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MULTIPLICATIVE HYPER IS-ALGEBRAS

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1. Introduction

The notion of BCK-algebras was proposed by Y. Imai and K. Iséki in 1966. In the same year, K. Iséki [I] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. In 1993, Y. B. Jun et al. [JHR] introduced a new class of algebras related to BCI-algebras and semigroups, called a BCI-semigroup. In 1998, Y. B. Jun, X. L. Lin and E. H. Roh [JXR] renamed the BCI-semigroup as the IS-algebra. They found the necessary and sufficient condition that an IS-algebra X is an IG-algebra. Y. B. Jun, E. H. Roh and X. L. Lin [JRX] characterized the $\mathcal{I}$-ideals in an IS-algebra and gave a description of the element of the left (resp. right) $\mathcal{I}$-ideal generated by the union of left (resp. right) $\mathcal{I}$-ideals A and B in an IS-algebra, which is a generalization of [AK, Theorem 2.5]. They also proved that if an IS-algebra is finite, then every $\mathcal{I}$-ideal is closed.

In this paper, we define multiplicative hyper IS-algebras. A necessary and sufficient condition for a strongly distributive hyper IS-algebra to be an IS-algebra is stated. We also show that every unitary strongly distributive hyper IS-algebra is an IS-algebra.

DEFINITION 1.1. A *BCI-algebra* is an algebra $(X; *, 0)$ of type (2,0) is satisfying the following axioms for all $x, y, z \in X$:

(a) $((x*y)*(x*z))*(z*y) = 0$
(b) $(x*(x*y))*y = 0$

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(c) $x * x = 0$
(d) $x * y = 0$ and $y * x = 0$ imply $x = y$.

For any BCI-algebra X, the relation $\leq$ defined by $x \leq y$ if and only if $x * y = 0$ is a partial order on X.

A BCI-algebra X has the following properties for all $x, y, z \in X$:

(1) $x * 0 = x$
(2) $(x * y) * z = (x * z) * y$
(3) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

DEFINITION 1.2. A non-empty subset I of a BCI-algebra X is called an *ideal of* X if it satisfies

(1) $0 \in I$,
(2) $x * y \in I$ and $y \in I$ imply $x \in I$

It is well known that an ideal I of a BCI-algebra X need not be a subalgebra.

DEFINITION 1.3. ([JXR]) An *IS-algebra* is a non-empty set X with two binary operations "$*$" and "$\cdot$" and constant 0 satisfying the axioms :

(1) $I(X) := (X, *, 0)$ is a BCI-algebra.
(2) $S(X) := (X, \cdot)$ is a semigroup.
(3) $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$ for all $x, y, z \in X$.

LEMMA 1.4. ([JHR]). *Let X be an IS-algebra. Then we have*

(1) $0 \cdot x = x \cdot 0 = 0$,
(2) $x \leq y$ *implies that* $x \cdot z \leq y \cdot z$ *and* $z \cdot x \leq z \cdot y$,

for all $x, y, z \in X$.

DEFINITION 1.5. ([AK]) A non-empty subset A of a semigroup $S(X) := (X, \cdot)$ is said to be *left* (resp. *right*) *stable* if $x \cdot a \in A$ (resp. $a \cdot x \in A$) whenever $x \in S(X)$ and $a \in A$. Both left and right stable is *two-sided stable* or simply *stable*.

DEFINITION 1.6. ([JXR]) A non-empty subset A of an IS-algebra X is called a *left* (resp. *right*) *$\mathcal{I}$-ideal of* X if

(1) A is a left (resp. right) stable subset of $S(X)$,
(2) for any $x, y \in I(X)$, $x * y \in A$ and $y \in A$ imply that $x \in A$.

Both a left and right $\mathcal{I}$-ideal is called a *two-sided $\mathcal{I}$-ideal* or simply an *$\mathcal{I}$-ideal.* It is clear that if A is a left (resp. right) $\mathcal{I}$-ideal of an IS-algebra X, then $0 \in A$. Thus A is an ideal of $I(X) := (X, *, 0)$.

2. Multiplicative hyper IS-algebras

DEFINITION 2.1. A *multiplicative hyper IS-algebra* is a non-empty set X with a binary operation " $*$ " and a hyperoperation " $\cdot$ " and constant 0 satisfying the following axioms :

(i) $I(X) := (X, *, 0)$ is a BCI-algebra,
(ii) multiplication is a hyperoperation $\cdot : X \times X \to \mathcal{P}_0(X)$, where $\mathcal{P}_0(X)$ is the power set of X with the empty set removed, such that $\cdot$ is associative,
(iii) for all $x, y, z \in X$, $x \cdot (y * z) \subseteq (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z \subseteq (x \cdot z) * (y \cdot z)$.

If in (iii) both containments are equality, we say that the multiplicative hyper IS-algebra is *strongly distributive.* All hyper IS-algebras in this paper will be multiplicative hyper IS-algebras and will be strongly distributive only if explicitly stated.

First, we prove the following.

THEOREM 2.2 *In a strongly distributive hyper IS-algebra X, we have $0 \in x \cdot 0$ and $0 \in 0 \cdot x$ for every $x \in X$.*

PROOF. We obtain that $x \cdot 0 = x \cdot (x * x) = x \cdot x * x \cdot x$, for every $x \in X$. Thus $0 \in x \cdot 0$, for every $x \in X$. Similarly, we have $0 \in 0 \cdot x$, for each $x \in X$

The above theorem is not true if the hyper IS-algebra is not strongly distributive, as is shown by the following example.

Example 2.3. Let $(X, *, \cdot, 0)$ be an IS-algebra such that $|X| > 1$. Define hypermultiplication as $x \bullet y = X - \{0\}$, for every $x, y \in X$. Then, by routine calculations, we can check that $(X, *, \bullet, 0)$ is a multiplicative hyper IS-algebra but not strongly distributive. For every $x \in X$, we have $0 \notin x \bullet 0 = 0 \bullet x$.

In the following theorem, we find the necessary and sufficient conditions for the hyper IS-algebra to be an IS-algebra.

THEOREM 2.4. *Let X be a strongly distributive hyper IS-algebra. Then the following are equivalent :*

(1) *X is an IS-algebra,*
(2) *$|x \cdot y| = 1$ for every $x, y \in X$,*
(3) *$|x \cdot 0| = 1$ for every $x \in X$,*
(4) *$|0 \cdot x| = 1$ for every $x \in X$,*
(5) *$|0 \cdot 0| = 1$.*

PROOF. We only prove that (3) $\Rightarrow$ (2), (5) $\Rightarrow$ (4) and (5) $\Rightarrow$ (3). Other cases are straightforward.

(3) $\Rightarrow$ (2) : Let $x, y \in X$. Then $x \cdot 0 = x(y * y) = (x \cdot y) * (x \cdot y)$. Assume that $a, b \in x \cdot y$. Then $a * b, b * a \in x \cdot 0$. Since $x \cdot 0 = \{0\}$, we have $a * b = 0$ and $b * a = 0$. Hence, $a = b$. Thus, $x \cdot y$ is a singleton set, i.e., $|x \cdot y| = 1$.

(5) $\Rightarrow$ (4) : Assume that $|0 \cdot 0| = 1$. If $x \in X$, then $0 \cdot 0 = 0 \cdot (x * x) = (0 \cdot x) * (0 \cdot x)$. Let us suppose that $a, b \in 0 \cdot x$. Then $a * b, b * a \in 0 \cdot 0$. It follows from $0 \cdot 0 = \{0\}$ that $a * b = 0$ and $b * a = 0$. Thus, $a = b$. So, we have $|0 \cdot x| = 1$.

(5) $\Rightarrow$ (3) : Similarly, if $|0 \cdot 0| = 1$, then $|x \cdot 0| = 1$, for each $x \in X$.

THEOREM 2.5. *Let X be a strongly distributive hyper IS-algebra. Then it is an IS-algebra if and only if there exists an element x in X such that $|x \cdot 0| = 1$.*

PROOF. Necessity is obvious. Conversely, assume that $|x \cdot 0| = 1$ for some $x \in X$. If $x = 0$, then we are done by Theorem 2.4. Suppose that $x \neq 0$. Then $0 \cdot 0 = (x * x) \cdot 0 = (x \cdot 0) * (x \cdot 0)$. But $x \cdot 0 = \{0\}$ by Theorem 2.2. Hence $(x \cdot 0) * (x \cdot 0)$ contains only $0 * 0 = 0$. Thus $|0 \cdot 0| = 1$ and in view of the equivalence in Theorem 2.4, X is an IS-algebra. This completes the proof.

Clearly the proof from Theorem 2.5 is valid if we replace $|x \cdot 0| = 1$ with $|0 \cdot x| = 1$. To show that the above Theorem is not valid in general if X is not strongly distributive, we give the following example.

Example 2.6. Let $X = \{0, a, b, c\}$. Define $*$-operation and multiplication "$\cdot$" by the following tables

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

Then we can easily check that X is an IS-algebra ([JXR]). Define hypermultiplication as

$$x \bullet y = \{x \cdot y, (x * (0 * x)) \cdot y, (x * (0 * (x * (0 * x)))) \cdot y, \cdots\}.$$

Hence, we have the following $\bullet$-table :

$\bullet$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{0, b\}$	$\{0, c\}$
b	$\{0\}$	$\{0, a\}$	$\{0, b\}$	$\{0, c\}$
c	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$

Then, by routine calculations, we can show that $(X, *, \bullet, 0)$ is a multiplicative hyper IS-algebra that is not strongly distributive. And the algebra $(X, *, \bullet, 0)$ is not an IS-algebra, since $a \bullet (b * c) = \{0, a\}$ and $(a \bullet b) * (a \bullet c) = \{0, a, b, c\}$. However, for every $x \in X$,

$$x \bullet 0 = \{x \cdot 0, (x * (0 * x)) \cdot 0, \cdots\} = \{0\} = 0 \bullet x.$$

THEOREM 2.7. *Let X be a strongly distributive hyper IS-algebra. Then it is an IS-algebra if and only if there exist elements x and y in X such that $|x \cdot y| = 1$.*

PROOF. Necessity is obvious. Conversely, suppose that there exist elements x and y in X such that $|x \cdot y| = 1$. Then $x \cdot 0 = x \cdot (y * y) = (x \cdot y) * (x \cdot y) = \{0\}$. By Theorem 2.5, X is an IS-algebra. This completes the proof.

DEFINITION 2.8. A hyper IS-algebra X is said to be *unitary* if it contains an element v in X such that $x \cdot v = v \cdot x = \{x\}$ for all $x \in X$.

Example 2.9. In the Example 2.6, the hyper IS-algebra X is then a unitary multiplicative hyper IS-algebra.

From Theorem 2.7, we have the following result.

THEOREM 2.10. *Every unitary strongly distributive hyper IS-algebra X is an IS-algebra.*

THEOREM 2.11. *Let X be a hyper IS-algebra. If there exist elements x and y in X such that $|x \cdot y| = 1$, then $0 \cdot 0 = \{0\}$.*

PROOF. Suppose that $|x \cdot y| = 1$ for some $x, y \in X$ and consider $x \cdot 0$. Then $x \cdot 0 = x \cdot (y * y) \subseteq (x \cdot y) * (x \cdot y) = \{0\}$. But then $0 \cdot 0 = (x * x) \cdot 0 \subseteq (x \cdot 0) * (x \cdot 0)$. Since $x \cdot 0$ is a singleton set, we have $0 \cdot 0 = \{0\}$, completing the proof.

COROLLARY 2.12. *In any unitary hyper IS-algebra X, we have $0 \cdot 0 = \{0\}$.*

3. Hyper $\mathcal{I}$-ideals

Let I be an ideal of a BCI-algebra $(X, *, 0)$. For any x, y in X, we define $x \sim y$ by $x * y \in I$ and $y * x \in I$. Then $\sim$ is a congruence relation on X. Let C_x denote the equivalence class containing x. Let $X/I := \{C_x | x \in X\}$ and define that $C_x * C_y = C_{x*y}$. Since $\sim$ is a congruence relation on X, the operation "$*$" is well-defined. Moreover, if $(X, *, 0)$ is a BCI-algebra and if I is an ideal of X, then $(X/I, *, C_0)$ is also a BCI-algebra, which is called *the quotient algebra via I*, and $C_0 = I$ [MX].

Let I be an $\mathcal{I}$-ideal of an IS-algebra X. Since I is an ideal of the BCI-algebra X, $(X/I, *, C_0)$ is a BCI-algebra. Define $C_x \cdot C_y = C_{x\,y}$ on X/I. Then the operation "$\cdot$" is well-defined. It is known that the operation "$\cdot$" is distributive on both sides over the operation "$*$".

Thus we have the following Theorem.

THEOREM 3.1. ([AK2]). *Let $(X, *, \ , 0)$ be an IS-algebra and let I be an $\mathcal{I}$-ideal of X. Then $(X/I, *, \cdot, C_0)$ is also an IS-algebra, which is called the quotient algebra via I, and $C_0 = I$.*

DEFINITION 3.2. A non-empty subset A of a multiplicative hyper IS-algebra X is called a *left* (resp. *right*) *hyper $\mathcal{I}$-ideal of X* if

(HI1) A is an ideal of $I(X)$,
(HI2) if $x \in X$ and $a \in A$, then $xa \subseteq A$ (resp. $ax \subseteq A$).

Both a left and a right hyper $\mathcal{I}$-ideal is called a *two-sided hyper $\mathcal{I}$-ideal* or simply *hyper $\mathcal{I}$-ideal.*

Let I be a hyper $\mathcal{I}$-ideal of a strongly distributive hyper IS-algebra X. Then $(X/I, *, C_0)$ is a BCI-algebra as above. Define $C_x \cdot C_y = C_{x\,y} = \{C_a \mid a \in x \cdot y\}$ on X/I. Then the binary operation "$\cdot$" is well-defined. Indeed, if $C_x = C_y$ and $C_s = C_t$, then $x * y, y * x \in I$ and $s * t, t * s \in I$. Since I is a hyper $\mathcal{I}$-ideal of a strongly distributive hyper IS-algebra X, $y(t * s) = yt * ys \subseteq I$ for every $y \in X$, and $(y * x)s = ys * xs \subseteq I$ for every $s \in X$. To show that $yt * xs \subseteq I$, let $a \in yt$, $b \in xs$ and take any $p \in ys$. Then $(a * b) * (p * b) \leq a * p \in yt * ys \subseteq I$ and hence $(a * b) * (p * b) \in I$. Since $p * b \in ys * xs \subseteq I$ and I is an $\mathcal{I}$-ideal of $I(X)$, we have $a * b \in I$. Thus $yt * xs \subseteq I$. Similarly, we also have $xs * yt \subseteq I$. It follows that $C_{x\,s} = C_{y\,t}$. Thus $C_x \cdot C_s = C_y \cdot C_t$. This shows that the operation "$\cdot$" is well-defined. Moreover, we have the following theorem.

THEOREM 3.3. *Let X be a strongly distributive hyper IS-algebra and let I be a hyper $\mathcal{I}$-ideal. Then $(X/I, *, \cdot, C_0)$ is a strongly distributive hyper IS-algebra, which is called the quotient hyper IS-algebra via I, and $C_0 = I$.*

PROOF. First, we can show that the operation "$\cdot$" is associative, i.e., $(C_x \cdot C_y) \cdot C_z = C_x \cdot (C_y \cdot C_z)$, for any $x, y, z \in X$.

Next, we prove that " $\cdot$ " is distributive on both sides over the operation "$*$". For any $C_x, C_y, C_z \in X$, let $C_a \in (C_x * C_y) \cdot C_z = C_{x*y} \cdot C_z$. Then $a \in (x * y) \cdot z = (x \cdot z) * (y \cdot z)$. Hence there exist $p \in x \cdot z$ and $q \in y \cdot z$ such that $a = p * q$. Since $C_p \in \{C_s \mid s \in x \cdot z\} = C_{x \cdot z} = C_x \cdot C_z$ and $C_q \in \{C_t \mid t \in y \cdot z\} = C_{y \cdot z} = C_y \cdot C_z$, $C_a = C_{p*q} = C_p * C_q \in C_x \cdot C_z * C_y \cdot C_z$. Thus $C_a \in C_x \cdot C_z * C_y \cdot C_z$. This proves that $(C_x * C_y) \cdot C_z \subseteq C_x \cdot C_z * C_y \cdot C_z$.

To show that the reverse inclusion, let $C_a * C_b \in C_x \cdot C_z * C_y \cdot C_z$. Then $C_a \in C_x \cdot C_z$ and $C_b \in C_y \cdot C_z$. Hence $a \in x \cdot z$ and $b \in y \cdot z$. Since $a * b \in (x \cdot z) * (y \cdot z) = (x * y) \cdot z$, $C_a * C_b = C_{a*b} \in C_{(x*y) \cdot z} = C_{x*y} \cdot C_z = (C_x * C_y) \cdot C_z$. Thus $(C_x * C_y) \cdot C_z = C_x \cdot C_z * C_y \cdot C_z$. By a similar method, we obtain $C_x \cdot (C_y * C_z) = C_x \cdot C_y * C_x \cdot C_z$. Thus X/I is a strongly distributive hyper IS-algebra, completing the proof.

LEMMA 3.4. *Let I be a hyper $\mathcal{I}$-ideal of a strongly distributive hyper IS-algebra X. Then for any element $C_a \in X/I$, we have $|C_a \cdot C_0| = 1$.*

PROOF. Let C_a be an element of X/I. Then $C_a \cdot C_0 = C_{a \cdot 0} = \{C_x | x \in a \cdot 0\}$. Since I is a hyper $\mathcal{I}$-ideal of X and $0 \in X$, we have $a \cdot 0 \subseteq I$. Hence $C_a \cdot C_0$ contains just the zero element of X/I. This completes the proof.

THEOREM 3.5. *Let $(X, *, \cdot, 0)$ be a strongly distributive hyper IS-algebra and let I be a hyper $\mathcal{I}$-ideal of X. Then the quotient hyper IS-algebra X/I is actually an IS-algebra.*

PROOF. From Theorem 3.3, we know that $(X/I, *, \cdot, C_0)$ is strongly distributive. In view of Lemma 3.4 and Theorem 2.4, X/I is an IS-algebra.

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