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# COINCIDENCE POINTS IN $T_1$ TOPOLOGICAL SPACES

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ABSTRACT In this paper, we prove a few coincidence point theorems for two pairs of mappings in $T_1$ topological spaces. Our results extend, improve and unify the corresponding results in [1]-[3]

## 1. Introduction

Machuca [3] established a coincidence point theorem involving a pair of mappings in $T_1$ topological spaces. Khan [1] and Liu [2] extended Machuca's result to three mappings. The aim of this paper is to establish some coincidence point theorems for two pairs of mappings in $T_1$ topological spaces. Our results are the extension of the results due to Khan [1], Liu [2] and Machuca [3].

Let $X$ and $Y$ be topological spaces. A mapping $f : X \to Y$ is said to be *proper* if $f^{-1}(A)$ is compact for each compact subset $A$ of $Y$ with $A \subseteq f(X)$. For any subset $A \subseteq Y$, $\overline{A}$, denotes the closure of $A$. Let $R^+ = [0, \infty)$ and

$\Phi = \{\phi : \phi : (R^+)^5 \to R^+$ is upper semicontinuous and nondecreasing in each coordinate variable and satisfies (1.1)$\}$, where

$$(1.1) \qquad \bar{\phi}(t) = \max\{\phi(t,t,t,at,bt) : a+b=2,\ a,b,\in \{0,1,2\}\} < t$$

for all $t > 0$.

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LEMMA 1.1. [4] *Let $\psi : R^+ \to R^+$ be nondecreasing and upper semi-continuous. Then for each $t > 0$, $\psi(t) < t$ if and only if $\lim_{n\to\infty} \psi^n(t) = 0$, where $\psi^n$ denotes the composition of $\psi$ with itself $n$-times.*

## 2. Main Results

THEOREM 2.1. *Let $X$ be a $T_1$ topological space satisfying the first axiom of countability, $(Y,d)$ be a complete metric space and $A, B, S, T : X \to Y$ satisfy*

(i) $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$,

*and one of the following conditions:*

(ii) *$A$ and $S$ are continuous, $A$ is proper with $A(X)$ closed;*

(iii) *$A$ and $S$ are continuous, $S$ is proper with $S(X)$ closed;*

(iv) *$B$ and $T$ are continuous, $B$ is proper with $B(X)$ closed;*

(v) *$B$ and $T$ are continuous, $T$ is proper with $T(X)$ closed.*

*If there exists some $\phi \in \Phi$ such that*

$$\begin{aligned} d(Ax, By) \leq \phi(&d(Sx,Ty), d(Ax,Sx), d(By,Ty), \\ &d(Ax,Ty), d(By,Sx)) \end{aligned} \tag{2.1}$$

*for all $x, y \in X$, then there exist $u, v \in X$ such that $Au = Su = Bv = Tv$.*

PROOF. Given $x_0 \in X$. From (i) we can easily choose sequences $\{x_n\}_{n\geq 1} \subset X$ and $\{y_n\}_{n\geq 1} \subset Y$ such that

$$\begin{cases} y_{2n+1} = Tx_{2n+1} = Ax_{2n}, & n \geq 0, \\ y_{2n} = Sx_{2n} = Bx_{2n-1}, & n \geq 1. \end{cases} \tag{2.2}$$

Put $d_n = d(y_n, y_{n+1})$ for all $n \geq 1$. By virtue of (2.1) and (2.2), we infer that for any $n \geq 1$,

$$\begin{aligned} &d_{2n+1} \\ &= d(Ax_{2n}, Bx_{2n+1}) \\ &\leq \phi(d(Sx_{2n}, Tx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \\ &\qquad d(Ax_{2n}, Tx_{2n+1}), d(Bx_{2n+1}, Sx_{2n})) \\ &= \phi(d_{2n}, d_{2n}, d_{2n+1}, 0, d(y_{2n+2}, y_{2n})) \\ &\leq \phi(d_{2n}, d_{2n}, d_{2n+1}, 0, d_{2n} + d_{2n+1}). \end{aligned} \tag{2.3}$$

Suppose that $d_{2n+1} > d_{2n}$. Then (2.3) implies that

$$d_{2n+1} \le \phi(d_{2n+1}, d_{2n+1}, d_{2n+1}, 0, 2d_{2n+1}) \le \bar{\phi}(d_{2n+1}) < d_{2n+1},$$

which is a contradiction. Therefore, $d_{2n+1} \le d_{2n}$. It follows from (2.3) that

$$d_{2n+1} \le \phi(d_{2n}, d_{2n}, d_{2n}, 0, 2d_{2n}) \le \bar{\phi}(d_{2n}).$$

Similarly, we can also deduce that $d_{2n} \le \bar{\phi}(d_{2n-1})$. Consequently, we have

$$d_n \le \bar{\phi}(d_{n-1}) \le \bar{\phi}^2(d_{n-2}) \le \cdots \le \bar{\phi}^{n-1}(d_1) \quad \text{for all } n \ge 1,$$

which and Lemma 1.1 mean that

$$\lim_{n\to\infty} d_n = 0. \tag{2.4}$$

In order to show that $\{y_n\}_{n\ge1}$ is a Cauchy sequence, it is sufficient to show that $\{y_{2n}\}_{n\ge1}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\ge1}$ is not a Cauchy sequence. Then there exist an $\epsilon > 0$ such that for each even integer $2k$, there exists even integers $2m(k)$ and $2n(k)$ with

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon, \quad 2m(k) > 2n(k) > 2k. \tag{2.5}$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (2.5), that is,

$$d(y_{2n(k)}, y_{2m(k)-2}) \le \epsilon \quad \text{and} \quad d(y_{2n(k)}, y_{2m(k)}) > \epsilon. \tag{2.6}$$

Note that

$$\epsilon < d(y_{2n(k)}, y_{2m(k)}) \le d(y_{2n(k)}, y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}. \tag{2.7}$$

Using (2.4), (2.6) and (2.7), we conclude that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \tag{2.8}$$

It is easy to verify that

$$(2.9)\qquad \begin{aligned} |d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| &\le d_{2m(k)-1}, \\ |d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)-1})| &\le d_{2n(k)}. \end{aligned}$$

According to (2.4), (2.8) and (2.9), we know that

$$(2.10)\qquad \lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) = \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}) = \epsilon.$$

In view of (2.1), we have

$$\begin{aligned} & d(y_{2n(k)}, y_{2m(k)}) \\ & \le d_{2n(k)} + d(Ax_{2n(k)}, Bx_{2m(k)-1}) \\ & \le d_{2n(k)} + \phi(d(Sx_{2n(k)}, Tx_{2m(k)-1}), d(Ax_{2n(k)}, Sx_{2n(k)}), \\ & \quad d(Bx_{2m(k)-1}, Tx_{2m(k)-1}), d(Ax_{2n(k)}, Tx_{2m(k)-1}), \\ & \quad d(Bx_{2m(k)-1}, Sx_{2n(k)})) \\ & = d_{2n(k)} + \phi(d(y_{2n(k)}, y_{2m(k)-1}), d_{2n(k)}, d_{2m(k)-1}, \\ & \quad d(y_{2n(k)+1}, y_{2m(k)-1}), d(y_{2m(k)}, y_{2n(k)})). \end{aligned}$$

Letting $k \to \infty$ in the above inequalities, by (2.4), (2.8) and (2.10) we obtain that

$$\epsilon \le \phi(\epsilon, 0, 0, \epsilon, \epsilon) \le \bar{\phi}(\epsilon) < \epsilon,$$

which is impossible. Hence $\{y_n\}_{n\ge1}$ is a Cauchy sequence. Since $(Y, d)$ is complete, there exists some $z \in Y$ with $\lim_{n\to\infty} y_n = z$.

Assume that (ii) holds. Set $C = \{Ax_{2n} : n \ge 1\} \cup \{z\}$. Then $C = \overline{C} \subseteq \overline{A(X)} = A(X) \subseteq Y$ and $C$ is compact. It follows that $A^{-1}(C)$ also compact because $A$ is proper. Consequently, there exists a subsequence $\{x_{2n(k)}\}_{k\ge1}$ of $\{x_{2n}\}_{n\ge1}$ such that it converges to some point $u \in X$. The continuity of $A$ and $S$ ensures that

$$(2.11)\qquad \lim_{k\to\infty} Ax_{2n(k)} = Au = z = \lim_{k\to\infty} Sx_{2n(k)} = Su.$$

Since $Au \in A(X) \subseteq T(X)$, there exists some $v \in X$ such that $Au = Tv$. Now we claim that $Au = Bv$. Otherwise $Au \ne Bv$. From (2.1) and

(2.11) we get that

$$\begin{aligned} d(Au, Bv) &\leq \phi(d(Su, Tv), d(Au, Su), d(Bv, Tv), \\ &\qquad d(Au, Tv), d(Bv, Su)) \\ &= \phi(0, 0, d(Au, Bv), 0, d(Au, Bv)) \\ &\leq \bar{\phi}(d(Au, Bv)) \\ &< d(Au, Bv), \end{aligned}$$

which is a contradiction. Hence $Au = Bv$. Thus, $Au = Su = Bv = Tv$.

Assume that (iii) holds. Set $C = \{Sx_{2n} : n \geq 1\} \cup \{z\}$. It is easy to see that $S^{-1}(C)$ also compact because $C$ is compact and $S$ is proper. Consequently, there exists a subsequence $\{x_{2n(k)}\}_{k \geq 1}$ of $\{x_{2n}\}_{n \geq 1}$ such that it converges to some point $u \in X$. The continuity of $A$ and $S$ ensures that (2.11) holds. Similarly, we can prove that there exists some $v \in X$ with $Au = Su = Bv = Tv$

Assume that (iv) holds. Put $C = \{Bx_{2n-1} : n \geq 1\} \cup \{z\}$. Then $B^{-1}(C)$ is compact since $C$ is compact and $B$ is proper. Clearly, there exists a subsequence $\{x_{2n(k)-1}\}_{k \geq 1}$ of $\{x_{2n-1}\}_{n \geq 1}$ such that it converges to some point $v \in X$. It follows from the continuity of $B$ and $T$ that

$$\lim_{k \to \infty} Bx_{2n(k)-1} = Bv = z = \lim_{k \to \infty} Tx_{2n(k)-1} = Tv. \tag{2.12}$$

Notice that $Bv \in B(X) \subseteq S(X)$. Of course, there exists some $u \in X$ such that $Bv = Su$. Suppose that $Au \neq Bv$. According to (2.1) and (2.12), we deduce that

$$\begin{aligned} d(Au, Bv) &\leq \phi(d(Su, Tv), d(Au, Su), d(Bv, Tv), \\ &\qquad d(Au, Tv), d(Bv, Su)) \\ &= \phi(0, d(Au, Bv), 0, d(Au, Bv), 0) \\ &\leq \bar{\phi}(d(Au, Bv)) \\ &< d(Au, Bv), \end{aligned}$$

which is impossible. Therefore $Au = Bv$. Thus, $Au = Su = Bv = Tv$.

Assume that (v) holds. Put $C = \{Tx_{2n-1} : n \geq 1\} \cup \{z\}$. Then $T^{-1}(C)$ is compact since $C$ is compact and $T$ is proper. Clearly, there exists a subsequence $\{x_{2n(k)-1}\}_{k\geq 1}$ of $\{x_{2n-1}\}_{n\geq 1}$ such that it converges to some point $v \in X$. It follows from the continuity of $B$ and $T$ that (2.12) holds. Similarly, we have $Au = Su = Bv = Tv$ for some $u \in X$. This completes the proof.

THEOREM 2.2. *Let $X$ be a $T_1$ topological space satisfying the first axiom of countability, $(Y, d)$ be a complete metric space and $A, B, S : X \to Y$ satisfy*

(i) $A(X) \cup B(X) \subseteq S(X)$,

*and one of the following conditions:*

(ii) *$A$ and $S$ are continuous, $A$ is proper with $A(X)$ closed;*
(iii) *$A$ and $S$ are continuous, $S$ is proper with $S(X)$ closed;*
(iv) *$B$ and $S$ are continuous, $B$ is proper with $B(X)$ closed;*
(v) *$B$ and $S$ are continuous, $S$ is proper with $S(X)$ closed.*

*If there exists some $\phi \in \Phi$ such that*

$$d(Ax, By) \leq \phi(d(Sx, Sy), d(Ax, Sx), d(By, Sy), d(Ax, Sy), d(By, Sx)) \tag{2.13}$$

*for all $x, y \in X$, then there exist $u \in X$ such that $Au = Bu = Su$.*

PROOF. Let $x_0$ be an arbitrary element in $X$. Then (i) ensures that there exist sequences $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$ in $X$ such that

$$\begin{cases} y_{2n+1} = Sx_{2n+1} = Ax_{2n}, & n \geq 0, \\ y_{2n} = Sx_{2n} = Bx_{2n-1}, & n \geq 1. \end{cases}$$

As in the proof of Theorem 2.1, we conclude that $\{y_n\}_{n\geq 1}$ converges to some $z \in Y$.

Assume that (ii) holds. Set $C = \{Ax_{2n} : n \geq 1\} \cup \{z\}$. Then $C = \overline{C} \subseteq \overline{A(X)} = A(X) \subseteq Y$ and $C$ is compact. It follows that $A^{-1}(C)$ also compact because $A$ is proper. Consequently, there exists a subsequence $\{x_{2n(k)}\}_{k\geq 1}$ of $\{x_{2n}\}_{n\geq 1}$ such that it converges to some point $u \in X$. The continuity of $A$ and $S$ ensures that (2.11) holds. Now we claim that $Au = Bu$. Otherwise $Au \neq Bu$. From (2.11), (2.13) we get that

$$\begin{aligned} d(Au, Bu) &\leq \phi(d(Su, Su), d(Au, Su), d(Bu, Su), \\ &\qquad d(Au, Su), d(Bu, Su)) \\ &= \phi(0, 0, d(Au, Bu), 0, d(Au, Bu)) \\ &\leq \bar{\phi}(d(Au, Bu)) \\ &< d(Au, Bu), \end{aligned}$$

which is a contradiction. Hence $Au = Bu$. Thus, $Au = Bu = Su$. Similarly, we can complete the proof if one of (iii)-(v) holds.

REMARK 2.1. Theorem 2.2 extends Theorems 1 and 2 in [2] and the results in [1] and [3].

In case $\phi(x, y, z, u, v) = r \max\{x, y, z, \frac{1}{2}(u+v)\}$ for all $(x, y, z, u, v) \in (R^+)^5$, where $r$ is a constant in $(0, 1)$, then Theorems 2.1 and 2.2 yield the following:

COROLLARY 2.1. *Let $X$ be a $T_1$ topological space satisfying the first axiom of countability, $(Y, d)$ be a complete metric space and $A, B, S, T : X \to Y$ satisfy* (i) *and one of* (ii)-(v) *in Theorem 2.1. Suppose that*

$$\begin{aligned} d(Ax, By) \leq r \max\{&d(Sx, Ty), d(Ax, Sx), d(By, Ty), \\ &\frac{1}{2}(d(Ax, Ty) + d(By, Sx))\} \end{aligned} \tag{2.14}$$

*for all $x, y \in X$, where $r$ is a constant in $(0, 1)$. Then there exist $u, v \in X$ such that $Au = Su = Bv = Tv$.*

COROLLARY 2.2. *Let $X$ be a $T_1$ topological space satisfying the first axiom of countability, $(Y, d)$ be a complete metric space and $A, B, S : X \to Y$ satisfy* (i) *and one of* (ii)-(v) *in Theorem 2.2. Suppose that*

$$\begin{aligned} d(Ax, By) \leq r \max\{&d(Sx, Sy), d(Ax, Sx), d(By, Sy), \\ &\frac{1}{2}(d(Ax, Sy) + d(By, Sx))\} \end{aligned} \tag{2.15}$$

*for all $x, y \in X$, where $r$ is a constant in $(0, 1)$. Then there exists $u \in X$ such that $Au = Bu = Su$.*

REMARK 2.2. The results in [1] and [3] are special cases of Corollary 2.1.

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