing the notion of a space of homogeneous type [2]: Let X be a topological space endowed with Borel measure μ. Assume that d is a pseudo-metric on X, that is, a nonnegative function on $X \times X$ with the properties:

(i) $d(x,x) = 0; d(x,y) > 0$ if $x \neq y$,
(ii) $d(x,y) = d(y,x)$, and
(iii) $d(x,z) \leq K(d(x,y) + d(y,z))$, where K is some fixed constant.

Assume further that

(a) the balls $B(x,\rho) = \{y \in X : d(x,y) < \rho\}$, $\rho > 0$, form a basis of open neighborhoods at $x \in X$

and that μ satisfies the doubling property:

(b) $0 < \mu(B(x,2\rho)) \leq A\mu(B(x,\rho)) < \infty$, where A is some fixed constant.

Then we call (X, d, μ) a *space of homogeneous type.*

Property (iii) will be referred to as the "triangle inequality."

Now consider the space $X \times (0,\infty)$, which is a kind of generalized upper half-space over X. Suppose that there is a given set $\Omega_x \subset X \times (0,\infty)$ for each $x \in X$. Let Ω denote the family $\{\Omega_x\}_{x \in X}$. Thus at each $x \in X$, Ω determines a collection of balls, namely, $\{B(y,t) : (y,t) \in \Omega_x\}$.

For a measurable function f on $X \times (0,\infty)$ and $x \in X$, we define a maximal function of f, with respect to Ω as

$$A_\Omega^\infty(f)(x) = \sup_{(y,t) \in \Omega_x} |f(y,t)|.$$

Throughout this paper we will always assume that Ω is chosen so that $A_\Omega^\infty(f)$ is a measurable function on X, and that $\Omega = \{\Omega_x\}_{x \in X}$ is a symmetric family, i.e., if $x \in \Omega_y(t)$, then $y \in \Omega_x(t)$, where $\Omega_x(t) = \{y \in X : (y,t) \in \Omega_x\}$.

For any set $E \subset X$, the *tent* over E, with respect to Ω, is the set

$$T(E_\Omega) = (X \times (0,\infty)) \setminus \bigcup_{x \notin E} \Omega_x.$$

It is then very easy to check that

$$T(E_\Omega) = \{(y,t) \in X \times (0,\infty) : \Omega_y(t) \subset E\}.$$

The *tent space* $T^p_{\infty,\Omega}$ is defined as the space of functions f on $X \times (0,\infty)$, so that $A^{\infty}_{\Omega}(f) \in L^p(d\mu)$, $0 < p < \infty$, and set

$$||f||_{T^p_{\infty,\Omega}} = ||A^{\infty}_{\Omega}(f)||_{L^p(d\mu)}.$$

For a measure ν on $X \times [0,\infty)$ and $\beta \geq 1$, we say that ν is an (Ω,β)-*Carleson measure* of order β, with respect to Ω, and write $\nu \in V^{\beta}_{\Omega}$ if

$$\sup_B \frac{|\nu|(T(B_\Omega))}{[\mu(B)]^\beta} \leq C < \infty, \tag{1}$$

where the supremum is taken over all balls B in X. Note that we can make the space of (Ω,β)-Carleson measures into a Banach space with norm equal to the left side of (1).

3. Main results

We begin with a lemma which is of the type due to Whitney in [3].

LEMMA 1. *Let O be an open subset of X. Then there are positive constants $M, h_1 > 1, h_2 > 1$ and $h_3 < 1$ which depend only on the space X, and a sequence $\{B(x_i,\rho_i)\}$ of balls such that*

(i) $\cup_i B(x_i,\rho_i) = O$,
(ii) $B(x_i,h_2\rho_i) \subset O$ *and* $B(x_i,h_1\rho_i) \cap (X \setminus O) \neq \emptyset$,
(iii) *the balls* $B(x_i,h_3\rho_i)$ *are pairwise disjoint, and*
(iv) *no point in O lies in more than M of the balls* $B(x_i,h_2\rho_i)$.

LEMMA 2. *Suppose $\Omega = \{\Omega_x\}_{x\in X}$ is a symmetric family of sets such that $\Omega_x(t)$ is open for all $(x,t) \in X \times (0,\infty)$. Let f be a measurable function on $X \times (0,\infty)$. Then $A^{\infty}_{\Omega}(f)$ is lower semicontinuous, that is, for all $\lambda > 0$, the set $\{x \in X : A^{\infty}_{\Omega}(f)(x) > \lambda\}$ is open.*

PROOF. If $A^{\infty}_{\Omega}(f)(x) > \lambda$, then there is a point $(z,t) \in \Omega_x$ so that $|f(z,t)| > \lambda$. By hypothesis, we have $x \in \Omega_z(t)$ and there is an $\varepsilon > 0$ such that if $d(x,y) < \varepsilon$ then $y \in \Omega_z(t)$. Again, by symmetry, $(z,t) \in \Omega_y$ and so $A^{\infty}_{\Omega}(f)(y) > \lambda$ if $d(x,y) < \varepsilon$. Thus the proof is complete. □

THEOREM 3. *Suppose* $\Omega = \{\Omega_x\}_{x \in X}$ *is as the hypothesis of Lemma 2. Let* f *be a measurable function on* $X \times (0,\infty)$ *and* $\nu \in V_\Omega^\beta, \beta \geq 1$. *Then there is a constant* C *such that*

$$\begin{aligned}|\nu|(\{(x,t) \in X \times (0,\infty) : |f(x,t)| > \lambda\}) \\ \leq C[\mu(\{x \in X : A_\Omega^\infty(f)(x) > \lambda\})]^\beta\end{aligned}$$

for each $\lambda > 0$.

PROOF. For each $\lambda > 0$, we define the set O^λ by

$$O^\lambda = \{x \in X : A_\Omega^\infty(f)(x) > \lambda\}.$$

Since $A_\Omega^\infty(f)$ is lower semicontinuous by Lemma 2, O^λ is an open set. Let

$$O^\lambda = \bigcup_{j=1}^\infty B(x_j^\lambda, \rho_j^\lambda) \equiv \bigcup_{j=1}^\infty B_j^\lambda$$

be a Whitney decomposition of the open set O^λ. Let

$$\tilde{B}_j^\lambda = B(x_j^\lambda, Ch_1\rho_j^\lambda),$$

where h_1 is given in (ii) of Lemma 1, and C will be chosen sufficiently large in a moment. If $(x,t) \in T(O_\Omega^\lambda)$, then $B(x,t) \subset O^\lambda$, and $x \in B_j^\lambda$ for some j. Let $y \in B(x_j^\lambda, h_1\rho_j^\lambda) \cap (X \setminus O^\lambda)$. Then

$$\begin{aligned}t &\leq d(x,y) \\ &\leq K(d(x,x_j^\lambda) + d(x_j^\lambda, y)) \\ &\leq K(1+h_1)\rho_j^\lambda.\end{aligned} \tag{2}$$

Hence if $z \in B(x,t)$, then it follows from (2) that

$$\begin{aligned}d(x_j^\lambda, z) &\leq K(d(x_j^\lambda, x) + d(x,z)) \\ &< K(\rho_j^\lambda + t) \\ &< K(\rho_j^\lambda + K(1+h_1)\rho_j^\lambda) \\ &= K(1 + K(1+h_1))\rho_j^\lambda.\end{aligned}$$

Thus if we choose C so that $K(1+K(1+h_1)) < Ch_1$, then it follows that

$$B(x,t) \subset B(x_j^\lambda, Ch_1\rho_j^\lambda) \equiv \tilde{B}_j^\lambda,$$

and hence $(x,t) \in T(\tilde{B}^{\lambda}_{j,\Omega})$. Then we have

$$\{(x,t) \in X \times (0,\infty) : |f(x,t)| > \lambda\} \subset \bigcup_{j=1}^{\infty} T(\tilde{B}^{\lambda}_{j,\Omega}).$$

Since ν is an (Ω,β)-Carleson measure, we have

$$\begin{aligned} & |\nu|(\{(x,t) \in X \times (0,\infty) : |f(x,t)| > \lambda\}) \\ \leq & \sum_{j=1}^{\infty} |\nu|(T(\tilde{B}^{\lambda}_{j,\Omega})) \\ \leq & C \sum_{j=1}^{\infty} [\mu(\tilde{B}^{\lambda}_{j})]^{\beta} \\ \leq & C \sum_{j=1}^{\infty} [\mu(B^{\lambda}_{j})]^{\beta} \qquad \text{(by the doubling property)} \\ = & C[\mu(\{x \in X : A^{\infty}_{\Omega}(f)(x) > \lambda\})]^{\beta}. \end{aligned}$$

Thus

$$\begin{aligned} & |\nu|(\{(x,t) \in X \times (0,\infty) : |f(x,t)| > \lambda\}) \\ \leq & C[\mu(\{x \in X : A^{\infty}_{\Omega}(f)(x) > \lambda\})]^{\beta}. \end{aligned}$$

The proof is therefore complete. □

THEOREM 4. *Suppose $\Omega = \{\Omega_x\}_{x \in X}$ is as the hypothesis of Lemma 2. Let $f \in T^{p}_{\infty,\Omega}, 0 < p < \infty$, and $\nu \in V^{1}_{\Omega}$. Then there is a constant C such that*

$$\int_{X \times (0,\infty)} |f(x,t)|^{p} |d\nu(x,t)| \leq C||A^{\infty}_{\Omega}(f)||^{p}_{L^{p}(d\mu)}.$$

PROOF. Let $f \in T^{p}_{\infty.\Omega}, 0 < p < \infty$, and $\nu \in V^{1}_{\Omega}$. Then it follows from

Theorem 3 that

$$
\begin{aligned}
&\left(\int_{X\times(0,\infty)} |f(x,t)|^p |d\nu(x,t)|\right)^{1/p} \\
&= \left(p\int_0^\infty \lambda^{p-1}|\nu|(\{(x,t)\in X\times(0,\infty) : |f(x,t)|>\lambda\})d\lambda\right)^{1/p} \\
&\le C\left(p\int_0^\infty \lambda^{p-1}\mu(\{x\in X : A_\Omega^\infty(f)(x)>\lambda\})d\lambda\right)^{1/p} \\
&= C\left(\int_X [A_\Omega^\infty(f)(x)]^p d\mu(x)\right)^{1/p} \\
&= C||A_\Omega^\infty(f)||_{L^p(d\mu)}.
\end{aligned}
$$

Thus

$$
\int_{X\times(0,\infty)} |f(x,t)|^p |d\nu(x,t)| \le C||A_\Omega^\infty(f)||^p_{L^p(d\mu)}.
$$

The proof is therefore complete. □

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Department of Mathematics
Dongyang University
Youngju 750-711, Korea
E-mail: cssuh@phenix.dyu.ac.kr