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CONVERGENCE THEOREMS OF THREE-STEP ITERATION METHODS FOR QUASI-CONTRACTIVE MAPPINGS

JINBIAO HAO, LI WANG, SHIN MIN KANG AND SOO HAK SHIM

ABSTRACT We obtain the convergence of three-step iteration methods and generalized three-step iteration methods for quasi-contractive and generalized quasi-contractive mappings, respectively, in Banach spaces Our results extend the corresponding results in [1], [4]-[6].

1. Introduction and Preliminaries

Convergence results for several iteration methods of quasi-contractive mappings have been obtained by some researchers (see, for example, [1], [4]-[6]). Ding [2] introduced generalized quasi-contractive mappings. Ćirić [1] established first both the existence of fixed points and convergence of Picard iterations for quasi-contractive mappings in complete metric spaces. Liu [4] obtained convergence-theorem of Ishikawa iteration methods for quasi-contractive mappings in Hilbert spaces. Zhao [6] studied convergence of Ishikawa iteration methods for quasi-contractive mappings and generalized quasi-contractive mappings in Banach spaces, respectively.

Our aim in this paper is to establish convergence theorems of three-step iteration methods and generalized three-step iteration methods

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for quasi-contractive and generalized quasi-contractive mappings, respectively, in Banach spaces, which are the generalizations of the corresponding results in [1], [4]-[6].

Throughout this paper, let $\delta(A)$ and N denote the diameter of A for any $A \subset X$ and the set of all positive integers, respectively. Let E be a nonempty subset of a Banach space $(X, \|\cdot\|)$ and $T : E \to E$ be a mapping. Recall that T is *generalized quasi-contractive* on E if there exist a $q \in (0,1)$ and a function $n : X \to N$ such that

$$\begin{aligned} &\|T^{n(x)}x - T^{n(y)}y\| \\ &\leq q \max\{\|x-y\|, \|x - T^{n(x)}x\|, \|y - T^{n(y)}y\|, \\ &\qquad \|x - T^{n(y)}y\|, \|y - T^{n(x)}x\|\} \end{aligned} \tag{1.1}$$

for $x, y \in X$. A mapping $T : E \to E$ is called *quasi-contractive* if it satisfies

$$\begin{aligned} \|Tx - Ty\| \leq q \max\{&\|x-y\|, \|x - Tx\|, \|y - Ty\|, \\ &\|x - Ty\|, \|y - Tx\|\} \end{aligned} \tag{1.2}$$

for $x, y \in X$ and some $q \in (0,1)$.

Clearly, each quasi-contractive mapping is generalized quasi-contractive. Now we give an example to show that the converse is not true.

EXAMPLE 1.1. Let $X = (-\infty, +\infty)$ with the usual metric and $E = [0,1] \cup \{2\}$. Define a function $n : X \to N$ and a mapping $T : E \to E$ by $n(x) = [x] + 2$ for all $x \in X$ and $Tx = \frac{1}{2}x$ for $x \in [0,1)$, $T_1 = 2$ and $T_2 = \frac{1}{4}$, where $[x]$ means the greatest integer not exceeding x. For any $q \in (0,1)$, there exist $x = 1$ and $y = 0$ such that

$$\begin{aligned} |Tx - Ty| &= 2 > 2q = q|y - Tx| \\ &= q \max\{|x-y|, |x-Tx|, |y-Ty|, |x-Ty|, |y-Tx|\}, \end{aligned}$$

which implies that T is not a quasi-contractive mapping. Now we claim that T is generalized quasi-contractive with $q = \frac{1}{4}$. For any $x, y \in E$ with $x \neq y$, we consider the following cases:

Case 1. Let $x, y \in [0,1)$. Then

$$\begin{aligned}|T^{n(x)}x - T^{n(y)}y| &= \frac{1}{4}|x-y| \\ &\leq q\max\{|x-y|, |x-T^{n(x)}x|, |y-T^{n(y)}y|, \\ &\qquad |x-T^{n(y)}y|, |y-T^{n(x)}x|\}.\end{aligned}$$

Case 2. Let $x \in [0,1)$ and $y = 1$. It follows that

$$\begin{aligned}|T^{n(x)}x - T^{n(y)}y| &= \frac{1}{4}\left|x-\frac{1}{2}\right| \leq q|y-T^{n(y)}y| \\ &\leq q\max\{|x-y|, |x-T^{n(x)}x|, |y-T^{n(y)}y|, \\ &\qquad |x-T^{n(y)}y|, |y-T^{n(x)}x|\}.\end{aligned}$$

Case 3. Let $x \in [0,1)$ and $y = 2$. We have

$$\begin{aligned}|T^{n(x)}x - T^{n(y)}y| &= \frac{1}{4}\left|\frac{1}{8}-x\right| \leq q|x-y| \\ &\leq q\max\{|x-y|, |x-T^{n(x)}x|, |y-T^{n(y)}y|, \\ &\qquad |x-T^{n(y)}y|, |y-T^{n(x)}x|\}.\end{aligned}$$

Case 4. Let $x = 1$ and $y = 2$. Then

$$\begin{aligned}|T^{n(x)}x - T^{n(y)}y| &= \frac{1}{4}\left|\frac{1}{8}-\frac{1}{2}\right| \leq q|x-y| \\ &\leq q\max\{|x-y|, |x-T^{n(x)}x|, |y-T^{n(y)}y|, \\ &\qquad |x-T^{n(y)}y|, |y-T^{n(x)}x|\}.\end{aligned}$$

It follows that T is generalized quasi-contractive with $q = \frac{1}{4}$.

For any given $x_0 \in E$ and the function $n \quad X \to N$, the sequence $\{x_n\}_{n\geq 0}$ defined by

$$\begin{aligned}z_n &= (1-c_n)x_n + c_n T^{n(x_n)}x_n, \\ y_n &= (1-b_n)x_n + b_n T^{n(z_n)}z_n, \qquad (1.3)\\ x_{n+1} &= (1-a_n)x_n + a_n T^{n(y_n)}y_n, \quad n \geq 0,\end{aligned}$$

where $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ are any sequences in $[0,1]$ is called *generalized three-step iteration sequence.*

For $x_0 \in E$, the sequence $\{x_n\}_{n\geq 0}$ defined by

$$\begin{aligned} z_n &= (1-c_n)x_n + c_nTx_n, \\ y_n &= (1-b_n)x_n + b_nTz_n, \\ x_{n+1} &= (1-a_n)x_n + a_nTy_n, \quad n \geq 0, \end{aligned} \tag{1.4}$$

where $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ are any sequences in $[0,1]$ is called *three-step iteration sequence.*

It is easy to see that three-step iteration sequence is a special case of generalized three-step iteration sequence by taking $n(x) \equiv 1$ for $x \in X$.

Particularly, if $c_n = 0$ for all $n \geq 0$ in (1.4), then the sequence $\{x_n\}_{n\geq 0}$ defined by

$$\begin{aligned} &x_0 \in E, \\ &y_n = (1-b_n)x_n + b_nTx_n, \\ &x_{n+1} = (1-a_n)x_n + a_nTy_n, \quad n \geq 0, \end{aligned} \tag{1.5}$$

where $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are any sequences in $[0,1]$ is called *Ishikawa iteration sequence* which was introduced by Ishikawa[3].

2. Convergence Theorems

In this section, we establish convergence theorems of three-step iteration methods and generalized three-step iteration methods for quasi-contractive and generalized quasi-contractive mappings, respectively, in Banach spaces.

THEOREM 2.1. *Let E be a nonempty closed convex subset of a Banach space X and $T: E \to E$ be a quasi-contractive mapping satisfying* (1.2) *Suppose that $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ are any sequences in $[0,1]$ satisfying $\sum_{n=0}^{\infty} a_n = \infty$. Then the three-step iteration sequence $\{x_n\}_{n\geq 0}$ defined by* (1 4) *converges to a unique fixed point of T in E.*

PROOF. For any integers m, n with $0 \le n < m$, let

$$D_{n,m} = \cup_{j=n}^{m} \{x_j, y_j, z_j, Tx_j, Ty_j, Tz_j\}$$

and $d_{n,m} = \delta(D_{n,m})$. It is clear that

$$\begin{aligned} &\max\{\|Tx_k - Tx_l\|, \|Tx_k - Ty_l\|, \|Tx_k - Tz_l\|, Tz_k - Tz_l\|\} \\ &\le q d_{n,m} \end{aligned} \tag{2.1}$$

for $n \le k, l \le m$. Now we assert that

$$d_{n,m} = \max\{\|x_n - Tx_j\|, \|x_n - Ty_j\|, \|x_n - Tz_j\| : n \le j \le m\}. \tag{2.2}$$

We consider the following cases:

Case 1. Suppose that $d_{n,m} = \max\{\|x_i - Tx_j\| : n \le i, j \le m\}$. Without loss of generality, we assume that $d_{n,m} = \|x_{k+1} - Tx_l\|$ for some k, l with $n \le k < m$, $n \le l \le m$. From (1.4) and (2.1), we have

$$\begin{aligned} d_{n,m} &= \|(1 - a_k)x_k + a_k Ty_k - Tx_l\| \\ &\le (1 - a_k)\|x_k - Tx_l\| + a_k\|Ty_k - Tx_l\| \\ &\le (1 - a_k)d_{n,m} + a_k q d_{n,m} \\ &\le d_{n,m}, \end{aligned}$$

which implies that $a_k = 0$ In this way, we infer that $d_{n,m} = \|x_n - Tx_l\|$ for some l with $n \le l \le m$.

Case 2. Suppose that $d_{n,m} = \max\{\|x_i - Ty_j\| \cdot n \le i, j \le m\}$. As in the proof of Case 1, we conclude that $d_{n,m} = \|x_n - Ty_l\|$ for some l with $n \le l \le m$.

Case 3. Suppose that $d_{n,m} = \max\{\|x_i - Tz_j\| . n \le i, j \le m\}$. The proof is similar to that of Case 1, so we obtain that $d_{n,m} = \|x_n - Tz_l\|$ for some l with $n \le l \le m$

Case 4. Suppose that $d_{n,m} = \max\{\|y_k - Tx_l\| : n \le k, l \le m\}$ Then there exist some k, l with $n \le k, l \le m$ such that $d_{n,m} = \|y_k - Tx_l\|$. In view of (1.4) and (2 1), we get that

$$\begin{aligned} d_{n,m} &= \|(1 - b_k)x_k + b_k Tz_k - Tx_l\| \\ &\le (1 - b_k)\|x_k - Tx_l\| + b_k\|Tz_k - Tx_l\| \\ &\le (1 - b_k)d_{n,m} + b_k q d_{n,m} \\ &\le d_{n,m}, \end{aligned}$$

which means that $b_k = 0$. Then we deduce that $d_{n,m} = \|x_k - Tx_l\|$. The rest of the proof is similar to that of Case 1.

Case 5. Suppose that $d_{n,m} = \max\{\|y_k - Ty_l\| : n \le k, l \le m\}$. As in the proof of Case 4, we conclude that $d_{n,m} = \|x_k - Ty_l\|$ for some k, l with $n \le k, l \le m$. The rest of the proof is analogous to that of Case 2, so we omit it.

Case 6. Suppose that $d_{n,m} = \max\{\|y_k - Tz_l\| : n \le k, l \le m\}$. The proof is similar to that of Case 4, hence (2.2) holds.

Case 7. Suppose that $d_{n,m} = \max\{\|z_k - Tx_l\| : n \le k, l \le m\}$. Without loss of generality, we assume that $d_{n,m} = \|z_k - Tx_l\|$ for some k, l with $n \le k, l \le m$. In the light of (1.4) and (2.1), we infer that

$$\begin{aligned} d_{n,m} &= \|(1 - c_k)x_k + c_kTx_k - Tx_l\| \\ &\le (1 - c_k)\|x_k - Tx_l\| + c_k\|Tx_k - Tx_l\| \\ &\le (1 - c_k)d_{n,m} + c_kqd_{n,m} \\ &\le d_{n,m}, \end{aligned}$$

which implies that $c_k = 0$. Then $d_{n,m} = \|x_k - Tx_l\|$. The rest of the proof is analogous to that of Case 1, thus (2.2) holds.

Case 8. Suppose that $d_{n,m} = \max\{z_k - Ty_l\| : n \le k, l \le m\}$. As in the proof of Case 7, we get that $d_{n,m} = \|x_k - Ty_l\|$ for some k, l with $n \le k, l \le m$. The rest of the proof is similar to that of Case 2, we omit it.

Case 9 Suppose that $d_{n,m} = \max\{\|z_k - Tz_l\| : n \le k, l \le m\}$. Similarly, (2.2) holds.

Case 10. Suppose that $d_{n,m} = \max\{\|x_i - x_j\| : n \le i, j \le m$. It is easy to see that there exist k, l such that $n \le k < l < m$ and $d_{n,m} = \|x_k - x_{l+1}\| > \|x_k - x_l\|$ From (1.4) and (2.1), we have

$$\begin{aligned} d_{n,m} &= \|(1 - a_l)x_l + a_lTy_l - x_k\| \\ &\le (1 - a_l)\|x_l - x_k\| + a_l\|Ty_l - x_k\| \\ &\le d_{n,m}, \end{aligned}$$

which means that $a_l = 1$. Then we conclude that $d_{n,m} = \|x_k - Ty_l\|$. The rest of the proof is analogous to that of Case 2, so (2.2) holds.

Case 11. Suppose that $d_{n,m} = \max\{\|y_k - x_l\| : n \le k, l \le m\}$. Without loss of generality, we assume that $d_{n,m} = \|y_k - x_l\|$ for some k, l with $n \le k, l \le m$. In view of (1.4) and (2.1), we infer that

$$\begin{aligned} d_{n,m} &= \|(1 - b_k)x_k + b_k T z_k - x_l\| \\ &\le (1 - b_k)\|x_k - x_l\| + b_k\|Tz_k - x_l\| \\ &\le d_{n,m}, \end{aligned}$$

which yields that $d_{n,m} = \|x_k - x_l\|$ or $d_{n,m} = \|Tz_k - x_l\|$. The rest of the proof is similar to that of Case 10 or Case 3, hence (2.2) holds.

Case 12. Suppose that $d_{n,m} = \max\{\|z_k - x_l\| : n \le k, l \le m\}$. We assume that $d_{n,m} = \|z_k - x_l\|$ for some k, l with $n \le k, l \le m$. By (1.4) and (2.1), we obtain that

$$\begin{aligned} d_{n,m} &= \|(1 - c_k)x_k + c_k T x_k - x_l\| \\ &\le (1 - c_k)\|x_k - x_l\| + c_k\|Tx_k - x_l\| \\ &\le d_{n,m}, \end{aligned}$$

which means that $d_{n,m} = \|x_k - x_l\|$ or $d_{n,m} = \|Tx_k - x_l\|$. The rest of the proof is analogous to that of Case 10 or Case 1, thus (2 2) holds.

Case 13. Suppose that $d_{n,m} = \max\{\|y_k - y_l\| : n \le k, l \le m\}$. Without loss of generality, we assume that $d_{n,m} = \|y_k - y_l\|$ for some $n \le k, l \le m$. In view of (1 4) and (2.1), we deduce that

$$\begin{aligned} d_{n,m} &= \|(1 - b_k)x_k + b_k T z_k - y_l\| \\ &\le (1 - b_k)\|x_k - y_l\| + b_k\|Tz_k - y_l\| \\ &\le d_{n,m}, \end{aligned}$$

which yields that $d_{n,m} = \|x_k - y_l\|$ or $d_{n,m} = \|Tz_k - y_l\|$. The rest of the proof is similar to that of Case 11 or Case 6, so we omit it.

Case 14. Suppose that $d_{n,m} = \max\{\|y_k - z_l\| : n \le k, l \le m\}$. We assume that $d_{n,m} = \|y_k - z_l\|$ for some k, l with $n \le k, l \le m$. In the light of (1 4) and (2.1), we have

$$\begin{aligned} d_{n,m} &= \|(1 - b_k)x_k + b_k T z_k - z_l\| \\ &\le (1 - b_k)\|x_k - Tz_l\| + b_k\|Tz_k - z_l\| \\ &\le d_{n,m}, \end{aligned}$$

which implies that $d_{n,m} = \|x_k - z_l\|$ or $d_{n,m} = \|Tz_k - z_l\|$. The rest of the proof is analogous to that of Case 12 or Case 9, hence (2.2) holds.

Case 15 Suppose that $d_{n,m} = \max\{\|z_k - z_l\| : n \le k, l \le m\}$. Without loss of generality, we assume that $d_{n,m} = \|z_k - z_l\|$ for some k, l with $n \le k, l \le m$. By virtue of (1.4) and (2.1), we deduce that

$$\begin{aligned} d_{n,m} &= \|(1-c_k)x_k + c_kTx_k - z_l\| \\ &\le (1-c_k)\|x_k - z_l\| + c_k\|Tx_k - z_l\| \\ &\le d_{n,m}, \end{aligned}$$

hence $d_{n,m} = \|x_k - z_l\|$ or $d_{n,m} = \|Tx_k - z_l\|$. The rest of the proof is similar to that of Case 12 or Case 7, thus we omit it.

From Case 1 to Case 15, we obtain that (2.2) holds. Note that

$$\begin{aligned} d_{0,m} &= \max\{\|x_0 - Tx_j\|, \|x_0 - Ty_j\|, \|x_0 - Tz_j\| : 0 \le j \le m\} \\ &\le \|x_0 - Tx_0\| + \max\{\|Tx_0 - Tx_j\|, \|Tx_0 - Ty_j\|, \\ &\qquad \|Tx_0 - Tz_j\| : 0 \le j \le m\} \\ &\le \|x_0 - Tx_0\| + qd_{0,m}, \end{aligned}$$

which yields that

$$d_{0,m} \le \frac{1}{1-q}\|x_0 - Tx_0\| \quad \text{for} \quad m \ge 0. \tag{2.3}$$

It follows from (2.2) and (2 3) that

$$\begin{aligned} & d_{n,n+p} \\ &= \max\{\|x_n - Tx_j\|, \|x_n - Ty_j\|, \|x_n - Tz_j\| : n \le j \le n+p\} \\ &\le (1-a_{n-1})d_{n-1,n+p} + a_{n-1}qd_{n-1,n+p} \\ &= (1-(1-q)a_{n-1})d_{n-1,n+p} \\ &\le \prod_{j=0}^{n-1}(1-(1-q)a_j)d_{0,n+p} \qquad (2.4) \\ &\le \prod_{j=0}^{n-1}(1-(1-q)a_j)\frac{1}{1-q}\|x_0 - Tx_0\| \\ &\le \frac{1}{1-q}\|x_0 - Tx_0\| \exp\left(-(1-q)\sum_{j=0}^{n-1} a_j\right) \end{aligned}$$

for any $p \geq 0$. Letting $n \to \infty$ in (2 4), we have $d_{n,n+p} \to 0$. Thus $\{x_n\}_{n\geq 0}$ is a Cauchy sequence and hence $\{x_n\}_{n\geq 0}$ converges to some $u \in E$. On the other hand, (2 2) and (2.4) ensure that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Next we assert that u is the fixed point of T. Otherwise $Tu \neq u$. By (1 2), we have

$$\|Tx_n - Tu\| \leq q \max\{\|x_n - u\|, \|x_n - Tx_n\|, \|u - Tu\|, \\ \|x_n - Tu\|, \|u - Tx_n\|\},$$

letting $n \to \infty$ in the above inequality, we infer that

$$\|u - Tu\| \leq q\|u - Tu\| < \|u - Tu\|,$$

which is impossible. Hence $u = Tu$ Since T is quasi-contractive, it is easy to see that u is the unique fixed point of T in E. This completes the proof.

REMARK 2 1 It is clear that Theorem 1 of Zhao [6] is a special case of Theorem 2.1 by taking $c_n = 0$.

THEOREM 2.2. *Let E be as in Theorem 2.1, $T : E \to E$ be a generalized quasi-contractive mapping satisfying* (1.1) *and $n(x)|n(Tx)$ for $x \in X$ Suppose that $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ are any sequences in $[0, 1]$ satisfying $\sum_{n=0}^{\infty} a_n = \infty$. Then the generalized three-step iteration sequence $\{x_n\}_{n\geq 0}$ defined by* (1.3) *converges to a unique fixed point of T in E.*

PROOF. For $x \in E$, we write $\widetilde{T}x = T^{n(x)}x$. It is easy to show that $\widetilde{T}$ and $\{x_n\}_{n\geq 0}$ satisfy the conditions of Theorem 2.1. It follows from Theorem 2.1 that $\{x_n\}_{n\geq 0}$ converges to the unique fixed point u of $\widetilde{T}$ in E. Using $n(u)|n(Tu)$, we deduce that $T^{n(Tu)}u = T^{n(u)}u = \widetilde{T}u = u$, which $\widetilde{T}(Tu) = T^{n(Tu)}(Tu) = Tu$ So Tu is also a fixed point of $\widetilde{T}$ It follows from the uniqueness of fixed point of $\widetilde{T}$ that $Tu = u$. Clearly, u is the unique fixed point of T This completes the proof

REMARK 2.2 Theorem 2 2 generalizes the corresponding results of [1], [4]-[6].

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Jinbiao Hao and Li Wang
Department of Mathematics
Liaoning Normal University
Dalian, Liaoning 116029
People's Republic of China

Shin Min Kang and Soo Hak Shim
Department of Mathematics
Gyeongsang National University
Chinju 660-701, Korea
E-mail: smkang@nongae gsnu.ac.kr