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PRIME RADICALS IN ORE EXTENSIONS

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ABSTRACT. Let R be a ring with an endomorphism σ and a derivation δ. An ideal I of R is (σ, δ)-ideal of R if $\sigma(I) \subseteq I$ and $\delta(I) \subseteq I$. An ideal P of R is a (σ, δ)-prime ideal of R if P $(\neq R)$ is a (σ, δ)-*ideal* and for (σ, δ)-ideals I and J of R, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$. An ideal Q of R is (σ, δ)-semiprime ideal of R if Q is a (σ, δ)-ideal and for (σ, δ)-ideal I of R, $I^2 \subseteq Q$ implies that $I \subseteq Q$. The (σ, δ)-prime radical (resp. prime radical) is defined by the intersection of all (σ, δ)-prime ideals (resp. prime ideals) of R and is denoted by $P_{(\sigma,\delta)}(R)$ (resp. $P(R)$). In this paper, the following results are obtained: (1) $P_{(\sigma,\delta)}(R)$ is the smallest (σ, δ)-semiprime ideal of R; (2) For every extended endomorphism $\bar{\sigma}$ of σ, the $\bar{\sigma}$-prime radical of an Ore extension $P(R[x, \sigma, \delta])$ is equal to $P_{(\sigma,\delta)}(R)[x, \sigma, \delta]$.

1. Introduction and Some Definitions

Throughout this paper, R will denote an associative ring with identity. A *skew derivation* on a ring R is a pair (σ, δ) where σ is a ring endomorphism of R and δ is a *(left) σ-derivation* on R, that is, an additive map from R to itself shch that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. A left (right, two-sided) ideal I of R is called a left (right, two-sided) (σ, δ)-*ideal* if $\sigma(I) \subseteq I$ and $\delta(I) \subseteq I$. An ideal P of R is called (σ, δ)-*prime ideal* if P $(\neq R)$ is a (σ, δ)-*ideal* and for (σ, δ)-ideals I, J of R, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$. An ideal Q of R is called (σ, δ)-*semiprime ideal* if Q is a (σ, δ)-ideal of R and for

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any $(\sigma,\ \delta)$-*ideal* I of R, $I^2 \subseteq Q$ implies that $I \subseteq Q$. R is called a $(\sigma,\ \delta)$-*prime* (resp. $(\sigma,\ \delta)$-*semiprime*) ring if (0) is a $(\sigma,\ \delta)$-prime (resp. (σ, δ)-semiprime) ideal. For more things about these terminologies, we refer to [3], [4] and [6]. In particular, in [2], Goodearl has provided a criterion for $(\sigma,\ \delta)$ being a skew derivation on R, which enables us to construct easily many examples of skew derivations. Note that every (σ, δ)-prime ideal of R is (σ, δ)-semiprime ideal, and every prime (resp. semiprime) ring is (σ, δ)-prime (resp. (σ, δ)-semiprime).

Recall that the prime radical (in other words, lower nil radical) of R (denoted by $P(R)$) is the intersection of all prime ideals of R. We can define $(\sigma,\ \delta)$-prime radical (in other words, $(\sigma,\ \delta)$-lower nil radical) of R (denoted by $P_{(\sigma,\delta)}(R)$) by the intersection of all $(\sigma,\ \delta)$-prime ideals of R. In Section 2, we will investigate some properties of $P_{(\sigma,\delta)}(R)$, in particular, we will show that $P_{(\sigma,\delta)}(R)$ is the smallest $(\sigma,\ \delta)$-semiprime ideal of R.

EXAMPLE 1. Let $R = F[x]$ be a polynomial ring over a field F, let $P = xF[x]$ be an ideal of R generated by x and let $\delta = \frac{d}{dx}$ be the formal differential on R. Then P is a prime ideal of R but $\delta(P) \nsubseteq P$, and so P is not a $(\sigma,\ \delta)$-prime ideal of R for any skew derivation $(\sigma,\ \delta)$ of R.

EXAMPLE 2. Let $R = F[x]$ be a polynomial ring over a field F, let $P = xF[x]$ be an ideal of R generated by x (as given in Example 1). Let $\delta_1 = x\frac{d}{dx}$ be a derivation. Consider an ring endomorphism $\sigma : R \longrightarrow R$ by $\sigma(f(x)) = f(-x)$ for all $f(x) \in R$. Then P is a prime ideal of R and $\delta(P) \subseteq P$, and also P is a $(\sigma,\ \delta)$-ideal of R.

Recall that for a skew derivation of R, Ore extensions (skew polynomial rings) $P(R[x;\sigma,\delta])$ are rings of polynomials in x with coefficients in R in which the multiplication is given by $xa = \sigma(a)x + \delta(a)$ for all $a \in R$. In particular, if $\sigma = 1$, then $P(R[x;\sigma,\delta])$ is simply denoted by $P(R[x;\delta])$ (called differential polynomial ring). In [4], Irving has worked on prime ideals of Ore extensions over commutative rings, in [2], Goodearl has analyzed prime ideals of Ore extension over R in case that R is commutative noetherian ring). In [1], Ferrero, Kishimoto and Motose have shown that for a differential polynomial ring

$R[x;\delta]$, the prime radical of $R[x;\delta]$ is equal to $P_\delta[x;\delta]$ where P_δ is the δ-prime radical of R (which is the intersection of all δ-prime ideals of R). From the previous works on Ore extension, It is natural for us to try to find the prime radical of Ore extension. In Section 3, by considering an extended endomorphism $\bar{\sigma}$ of σ we will show that for every extended endomorphism $\bar{\sigma}$ of σ, R is a (σ, δ)-semiprime ring if and only if $P(R[x;\sigma,\delta])$ is a $\bar{\sigma}$-semiprime ring, and the $\bar{\sigma}$-prime radical of an Ore extension $P(R[x;\sigma,\delta])$ is equal to $P_{(\sigma,\delta)}(R)[x;\sigma,\delta]$, as corollary we can improve the result shown by Ferrero, Kishimoto and Motose; $P(R[x;\delta]) = P_{(\delta)}(R)[x;\delta]$.

2. σ-Prime Radical of a Ring R

The definitions and the resluts in this section are obtained by the similar arguments on prime radical of ring R in [5]. A nonempty subset S of a ring R is called a (σ, δ)-*m-system* if, for any $a, b \in S$ such that (a) and (b) are (σ, δ)-ideals of R, there exists $r \in R$ such that $arb \in S$.

LEMMA 2.1. *Let R be a ring with an automorphism σ and a derivation δ. Then $\sigma(a)$, $\delta(a) \in (a)$ if and only if (a) is a principal (σ, δ)-ideal of R.*

PROOF Suppose that $\sigma(a), \delta(a) \in (a)$ and let $b \in (a)$ be arbitrary Then $b = \sum_{i=1}^{m} r_i a s_i \in (a)$ for some r_i, $s_i \in R$ $(i = 1, \ldots, m)$. Since $\sigma(a) \in (a)$, $\sigma(r_i a s_i) = \sigma(r_i)\sigma(a)\sigma(s_i) \in (a)$ for each i, and then $\sigma(b) \in (a)$. Since $\delta(a) \in (a)$, $\delta(r_i a s_i) = \delta(r_i) a s_i + r_i \delta(a) s_i + r_i a \delta(s_i) \in (a)$ for each i, and then $\delta(b) \in (a)$. Hence (a) is a (σ, δ)-ideal of R. The converse is clear.

PROPOSITION 2 2. *Let R be a ring with an automorphism σ and a derivation δ. If $P \subsetneq R$ is any (σ, δ)-ideal of R, then the following are equivalent:*

(1) *P is (σ, δ)-prime;*
(2) *For any $a, b \in R$ such that (a) and (b) are (σ, δ)-ideals of R, $(a)\cdot(b) \subseteq P$ implies that $a \in P$ or $b \in P$.*
(3) *For any $a, b \in R$ such that (a) and (b) are (σ, δ)-ideals of R, $aRb \subseteq P$ implies that $a \in P$ or $b \in P$.*

(4) *For left (σ, δ)-ideals I, J of R, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$.*

(5) *For right (σ, δ)-ideals I, J of R, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$.*

PROOF. It is enough to show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Clear.

(2) $\Rightarrow$ (3): If $aRb \subseteq P$ such that (a) and (b) are (σ, δ)-ideals of R, then $(a)\cdot(b) = RaRRbR \subseteq RPR = P$. By (2), $a \in P$ or $b \in P$.

(3) $\Rightarrow$ (4): Assume that there exist two left (σ, δ)-ideals I, J of R such that $I \cdot J \subseteq P$ but $I \nsubseteq P$, $J \nsubseteq P$. Choose $a \in I \setminus P$ and $b \in J \setminus P$. Then $aRb \subseteq IJ \subseteq P$. By (3), $a \in P$ or $b \in P$, a contradiction.

(4) $\Rightarrow$ (1): Clear.

COROLLARY 2.3 *Let R be a ring with an automorphism σ and a derivation δ. Then P is a (σ, δ)-prime-ideal of R if and only if $R \setminus P$ is a (σ, δ)-m-system.*

PROOF. It follows from the definition of (σ, δ)-m-system and Proposition 2.2.

PROPOSITION 2.4. *Let R be a ring with an automorphism σ and a derivation δ and let $S \subseteq R$ be a (σ, δ)-m-system which is disjoint from a (σ, δ)-ideal I of R. Then there exists a (σ, δ)-ideal P which is maximal in the set of all (σ, δ)-ideals of R disjoint from S and containing I. Furthermore any such ideal P is a (σ, δ)-prime ideal of R.*

PROOF. Consider the set $\Gamma_{(\sigma,\delta)}$ of all (σ, δ)-ideals of R disjoint from S and containing I. Then $\Gamma_{(\sigma,\delta)}$ is nonempty since $I \in \Gamma_{(\sigma,\delta)}$. Since $\Gamma_{(\sigma,\delta)} \neq \emptyset$, every (σ, δ)-ideal in $\Gamma_{(\sigma,\delta)}$ is properly contained in R. Let $\Gamma_{(\sigma,\delta)}$ be partially ordered by inclusion. By Zorn's Lemma there is a (σ, δ)-ideal P of R which is maximal in $\Gamma_{(\sigma,\delta)}$. Let U, V be (σ, δ)-ideals of R such that $UV \subseteq P$. If $U \nsubseteq P$ and $V \nsubseteq P$. then each of the (σ, δ)-ideals $P + U$ and $P + V$ properly contains P and hence must meet S. Consequently, for some $p_1, p_2 \in P$, $u \in U$, $v \in V$, $p_1 + u = s_1 \in S$ and $p_2 + v = s_2 \in S$. Since S is a (σ, δ)-m-system, there exists an element $r \in R$ such that $s_1rs_2 \in S$. Thus $s_1rs_2 = p_1rp_2 + p_1rv + urp_2 + urv \in P + UV \subseteq P$, a contradiction since $s_1rs_2 \in S \cap P = \emptyset$. Therefore $U \subseteq P$ or $V \subseteq P$, and so P is a (σ, δ)- prime ideal of R.

For a (σ, δ)-ideal I in a ring R with an automorphism σ and a derivation δ, let $P_{(\sigma,\delta)}(R : I) = \{r \in R :$ every σ-m-system containing r meets $I\}$. Then we have the following theorem:

THEOREM 2.5. *Let R be a ring with an automorphism σ and a derivation δ. Then for any (σ, δ)-ideal I in a ring R, $P_{(\sigma,\delta)}(R : I)$ equals to the intersection of all the (σ, δ)-prime ideals containing I. In particular, $P_{(\sigma,\delta)}(R : I)$ is a (σ, δ)-ideal of R.*

PROOF. Let $a \in P_{(\sigma,\delta)}(R : I)$ and P be any (σ, δ)-prime ideal of R containing I. Then $R \setminus P$ is a (σ, δ)-m-system by Corollary 2.3 This (σ, δ)-m-system cannot contain a, for otherwise $(R \setminus P) \cap I \neq \emptyset$, a contradiction Therefore, we have $a \in P$. Conversely, assume that $a \notin P_{(\sigma,\delta)}(R : I)$. Then by definition, there exists a (σ, δ)-m-system S containing a which is disjoint from I. By Proposition 2.4, there exists a (σ, δ)-prime-ideal P which is maximal in the set of all (σ, δ)-ideals of R disjoint from S and containing I. Hence we have $a \notin P$, as desired.

A nonempty subset S of a ring R is called a (σ, δ)-*n-system* if, for any $a \in S$ such that (a) is (σ, δ)-ideal of R there exists $r \in R$ such that $ara \in S$.

PROPOSITION 2.6. *Let R be a ring with an automorphism σ and a derivation δ. For any (σ, δ)-ideal Q of R, the following are equivalent:*

(1) *Q is (σ, δ)-semiprime;*
(2) *For any $a \in R$ such that (a) is (σ, δ)-ideal of R, $(a)^2 \subseteq Q$ implies that $a \in Q$;*
(3) *For any $a \in R$ such that (a) is (σ, δ)-ideal of R, $aRa \subseteq Q$ implies that $a \in Q$;*
(4) *For left (σ, δ)-ideals I of R, $I^2 \subseteq Q$ implies that $I \subseteq Q$;*
(5) *For right (σ, δ)-ideals I of R, $I^2 \subseteq Q$ implies that $I \subseteq Q$.*

PROOF It is similar to the proof as given in the Proposition 2.2.

COROLLARY 2.7. *Let R be a ring with an automorphism σ and a derivation δ. Then P is a (σ, δ)-semiprime-ideal of R if and only if $R \setminus P$ is a (σ, δ)-n-system.*

PROOF. It follows from the definition of (σ, δ)-n-system and Proposition 2.6.

LEMMA 2.8. *Let R be a ring with an automorphism σ and a derivation δ. If N is a (σ, δ)-n-system in R and $a \in N$, then there exists a (σ, δ)-m-system $M \subseteq N$ such that $a \in M$.*

PROOF. Consider a subset $M = \{a_1, a_2, a_3, \dots\}$ of R defined inductivley as follows: $a_1 = a$, $a_{i+1} = a_i r_i a_i \in N$ for some $r_i \in R$ where $i = 1, 2, \dots$. We will show that M is a (σ, δ)-m-system. Let a_i, $a_j \in M$ be arbitrary. If $i < j+1$, then $a_{j+1} \in a_j R a_j \subseteq a_i R a_j$, which means $a_{j+1} \in M$. If $j < i+1$, then similarly $a_{i+1} \in M$. Hence there is a (σ, δ)-m-system $M \subseteq N$ such that $a \in M$.

THEOREM 2.9. *Let R be a ring with an automorphism σ and a derivation δ. For any (σ, δ)-ideal Q of R, the following are equivalent:*

(1) *Q is a (σ, δ)-semiprime ideal;*
(2) *Q is an intersection of (σ, δ)-prime ideals;*
(3) *$Q = P_{(\sigma,\delta)}(R : Q)$.*

PROOF. (3) $\Rightarrow$ (2). It follows from Theorem 2.5 since any (σ, δ)-prime ideal is (σ, δ)-semiprime.

(2) $\Rightarrow$ (1). It follows from the observation that every (σ, δ)-prime ideal is (σ, δ)-semiprime and the intersection of any (σ, δ)-semiprime ideals is (σ, δ)-semiprime

(1) $\Rightarrow$ (3). Suppose that Q is a (σ, δ)-semiprime ideal. By definition of (σ, δ)-n-system, $Q \subseteq P_{(\sigma,\delta)}(R : Q)$. We want to show that $P_{(\sigma,\delta)}(R : Q) \subseteq Q$. Let $a \notin Q$ and let $N = R \setminus Q$. Then N is a (σ, δ)-n-system containing a by Corollary 2.7. By Lemma 2.8, there exists a (σ, δ)-m-system $M \subseteq N$ such that $a \in M$. Since M is disjoint from Q, $a \notin P_{(\sigma,\delta)}(R : Q)$.

COROLLARY 2.10. *Let R be a ring with an automorphism σ and a derivation δ. Then $P_{(\sigma,\delta)}(R : I)$ is the smallest (σ, δ)-semiprime ideal of R which contains I.*

PROOF. If follows from the Theorem 2.9.

For a ring R with an automophism σ and a derivation δ, $P_{(\sigma,\delta)}(R : (0))$ (simply denoted by $P_{(\sigma,\delta)}(R)$).is called the (σ, δ)-prime-radical of R. We can note that $P_{(\sigma,\delta)}(R)$ is the intersection of all (σ, δ)-prime ideals of R by Theorem 2.9 and it is the smallest (σ, δ)-semiprime ideal of R by Corollary 2 10.

PROPOSITION 2.11. *Let R be a ring with an automorphism σ and a derivation δ. Then the following are equivalent:*

(1) *R is a (σ, δ)-semiprime ring;*
(2) *$P_{(\sigma,\delta)}(R) = (0)$;*
(3) *R has no nonzero nilpotent (σ, δ)-ideal;*
(4) *R has no nonzero nilpotent left (σ, δ)-ideal.*

PROOF. (1) $\Leftrightarrow$ (2), (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are clear. It remains to show the implication (1) $\Rightarrow$ (4). Suppose that R is a (σ, δ)-semiprime ring and let I be a nilpotent left (σ, δ)-ideal. Then $I^n = (0)$ and $I^{n-1} \neq (0)$ for some positive integer n. If $n \geqslant 2$, then $(I^{n-1})^2 = I^{2n-2} \subseteq I^{2n} = (0)$ implies $I^{n-1} = (0)$ since R is (σ, δ)-semiprime, a contradiction Thus $n = 1$ and so $I = (0)$.

3. Prime radicals of Ore Extensions

For a ring R with a (left) skew derivation (σ, δ), there exist an automomorphism and a derivation which extend σ and δ respectively. For example, consider $\bar{\sigma}$ and $\bar{\delta}$ on $A = R[x; \sigma, \delta]$ defined by $\bar{\sigma}(f(x)) = \sigma(a_0) + \sigma(a_1)x + \cdots + \sigma(a_n)x^n$ and $\bar{\delta}(f(x)) = xf(x) - f(x)x = \delta(a_0) + \delta(a_1)x + \cdots + \delta(a_n)x^n$ for all $f(x) = a_0 + a_1x + \cdots + a_nx^n \in A$. Then $\bar{\sigma}$ is automorphism on A and $\bar{\delta}$ is a derivation on A, and also $\bar{\sigma}(r) = \sigma(r)$, $\bar{\delta}(r) = \delta(r)$ for all $r \in R$, which means that $\bar{\sigma}$ (resp $\bar{\delta}$) is an extensions of σ (resp. δ) We call such an automorphism $\bar{\sigma}$

(resp. derivation $\bar{\delta}$) on A an *extended automorphism* of σ (resp. an *extended derivation* of δ). It is natural to determine whether $(\bar{\sigma}, \bar{\delta})$ is a skew derivation on A or not.

LEMMA 3.1. *Let R be a ring with a (left) skew derivation (σ, δ). If $\sigma\delta = \delta\sigma$, then $(\bar{\sigma}, \bar{\delta})$ is a skew derivation on $A = R[x;\sigma,\delta]$ i.e., $\bar{\sigma}$, $\bar{\delta}$ satisfies the following; (1) $\bar{\sigma}(fg) = \bar{\sigma}(f)\bar{\sigma}(g)$, (2) $\bar{\delta}(fg) = \bar{\sigma}(f)\bar{\delta}(g) + \bar{\delta}(f)g$ for all $f, g \in A$.*

PROOF. (1) Let $f = \sum_{j=0}^{m} a_j x^j$, $g = \sum_{k=0}^{n} b_k x^k \in A$ be arbitrary. Then $fg = \sum_{j=0}^{m}\sum_{k=0}^{n} a_j x^j b_k x^k$. Since $\bar{\sigma}(fg) = \sum_{j=0}^{m}\sum_{k=0}^{n} \bar{\sigma}(a_j x^j b_k x^k)$, it is enough to show that $\bar{\sigma}(a_j x^j b_k x^k) = \sigma(a_j)x^j\sigma(b_k)x^k$.
$\bar{\sigma}(a_j x^j b_k x^k) = \bar{\sigma}(a_j \sum_{i=0}^{j} \binom{j}{i}\sigma^i\delta^{j-i}b_k x^i)x^k = \sigma(a_j)\sum_{i=0}^{j}\binom{j}{i}\sigma(\sigma^i\delta^{j-i}b_k)x^{i+k}$
$= \sigma(a_j)\sum_{i=0}^{j}\binom{j}{i}\sigma\sigma^i\delta^{j-i}\sigma(b_k)x^{i+k} = \sigma(a_j)x^j\sigma(b_k)x^k$.
(2) It is also enough to show that $\bar{\delta}(a_j x^j b_k x^k) = \sigma(a_j)x^j\delta(b_k)x^k + \delta(a_j)x^j b_k x^k$ as in the proof of (1).
$\bar{\delta}(a_j x^j b_k x^k) = \bar{\delta}(a_j \sum_{i=0}^{j}\binom{j}{i}\sigma^i\delta^{j-i}b_k x^{i+k}) = \sum_{i=0}^{j}\binom{j}{i}\delta(a_i\sigma^i\delta^{j-i}b_k)x^{i+k}$
$= \sum_{i=0}^{j}\binom{j}{i}[\sigma(a_j)\delta(\sigma^i\delta^{j-i}(b_k) + \delta(a_j)\sigma^i\delta^{j-i}(b_k)]x^{i+k}$
$= \sigma(a_j)\sum_{i=0}^{j}\binom{j}{i}\sigma^i\delta^{j-i}(\delta(a_j))x^{i+k} + \delta(a_j)\sum_{i=0}^{j}\sigma^i\delta^{j-i}(b_k)x^{i+k}$
$= \sigma(a_j)x^j\delta(b_k)x^k + \delta(a_j)x^j b_k x^k$.

We call $(\bar{\sigma}, \bar{\delta})$ an extended skew derivation on an Ore extension $R[x;\sigma,\delta]$.

If J is an ideal of $R[x;\ \sigma,\ \delta]$, we denote by J_0 the set of all leading coefficients of all $f \in J$.

LEMMA 3.2. *Let R be a ring with a (left) skew derivation (σ, δ) such that $\sigma\delta = \delta\sigma$. Let $A = R[x;\ \sigma,\ \delta]$. Then*

(1) *If J is a $\bar{\sigma}$-ideal of A, then J is a $\bar{\delta}$-ideal of A.*
(2) *If J is a $\bar{\sigma}$-ideal of A, then J_0 is a (σ, δ)-ideal of R.*
(3) *If J is a $\bar{\sigma}$-ideal of A, then $J \cap R$ is a (σ, δ)-ideal of R.*
(4) *If I is a (σ, δ)-ideal of R, then IA is a $\bar{\sigma}$-ideal of A.*
(5) *If P is a $\bar{\sigma}$-prime ideal of A, then $P \cap R$ is a (σ, δ)-prime ideal of R.*
(6) *If Q is a (σ, δ)-prime ideal of R, then QA is a $\bar{\sigma}$-prime ideal of A.*

PROOF. (1) Let J be a $\bar{\sigma}$-ideal of A. Then for any $f \in J$, $xf - \bar{\sigma}(fx) = \bar{\delta}(f) \in J$, and so J is a $\bar{\delta}$-ideal of A.

(2) Let J be a $\bar{\sigma}$-ideal of A. For any $f \in J$, let $f = a_n x^n + \{\text{terms of lower degrees}\}$ where $a_n \in J_0$. Consider xf and $xf - \bar{\sigma}(fx) \in J$. Then $xf = a(a_n x^n + \{\text{terms of lower degrees}\}) = \sigma(a_n)x^{n+1} + \{\text{terms of lower degrees}\}$, and $xf - \bar{\sigma}(fx) = \bar{\delta}(f)$, and so $\sigma(a_n)$, $\delta(a_n) \in J_0$. Hence J_0 is a (σ, δ)-ideal of R.

(3) Let J be a $\bar{\sigma}$-ideal of A. Clearly $J \cap R$ is an ideal of R. Let $a \in J \cap R$ be arbitrary. Then $\bar{\sigma}(a) = \sigma(a)$, $xa - \bar{\sigma}(ax) = \delta(a) \in J \cap R$, and so $J \cap R$ is (σ, δ)-ideal of R.

(4) Clear.

(5) Suppose that P is a $\bar{\sigma}$-prime ideal of A. Let I, J be (σ, δ)-ideals of R such that $IJ \subseteq P \cap R$. Since IA, JA are $\bar{\sigma}$-ideals of A by (4), $(IA)(JA) \subseteq (IJ)A \subseteq (P \cap R)A \subseteq PA \subseteq P$. Since P is a $\bar{\sigma}$-prime ideal of A, $IA \subseteq P$ or $JA \subseteq P$, say $IA \subseteq P$. Hence $I \subseteq (IA) \cap R \subseteq P \cap R$, and so $P \cap R$ is a (σ, δ)-prime ideal of R.

(6) Suppose that Q is a (σ, δ)-prime ideal of R. Let S, T be $\bar{\sigma}$-ideals of A such that $ST \subseteq QA$, $QA \subseteq S$ and $QA \subseteq T$. Then S_0 and T_0 are (σ, δ)-ideals of R and $S_0 T_0 \subseteq (QA)_0 = Q$. Since Q is a (σ, δ)-prime ideal of R, $S_0 \subseteq Q$ or $T_0 \subseteq Q$, say $S_0 \subseteq Q$. Let $\sum_{i=0}^{n} a_i x^i \in S$. Then $a_n \in J_0 \subseteq Q$ so that $a_n x^n \in QA \subseteq S_0$. Thus $\sum_{i=0}^{n-1} a_i x^i \in S$, and then $a_{n-1} \in S_0 \subseteq Q$ Continuing in this way, we have $a_i \in Q$ for all i, and so $S \subseteq QA$.

PROPOSITION 3.3. *Let R be a ring with a (left) skew derivation (σ, δ) such that $\sigma\delta = \delta\sigma$. Let $A = R[x; \sigma, \delta]$ Then the following are equivalent:*

(1) *R is (σ, δ)-semiprime;*
(2) *A is $\bar{\sigma}$-semiprime for every extended endomorphism $\bar{\sigma}$ on A of σ.*

PROOF. (1) $\Rightarrow$ (2). Suppose that R is (σ, δ)-semiprime. Let J be a $\bar{\sigma}$-ideal of A such that $J^2 = 0$. Consider J_0, the set of all leading coefficients of every $f(x) \in J$. Then J_0 is a (σ, δ)-ideal of R by Lemma 3.2 Since $J^2 = 0$, $J_0^2 = 0$, and so $J_0 = 0$ by the assumption. Continuing in this way, every coefficient of $f(x)$ is equal to 0 for all $f(x) \in J$. Thus $J = 0$, and so A is $\bar{\sigma}$-semiprime.

(2) $\Rightarrow$ (1). Suppose that A is $\bar{\sigma}$-semiprime. Let I be a nonzero (σ, δ)-ideal of R. Then IA is a nonzero $\bar{\sigma}$-ideal of A by Lemma 3.2. Since A is $\bar{\sigma}$-semiprime, $(IA)^2 = I^2A \neq 0$, and then $I^2 \neq 0$. Hence R is (σ, δ)-semiprime.

For any (σ, δ)-ideal I of a ring R with a skew derivation (σ, δ), we can have a reduced endomorphism σ' and a reduced derivation δ' on R/I defined by $\sigma'(a+I) = \sigma(a) + I$ and $\delta'(a+I) = \delta(a) + I$ for all $a + I \in R/I$. It is also natural to determine whether (σ', δ') is a skew derivation on R/I or not Observe that if $\sigma\delta = \delta\sigma$, then (σ', δ') is a skew derivation on R/I i.e., (σ', δ') satisfies the following; (1) $\sigma'(\bar{a}\bar{b}) = \sigma'(\bar{a})\sigma'(\bar{b})$, (2) $\delta'(\bar{a}\bar{b}) = \bar{\sigma}(f)\bar{\delta}(g) + \bar{\delta}(f)g$ for all $\bar{a} = a + I, \bar{b} = b + I \in R/I$ We call (σ', δ') a reduced skew derivation on R/I. Hence we can consider an Ore extension $(R/I)[x; \sigma', \delta']$ with multiplication subject to the relation $x\bar{a} = \sigma'(\bar{a})x + \delta'(\bar{a})$ for all $\bar{a} = a + I \in R/I$. Observe that for any ideal K of R such that $R \supseteq K \supseteq I$. Then K is a (σ, δ)-ideal of R if and only if K/I is a σ'-ideal of R/I.

LEMMA 3.4 *Let R be a ring with a (left) skew derivation (σ, δ) such that $\sigma\delta = \delta\sigma$. Let K, I be ideals of R such that $R \supseteq K \supseteq I$ Then K is a (σ, δ)-ideal of R if and only if K/I is a (σ', δ')-ideal of R/I.*

PROOF. It follows from the definition of a reduced skew derivation (σ', δ').

LEMMA 3 5. *Let R be a ring with a (left) skew derivation (σ, δ) such that $\sigma\delta = \delta\sigma$. Let I be an ideal of R. Then I is a (σ, δ)-semiprime ideal of R if and only if R/I is a (σ', δ')-semiprime ring.*

PROOF. $(\Rightarrow)$ Suppose that I is a (σ, δ)-semiprime ideal of R. If K/I is any (σ', δ')-ideal of R/I such that $(K/I)^2 = (\bar{0})$, the zero ideal of R/I. Then $K^2 = I$. By Lemma 3.4, K is (σ, δ)-ideal of R. Since I is a (σ, δ)-semiprime ideal, $K = I$ and so $K/I = (\bar{0})$, which means that

R/I is a (σ', δ')-semiprime ring. Hence R/I is a (σ', δ')-semiprime ring.

($\Leftarrow$) Suppose that R/I is a (σ', δ')-semiprime ring. If Q is any (σ, δ)-ideal of R such that $Q^2 \subseteq I$, then $(\bar{0}) = Q^2/I = (Q/I)^2$. Since R/I is a (σ', δ')-semiprime ring, $Q/I = (\bar{0})$, so $Q = I \subseteq I$ Hence I is a (σ, δ)-semiprime ideal of R and so I is a (σ, δ)-semiprime ideal of R.

LEMMA 3.6. *Let R be a ring with a (left) skew derivation (σ, δ) such that $\sigma\delta = \delta\sigma$. Let I be a (σ, δ)-ideal of R. Then for such a reduced skew derivation (σ', δ') on R/I, $R[x; \sigma, \delta]/I[x; \sigma, \delta] \simeq (R/I)[x; \sigma', \delta']$.*

PROOF. Define $\theta : R[x; \sigma, \delta] \longrightarrow (R/I)[x; \sigma', \delta']$ by $\theta(f(x)) = \sum_{i=0}^{n}(\bar{a}_i)x^i$ for all $f(x) = \sum_{i=m}^{n} a_i x^i \in R[x; \sigma, \delta]$. It is straightforward to show that θ is an epimorphism and the kernel of θ is equal to $I[x; \sigma, \delta]$. Hence we have the result by the First Homomorphism Theorem

THEOREM 3 7. *Let R be a ring with a (left) skew derivation (σ, δ) such that $\sigma\delta = \delta\sigma$. Then $P_{\bar{\sigma}}(R[x;\sigma,\delta]) = P_{(\sigma,\delta)}(R)[x,\sigma,\delta]$.*

PROOF. Let $I = P_{(\sigma,\delta)}(R)$. Then I is the smallest (σ, δ)-semiprime ideal of R by Corollary 2.10 and then R/I is (σ', δ')-semiprime by Lemma 3.5 Thus $(R/I)[x; \sigma', \delta']$ is $(\bar{\sigma'})$-semiprime by Proposition 3.3. Since $(\bar{\sigma'}) = (\bar{\sigma})'$, $I[x; \sigma, \delta]$is a $\bar{\sigma}$-semiprime ideal of $R[x; \sigma, \delta]$. Hence we have $I[[x; \sigma, \delta] \supseteq P_{\bar{\sigma}}(R[x,\sigma,\delta])$. To show the converse inclusion $I[x; \sigma, \delta] \subseteq P_{\bar{\sigma}}(R[x;\sigma,\delta])$, let P be any $\bar{\sigma}$-prime ideal of $R[x; \sigma, \delta]$ Then $P \cap R$ is a (σ, δ)-prime ideal of R by Lemma 3.2. Since $P \cap R$ is a (σ, δ)-prime ideal of R, $I \subseteq P \cap R \subseteq P$, which implies that $I[x; \sigma, \delta] \subseteq P$, and so $I[x; \sigma, \delta] \subseteq P_{\bar{\sigma}}(R[x;\sigma,\delta])$

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