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CONVERGENCE THEOREMS IN MEASURES FOR THE OPERATOR-VALUED FEYNMAN INTEGRAL

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Abstract. The existence of the operator-valued Feynman integral was established when a Wiener functional is given by a Fourier transform of complex Borel measures. In this paper, we investigate the stability of the Feynman integral with respect to the measures.

1. Introduction

Let $\mathbb{C}$, $\mathbb{C}_+$ and $\tilde{\mathbb{C}}_+$ be the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part, respectively. For a given $t > 0$ and an integer $N \geq 1$ let C^t be the space of $\mathbb{R}^N$-valued continuous functions x on $[0, t]$. C_0^t denotes the Wiener space, that is, the set of all $x \in C^t$ which vanish at 0. m denotes Wiener measure on C_0^t.

Let F be a function from C^t to $\mathbb{C}$. Given $\lambda > 0$, $\psi \in L^2$ $(\mathbb{R}^N)$ and $\xi \in \mathbb{R}^N$, let

$$(1.1) \qquad (K_\lambda(F)\psi)(\xi) = \int_{C_0^t} F(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(t) + \xi)\, dm(x).$$

Definition. The operator-valued function space integral $K_\lambda(F)$ exists for $\lambda > 0$ if (1.1) defines $K_\lambda(F)$ as a bounded linear operator on $L^2(\mathbb{R}^N)$. If, in addition, the operator-valued function $K_\lambda(F)$, as a function of λ, has an extension to an analytic function in $\mathbb{C}_+$ and a strongly continuous function in $\tilde{\mathbb{C}}_+$, we say that $K_\lambda(F)$ exists for $\lambda \in \tilde{\mathbb{C}}_+$. When

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λ is purely imaginary, $K_\lambda(F)$ is called the operator-valued Feynman integral of F.

For $s > 0$, $\lambda \in \tilde{\mathbb{C}}_+$ and $\psi \in L^2(\mathbb{R}^N)$, let

$$(1.2)\qquad \begin{aligned}&(\exp[-s(H_0/\lambda)]\psi)(\xi)\\ &= \Big(\frac{\lambda}{2\pi s}\Big)^{\frac{N}{2}} \int_{\mathbb{R}^N} \psi(u)\ \exp\Big(-\frac{\lambda\|u-\xi\|^2}{2s}\Big)\,du.\end{aligned}$$

The integral in (1.2) exists as an ordinary Lebesgue integral for $\lambda \in \mathbb{C}_+$, but, when λ is purely imaginary and ψ is not integrable, the integral should be interpreted in the mean as in the theory of the Fourier-Plancherel transform.

$M(0,t)$ will denote the space of complex Borel measures η on $(0,t)$. Then every measure $\eta \in M(0,t)$ has a unique decomposition $\eta = \mu + \eta_d$ into a continuous part μ and a discrete part η_d [8]. The case where η_d has a finite support is most likely to be of interest. So, let $\eta_d = \sum_{j=1}^h \omega_j \delta_{\tau_j}$ where δ_{τ_j} is as usual the Dirac measure at $\tau_j \in (0,t)$, $0 < \tau_1 < \cdots < \tau_h < t$ and $\omega_j \in \mathbb{C}$ for $j = 1, 2, \cdots, h$.

Let $L_{\infty 1;\eta}$ be the space of $\mathbb{C}$-valued, Borel measurable functions θ on $(0,t) \times \mathbb{R}^N$ such that $\|\theta\|_{\infty 1;\eta} := \int_{(0,t)} \|\theta(s,\cdot)\|_\infty \, d|\eta|(s) < \infty$.

Let $M(\mathbb{C})$ be the space of complex Borel measures on $\mathbb{C}$. The Fourier transform of $\nu \in M(\mathbb{C})$ is the function $\hat{\nu}$ defined by $\hat{\nu}(u) = \int_{\mathbb{C}} e^{-iuv}\, d\nu(v)$ $u \in \mathbb{C}$.

Consider the functional for $\nu \in M(\mathbb{C})$, $\theta \in L_{\infty 1;\eta}$ and $\eta \in M(0,t)$.

$$(1.3)\qquad F(x) = \hat{\nu}\Big(\int_{(0,t)} \theta(s, x(s))\, d\eta(s)\Big), \qquad x \in C^t.$$

Then, the following lemmas are contained in [2].

LEMMA 1. *$K_\lambda(F)$ exists for $\lambda > 0$.*

LEMMA 2. *$K_\lambda(F)$ exists for $\lambda \in \tilde{\mathbb{C}}_+$ and is given by the generalized Dyson series, provided that $\nu \in M(\mathbb{C})$ satisfies*

$$\int_{\mathbb{C}} e^{\|\theta\|_{\infty 1;\eta}|u|}\, d|\nu|(u) < \infty$$

i.e., for all $\lambda \in \tilde{\mathbb{C}}_+$*, the following expansion of* $K_\lambda(F)$ *holds:*

$$K_\lambda(F) = \sum_{n=0}^{\infty} n! a_n \sum_{q_0+\cdots+q_h=n} \frac{\omega_1^{q_1}\cdots\omega_h^{q_h}}{q_1!\cdots q_h!} \times \sum_{k_1+\cdots+k_{h+1}=q_0} \int_{\Delta_{q_0;k_1,\cdots,k_{h+1}}} L_0 L_1 \cdots L_h \, d\mu(s_1)\cdots d\mu(s_{q_0})$$

where $q_0, \cdots, q_h, k_1, \cdots, k_{h+1}$ *are nonnegative integers,*

$$\begin{aligned}\Delta_{q_0;k_1,\cdots,k_{h+1}} = \{(s_1,\cdots,s_{q_0}) \in (0,t)^{q_0} : 0 < s_1 < \cdots < s_{k_1} \\ < \tau_1 < s_{k_1+1} < \cdots < s_{k_1+k_2} < \tau_2 < s_{k_1+k_2+1} < \cdots \\ < s_{k_1+\cdots+k_h} < \tau_h < s_{k_1+\cdots+k_h+1} < \cdots < s_{q_0} < t\}\end{aligned}$$

and, for $(s_1,\cdots,s_{q_0}) \in \Delta_{q_0;k_1,\cdots,k_{h+1}}$ *and* $r \in \{0,1,\cdots,h\}$

$$\begin{aligned}L_r =& [\theta(\tau_r)]^{q_r} e^{-(s_{k_1+\cdots+k_r+1}-\tau_r)(H_0/\lambda)}\theta(s_{k_1+\cdots+k_r+1}) \\ & e^{-(s_{k_1+\cdots+k_r+2}-s_{k_1+\cdots+k_r+1})(H_0/\lambda)}\theta(s_{k_1+\cdots+k_r+2})\cdots \\ & \theta(s_{k_1+\cdots+k_{r+1}})e^{-(\tau_{r+1}-s_{k_1+\cdots+k_{r+1}})(H_0/\lambda)}\end{aligned}$$

and $a_n = \frac{1}{n!}\int_{\mathbb{C}}(-i)^n u^n \, d\nu(u)$.

We use the conventions $\tau_0 = 0, \tau_{h+1} = t$ and $[\theta(\tau_0)]^{q_0} = 1$.

2. Stability theorems

We begin with a lemma which is easily proved and is essentially contained in [1].

LEMMA 3. *Let* $\{F_n(x)\}$ *be a sequence of Borel measurable functionals such that* $|F_n(x)| \le B$ *for some constant* $B > 0$ *and for all* $n = 1, 2, 3, \cdots$ *. Further suppose that for every* $\lambda > 0$

$$F_n(\lambda^{-\frac{1}{2}}x + \xi) \to F(\lambda^{-\frac{1}{2}}x + \xi) \quad \text{as} \quad n \to \infty$$

for $m \times$ *Leb. – a.e.* (x, ξ)*. Then for every* $\lambda > 0$

$$K_\lambda(F_n) \to K_\lambda(F) \quad \text{strongly as} \quad n \to \infty.$$

We consider stability in the measure η first. Let $\eta,\ \eta_n,\ n=1,2,\cdots$ be in $M(0,t)$. We say that η_n converges weakly to η provided that

$$\int_{(0,t)} b(u)\,d\eta_n(u) \to \int_{(0,t)} b(u)\,d\eta(u)$$

for every bounded continuous function b on $(0,t)$.

THEOREM 1. *Let θ be a continuous function bounded by a constant C on all of $(0,t)\times\mathbb{R}^N$. Let $\eta,\ \eta_n,\ n=1,2,\cdots$ be in $M(0,t)$. Assume that*

$$\eta_n \to \eta \quad \text{weakly as} \quad n\to\infty. \tag{2.1}$$

Let F be defined as (1.3) and F_n be defined as (1.3) except with η replaced by η_n. Then for all $\lambda>0$, $K_\lambda(F_n)\to K_\lambda(F)$ strongly as $n\to\infty$.

Further, if $\int_{\mathbb{C}} e^{\|\theta\|_{\infty 1:\eta}|u|}\,d|\nu|(u)<\infty$ and if there exists $N_0<\infty$ such that

$$\int_{\mathbb{C}} e^{\|\theta\|_{\infty 1:\eta_n}|u|}\,d|\nu|(u) < N_0 \tag{2.2}$$

for all $n=1,2,\cdots$, then for $\lambda\in\mathbb{C}_+$,

$$K_\lambda(F_n)\to K_\lambda(F) \quad \text{strongly as} \quad n\to\infty. \tag{2.3}$$

Further, the convergence in (2.3) is uniform on all compact subsets of $\mathbb{C}_+$.

PROOF. Let $\lambda>0$ and $\xi\in\mathbb{R}^N$ be given. Given $x\in C_0^t$, the function $\theta(s,\lambda^{-1/2}\,x(s)+\xi)$ is bounded by C and is continuous as a function of s. Hence, by (2.1),

$$\int_{(0,t)} \theta(s,\lambda^{-1/2}x(s)+\xi)\,d\eta_n(s) \to \int_{(0,t)} \theta(s,\lambda^{-1/2}x(s)+\xi)\,d\eta(s).$$

Since $\hat\nu$ is continuous

$$\begin{aligned}&\hat\nu\Big(\int_{(0,t)} \theta(s,\lambda^{-1/2}x(s)+\xi)\,d\eta_n(s)\Big)\\ &\quad\to \hat\nu\Big(\int_{(0,t)} \theta(s,\lambda^{-1/2}x(s)+\xi)\,d\eta(s)\Big)\end{aligned} \tag{2.4}$$

i.e. $F_n(\lambda^{-1/2}x+\xi) \to F(\lambda^{-1/2}x+\xi)$. Note that

$$(2.5) \qquad |F_n(x)| = \Big|\hat{\nu}\Big(\int_{(0,t)} \theta(s,x)\,d\eta_n(s)\Big)\Big| \le \|\nu\|$$

for all $x \in C^t$ and all $n = 1, 2, \cdots$. In a view of (2.4), (2.5) and Lemma 3 yields for all $\lambda > 0$,

$$(2.6) \qquad K_\lambda(F_n) \to K_\lambda(F) \quad \text{strongly as} \quad n \to \infty.$$

Let $\psi \in L^2(\mathbb{R}^N)$. We know from Lemma 2 that $K_\lambda(F_n)(\psi)$ and $K_\lambda(F)(\psi)$ are analytic for $\lambda \in \mathbb{C}_+$ and from the norm estimate [2],

$$\|K_\lambda(F_n)\psi\| \le \int_{\mathbb{C}} e^{\|\theta\|_{\infty 1;\eta_n}|u|}\,d|\nu|(u)\,\|\psi\|.$$

Hence

$$(2.7) \qquad \|K_\lambda(F_n)\psi\| \le N_0\,\|\psi\|$$

for all $\lambda \in \mathbb{C}_+$ and for all $n = 1, 2, \cdots$.

Hence, by (2.6) and (2.7), Vitali's theorem for operator-valued function [4, Theorem 3.14.1] assures us that $K_\lambda(F_n) \to K_\lambda(F)$ uniformly on all compact subsets of $\mathbb{C}_+$. □

Under the assumption $\eta_n \to \eta$ in norm, we can show that $K_\lambda(F_n) \to K_\lambda(F)$ in the norm topology for all $\lambda \in \mathbb{C}_+$.

THEOREM 2. *Assume that $\eta_n \to \eta$ in norm as $n \to \infty$ (i.e. $\|\eta_n - \eta\| \to 0$ as $n \to \infty$). Then, under the hypotheses of Theorem 1,*

$$(2.8) \qquad K_\lambda(F_n) \to K_\lambda(F) \quad \textit{in the operator norm topology}$$

as $n \to \infty$ uniformly in λ on the all compact subsets in $\mathbb{C}_+$. Moreover, for all $\lambda > 0$ we have the norm estimate

$$(2.9) \qquad \|K_\lambda(F_n) - K_\lambda(F)\| \le M_0\,\|\eta_n - \eta\|,$$

where $M = \|\theta\|_\infty$ and $M_0 = 2M \int_{\mathbb{C}} |u|\,d|\nu|(u)$.

PROOF. Clearly for all $\lambda > 0$, $x \in C_0^t$ and $\xi \in \mathbb{R}^N$

$$\Big| \int_{(0,t)} \theta(s, \lambda^{-1/2}x(s) + \xi)\, d\eta_n(s) - \int_{(0,t)} \theta(s, \lambda^{-1/2}x(s) + \xi)\, d\eta(s) \Big| \le M\, \|\eta_n - \eta\|.$$

Then, for $\lambda > 0$ and $\xi \in \mathbb{R}^N$,

$$\begin{aligned} &\Big| \exp\Big[-iu\Big(\int_{(0,t)} \theta(s, \lambda^{-1/2}x(s) + \xi)\, d\eta_n(s) \Big) \Big] \\ &\quad - \exp\Big[-iu\Big(\int_{(0,t)} \theta(s, \lambda^{-1/2}x(s) + \xi)\, d\eta(s) \Big) \Big] \Big| \\ &\le 2|u|\, M\, \|\eta_n - \eta\| \end{aligned}$$

since $|e^{-iuv} - e^{-iuv'}| \le 2|u|\,|v - v'|$ for $u, v, v' \in \mathbb{C}$. Thus

$$\begin{aligned} &\Big| \int_{\mathbb{C}} \exp\Big[-iu\Big(\int_{(0,t)} \theta(s, \lambda^{-1/2}x(s) + \xi)\, d\eta_n(s) \Big) \Big]\, d\nu(u) \\ &\quad - \int_{\mathbb{C}} \exp\Big[-iu\Big(\int_{(0,t)} \theta(s, \lambda^{-1/2}x(s) + \xi)\, d\eta(s) \Big) \Big]\, d\nu(u) \Big| \\ &\le 2M \int_{\mathbb{C}} |u|\, d|\nu|(u)\ \|\eta_n - \eta\| \\ &= M_0\, \|\eta_n - \eta\|. \end{aligned}$$

So,

$$\begin{aligned} &\Big| \int_{\mathbb{C}} \exp\Big[-iu\Big(\int_{(0,t)} \theta(s, \lambda^{-1/2}x(s) + \xi)\, d\eta_n(s) \Big) \Big]\, d\nu(u) \\ &\qquad \times \psi(\lambda^{-1/2}x(t) + \xi) \\ &\quad - \int_{\mathbb{C}} \exp\Big[-iu\Big(\int_{(0,t)} \theta(s, \lambda^{-1/2}x(s) + \xi)\, d\eta(s) \Big) \Big]\, d\nu(u) \\ &\qquad \times \psi(\lambda^{-1/2}x(t) + \xi) \Big| \\ &\le M_0\, \|\eta_n - \eta\|\, \Big| \psi(\lambda^{-1/2}x(t) + \xi) \Big| \end{aligned}$$

for all $\psi \in L^2(\mathbb{R}^N)$. Hence

$$\begin{aligned} &|(K_\lambda(F_n)\psi)(\xi) - (K_\lambda(F)\psi)(\xi)| \\ &\le M_0\, \|\eta_n - \eta\|\, (e^{-t(H_0/\lambda)}|\psi|)(\xi) \end{aligned}$$

and thus $\|K_\lambda(F_n)\psi - K_\lambda(F)\psi\| \le M_0 \, \|\eta_n - \eta\| \, \|\psi\|$. Since ψ is arbitrary in $L^2(\mathbb{R}^N)$, (2.8) and (2.9) follow for all $\lambda > 0$.

By Lemma 2, $K_\lambda(F_n)$ and $K_\lambda(F)$ are analytic in $\mathbb{C}_+$. By (2.7) $\|K_\lambda(F_n)\| < N_0$ for all $\lambda \in \mathbb{C}_+$ and $n = 1, 2, \cdots$ where N_0 is given by (2.2). Also for $\lambda > 0$, $K_\lambda(F_n) \to K_\lambda(F)$ in the operator norm topology. Hence Vitali's theorem gives the result $\lambda \in \mathbb{C}_+$. □

We next turn to the question of stability in the measure ν. Let $\nu, \nu_n, n = 1, 2, \cdots$ be in $M(\mathbb{C})$. Of course, ν_n converges weakly to ν provided that

$$\int_{\mathbb{C}} \phi(u) \, \nu_n(u) \to \int_{\mathbb{C}} \phi(u) \, \nu(u)$$

for every bounded continuous function ϕ on $\mathbb{C}$.

THEOREM 3. *Let $\theta \in L_{\infty 1;\eta}$ and let $\nu, \nu_n, n = 1, 2, \cdots$ be in $M(\mathbb{C})$. Assume that $\nu_n \to \nu$ weakly. Let F_n be defined as in* (1.3) *except with ν replaced by ν_n. Then for every $\lambda > 0$,*

$$K_\lambda(F_n) \to K_\lambda(F) \quad \textit{strongly as} \quad n \to \infty.$$

Further, if $\int_{\mathbb{C}} e^{\|\theta\|_{\infty 1;\eta}|u|} \, d|\nu|(u) < \infty$ and if there exists $N_1 < \infty$ such that $\int_{\mathbb{C}} e^{\|\theta\|_{\infty 1;\eta}|u|} \, d|\nu_n|(u) < N_1$ for all $n = 1, 2, \cdots$, then for $\lambda \in \mathbb{C}_+$

$$K_\lambda(F_n) \to K_\lambda(F) \quad \textit{strongly as} \quad n \to \infty. \tag{2.10}$$

Further, the convergence in (2.10) *is inform on all compact subsets of $\mathbb{C}_+$.*

PROOF. Let $\lambda > 0$, $x \in C_0^t$ and $\xi \in \mathbb{R}^N$ be given. The function e^{iuv} is bounded and continuous as a function of u. Hence,

$$\begin{aligned} &\int_{\mathbb{C}} e^{-iu \int_{(0,t)} \theta(s, \lambda^{-1/2} x(s) + \xi) \, d\eta(s)} \, d\nu_n(u) \\ &\to \int_{\mathbb{C}} e^{-iu \int_{(0,t)} \theta(s, \lambda^{-1/2} x(s) + \xi) \, d\eta(s)} \, d\nu(u), \end{aligned}$$

i.e., $F_n(\lambda^{-1/2}x + \xi) \to F(\lambda^{-1/2}x + \xi)$. Note that for all $x \in C^t$ and all $n = 1, 2, \cdots$,

$$\begin{aligned} |F_n(x)| &= \left| \hat{\nu}_n \Big(\int_{(0,t)} \theta(s, x(s)) \, d\eta(s) \Big) \right| \\ &\le \|\nu_n\| \le d \end{aligned}$$

where $d = \sup_n \|\nu_n\|$, a finite number [7, Theorem 9.3.5]. Hence by Lemma 3, $K_\lambda(F_n) \to K_\lambda(F)$ strongly as $n \to \infty$.

Let $\psi \in L^2(\mathbb{R}^N)$, then $K_\lambda(F_n)\psi$ and $K_\lambda(F)\psi$ are analytic for $\lambda \in \mathbb{C}_+$ and $\|K_\lambda(F_n)\psi\| \le \int_{\mathbb{C}} e^{\|\theta\|_{\infty 1;\eta}|u|}\, d|\nu_n|(u)\, \|\psi\|$. Hence

$$\|K_\lambda(F_n)\psi\| \le N_1 \|\psi\| \tag{2.11}$$

for all $\lambda \in \mathbb{C}_+$ and for all $n = 1, 2, \cdots$. Hence, Vitali's theorem yields the result as in the proof of Theorem 1. □

THEOREM 4. *Assume that $\nu_n \to \nu$ in norm. Then, the under the hypotheses of Theorem 3, $K_\lambda(F_n) \to K_\lambda(F)$ in the operator norm topology uniformly in λ on all compact subsets of $\mathbb{C}_+$. Moreover, for all $\lambda > 0$ we have the norm estimate $\|K_\lambda(F_n) - K_\lambda(F)\| \le \|\nu_n - \nu\|$, $n = 1, 2, \cdots$.*

PROOF. Clearly for all $\lambda > 0, x \in C_0^t$ and $\xi \in \mathbb{R}^N$

$$\begin{aligned}
&\Big| \int_{\mathbb{C}} e^{-iu \int_{(0,t)} \theta(s, \lambda^{-1/2} x(s) + \xi)\, d\eta(s)}\, d\nu_n(u) \\
&\quad - \int_{\mathbb{C}} e^{-iu \int_{(0,t)} \theta(s, \lambda^{-1/2} x(s) + \xi)\, d\eta(s)}\, d\nu(u) \Big| \\
&\le \|\nu_n - \nu\|.
\end{aligned}$$

Then, for $\lambda > 0$, $x \in C_0^t$ and $\psi \in L^2(\mathbb{R}^N)$

$$\begin{aligned}
&\Big| \int_{\mathbb{C}} e^{-iu \int_{(0,t)} \theta(s, \lambda^{-1/2} x(s) + \xi)\, d\eta(s)}\, d\nu_n(u)\ \psi(\lambda^{-1/2} x(t) + \xi) \\
&\quad - \int_{\mathbb{C}} e^{-iu \int_{(0,t)} \theta(s, \lambda^{-1/2} x(s) + \xi)\, d\eta(s)}\, d\nu(u)\ \psi(\lambda^{-1/2} x(t) + \xi) \Big| \\
&\le \|\nu_n - \nu\|\, |\psi(\lambda^{-1/2} x(t) + \xi)|.
\end{aligned}$$

Hence

$$\begin{aligned}
&|(K_\lambda(F_n)\psi)(\xi) - (K_\lambda(F)\psi(\xi)| \\
&\le \int_{C_0} \|\nu_n - \nu\|\, |\psi(\lambda^{-1/2} x(t) + \xi)|\, dm(x) \\
&= \|\nu_n - \nu\|\, (e^{-t(H_0/\lambda)} |\psi|)(\xi)
\end{aligned}$$

and thus

$$\|K_\lambda(F_n)\psi - K_\lambda(F)\psi\| \le \|\nu_n - \nu\|\,\|\psi\|.$$

Since ψ is arbitrary in $L^2(R^N)$, $\|K_\lambda(F_n) - K_\lambda(F)\| \le \|\nu_n - \nu\|$, $n = 1, 2, \cdots$ and $K_\lambda(F_n) \to K_\lambda(F)$ in the operator norm topology for $\lambda > 0$.

By [2], $K_\lambda(F_n)$ and $K_\lambda(F)$ are analytic in $\mathbb{C}_+$. By (2.11), $\|K_\lambda(F_n)\| \le N_1$ for all $\lambda \in \mathbb{C}_+$ and $n = 1, 2, \cdots$. Hence the conclusion follows as in the proof in Theorem 2. □

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