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SOME FIXED POINTS FOR EXPANSIVE MAPPINGS AND FAMILIES OF MAPPINGS

Z. LIU, C. FENG, S. M. KANG AND Y. S. KIM

ABSTRACT. In this paper we obtain some fixed points theorems of expansive mappings and several necessary and sufficient conditions for the existence of common fixed points of families of self-mappings in metric spaces. Our results generalize and improve the main results of Fisher [1]-[5], Furi-Vignoli [6], Iséki [7], Jungck [8], [9], Kashara-Rhoades [10], Liu [13], [14] and Sharma and Strivastava [16].

1. Introduction

Jungck [9] proved a fixed point theorem of expansive mappings which satisfy

$$d(fx, gy) > d(hx, hy) \tag{1.1}$$

for some $h \in C_f \cap C_g$. Liu [13] extended this result to a more general case. On the other hand, Jungck [8] first gave a necessary and sufficient condition for the existence of fixed points of a continuous self-mapping of complete metric spaces. Park [15] and Khan-Fisher [11] established a few results similar to that of Jungck [8]. In 1969 Furi-Vignoli [6] proved several fixed point theorems for densifying mappings. Afterwards, Iséki [7], Sharma and Strivastava [16] also established fixed point theorems for densifying mappings respectively. Fisher [1], [2] gave a few fixed point theorems for compact mappings.

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The purpose of this paper is to give some fixed point theorems for expansive mappings and establish some characterizations for the existence of common fixed points of families of self-mappings in metric spaces by using compact mappings and densifying mappings, respectively. Our results generalize and improve the results of Fisher [1]-[5], Furi-Vignoli [6], Iséki [7], Jungck [8], [9], Kashara-Rhoades [10], Liu [13], [14] and Sharma-Strivastava [16].

Throughout this paper N denotes the set of positive integers and I denotes the identity mapping in X. Let (X, d) be a metric space and f a self-mapping of X. $\Im$ stands for a family of self-mappings of X. We need the following known definitions and result. The mapping f is said to be *compact* if there exists a compact subset $Y \subseteq X$ such that $fX \subseteq Y$. The mapping f is said to be *densifying* if for every bounded subset A of X with $\alpha(A) > 0$, we have $\alpha(fA) < \alpha(A)$, where $\alpha(A)$ denotes the measure of noncompactness in the sense of Kuratowski. Define

$C_f = \{h \mid h : X \to X \text{ and } fh = hf\}$,
$H_f = \{t \mid t : X \to X \text{ and } t(\cap_{n\in N} f^n X) \subseteq \cap_{n\in N} f^n X\}$,
$C_\Im = \{f \mid f : X \to X \text{ and } fp = pf \text{ for all } p \in \Im\}$,
$O(x, f) = \{f^n x \mid n \in N\}$ for $x \in X$ and
$O(x, y, f) = O(x, f) \cup O(y, f)$ for $x, y \in X$.
For $A, B \subseteq X$, $\delta(A)$ denotes the diameter of A and

$$\delta(A, B) = \sup\{d(x, y) \mid x \in A, y \in B\}.$$

A point $x \in X$ is said to be a *fixed point* of $\Im$ if $fx = x$ for all $f \in \Im$.

LEMMA 1.1. [12] *Let f and g be commuting self-maps of a compact metric space (X, d) such that gf is continuous. If $A = \cap_{n=1}^{\infty}(gf)^n X$, then*

(i) $hA \subseteq A$ *for* $h \in C_{gf}$;
(ii) $fA = gA = A \neq \emptyset$ *and*
(iii) A *is compact.*

2. Main Results

THEOREM 2.1. *Let f and g be continuous commuting self-mappings of a metric space (X,d) and there exist $k,s \in N$ such that f^k and g^s are compact. Suppose that*

$$\begin{aligned} d(fx,gy) > \inf\{&d(x,y), d(f^{n+1}x, f^n x), d(f^{n+1}y, f^n y), \\ &d(g^{n+1}x, g^n x), d(g^{n+1}y, g^n y), d(hx,hy) \\ &: n \geq 0,\ h \in C_f \cap C_g\} \end{aligned} \tag{2.1}$$

for any $x,y \in X$ with $fx \neq gy$. Then at least one of f and g has a fixed point in X.

PROOF. Since f^k and g^s are compact, it follows that $g^s f^k$ is also compact. Therefore there exists a compact subset Y of X such that $g^s f^k X \subseteq Y$. Let $A = \cap_{n\in N}(g^s f^k)^n X$, $B = \cap_{n \in N}(g^s f^k)^n Y$. It is easy to see that $A = B$. From Lemma 1.1 we conclude that $hB \subseteq B$ for $h \in C_{g^s f^k}$, B is compact and $fB = gB = B \neq \emptyset$. Since f and g are continuous and A is compact, there exist $a, b \in A$ such that

$$d(fa,a) \leq d(fx,x) \quad \text{and} \quad d(gb,b) \leq d(gx,x) \tag{2.2}$$

for any $x \in A$. Without loss of generality we can assume that

$$d(fa,a) \leq d(gb,b). \tag{2.3}$$

Since $gA = A$, there exists some $z \in A$ such that $gz = a$. Suppose that $fa \neq a$, that is, $fg^s z \neq g^s z$. From (2.1)-(2.3) we obtain that

$$\begin{aligned} d(fa,a) &= d(gfz,gz) \\ &> \inf\{d(gz,z), d(f^{n+1}gz, f^n gz), d(f^{n+1}z, f^n z), \\ &\quad d(g^{n+1}gz, g^n gz), d(g^{n+1}z, g^n z), d(hgz,hz) \\ &\quad : n \geq 0,\ h \in C_f \cap C_g\} \\ &= \inf\{d(gz,z), d(f^{n+1}a, f^n a), d(f^{n+1}z, f^n z), d(g^{n+1}a, g^n a), \\ &\quad d(g^{n+1}z, g^n z), d(ghz,hz) : n \geq 0,\ h \in C_f \cap C_g\} \\ &\geq d(fa,a), \end{aligned}$$

which is impossible. Hence $fa = a$ and this completes the proof.

As immediate consequences of Theorem 2.1, we have the following corollaries.

COROLLARY 2.1. *Let f and g be continuous commuting self-mappings of a compact metric space (X, d). If $fx \neq gy$ implies there exists some $u \in C_f \cap C_g$ such that*

$$d(fx, gy) > \inf\{d(x,y), d(f^{n+1}x, f^n x), d(f^{n+1}y, f^n y), \quad d(g^{n+1}x, g^n x), d(g^{n+1}y, g^n y), d(ux, uy) : n \geq 0\}. \tag{2.4}$$

Then at least one of f and g has a fixed point in X.

COROLLARY 2.2. *Let f and g be continuous commuting self-mappings of a compact metric space (X, d) satisfying*

$$d(fx, gy) > \inf\{d(x,y), d(fx,x), d(fy,y), d(gx,x), \quad d(gy,y), d(hx,hy) : h \in C_f \cap C_g\} \tag{2.5}$$

for any $x, y \in X$ with $fx \neq gy$. Then at least one of f and g has a fixed point in X.

REMARK 2.1. Corollaries 2.1 and 2.2 generalize Theorem 4.4 of Jungck [9]. The following example reveals that Corollaries 2.1 and 2.2 are more general than the result of Jungck [9]. We can obtain that not both f and g of Corollaries 2.1 and 2.2 have a fixed point and the fixed point may be nonunique.

EXAMPLE 2.1. Let $X = \{2, 3, 8\}$ with the usual metric d. Define $f, g : X \to X$ by $f2 = 3$, $f3 = 8$, $f8 = 2$ and $g = I$. Clearly g has three fixed points while f has none. It is easy to see that f and g are continuous commuting self-mappingss of the compact metric space (X, d) and $C_f \cap C_g = C_f = \{f, f^2, I\}$. Given $x, y \in X$ with $fx \neq gy$, we have

$$\begin{aligned} d(fx, gy) > 0 = d(gy, y) \\ &= \inf\{d(ux, uy), d(x,y), d(f^{n+1}x, f^n x), d(f^{n+1}y, f^n y), \\ &\quad d(g^{n+1}x, g^n x), d(g^{n+1}y, g^n y) : n \geq 0\} \end{aligned}$$

for any $u \in C_f \cap C_g$. We also have

$$\begin{aligned} d(fx, gy) > 0 = d(gy, y) \\ &= \inf\{d(x,y), d(fx,x), d(fy,y), d(gx,x), \\ &\quad d(gy,y), d(hx,hy) : h \in C_f \cap C_g\}. \end{aligned}$$

Hence the conditions of Corollaries 2.1 and 2.2 hold, but Theorem 4.4 of Jungck [9] is not applicable since

$$d(f8, g3) = d(2, 3) = 1 \leq d(h8, h3), \quad h \in C_f \cap C_g.$$

REMARK 2 2. Taking $n = 0$ in Corollary 2.1, we get Theorem 1 of Liu [13].

Next we give some common fixed point theorems of families of self-mappings in a metric space.

THEOREM 2.2. *Let $\Im$ and $\Re$ be families of self-mappings of a bounded metric space (X, d). Then $\Im$ and $\Re$ have a common fixed point in X if and only if there exist continuous compact self-mappings $f \in C_\Im$, $g \in C_\Re$ and $r, s \in N$ such that*

$$d(f^r x, g^s y) < \delta(\cup_{h \in H_f} O(x, y, h), \ \cup_{t \in H_g} O(x, y, t)) \tag{2.6}$$

for all $x, y \in X$ for which the right-hand side of (2.6) *is positive.*

PROOF. Suppose that $\Im$ and $\Re$ have a common fixed point $w \in X$. Define mappings $f, g : X \to X$ by $fx = gx = w$ for all $x \in X$. Then $fpx = w = pw = pfx$ for all $x \in X$ and $p \in \Im$. That is $f \in C_\Im$. Similarly we have $g \in C_\Re$. Clearly (2.6) holds.

Conversely, suppose that there exist continuous compact self-mappings $f \in C_\Im$, $g \in C_\Re$ and $r, s \in N$ satisfying (2.6). Since f is compact, there exists a compact subset $Y \subseteq X$ such that $fX \subseteq Y$. Set $A = \cap_{n \in N} f^n X$, $B = \cap_{n \in N} f^n Y$. Then $A = B$. As in the proof of Theorem 2.1, we conclude that B is a nonempty compact set and $fB \subseteq \cap_{n \in N} f^{n+1} Y = B$. We next show that $fB \supseteq B$. Given $b \in B = \cap_{n \in N} f^n Y$, there exists $x_n \in f^n Y$ with $b = f x_n$ for $n \in N$. Because $\{x_n\}_{n \in N} \subset Y$, we can extract a subsequence $\{x_{n_i}\}_{i \in N}$ converging to $p \in Y$. For every $m \in N$, there exists $i > m$ such that $\{x_{n_i}, x_{n_i+1}, x_{n_i+2}, \cdots\} \subset f^n Y$ By the compactness of $f^n Y$, we have $x_{n_i} \to p \in f^n Y$ as $i \to \infty$. Therefore $p \in \cap_{n \in N} f^n Y = B$. Since f is continuous, we know that $b = x_{n_i} \to fp \in f^n Y$ as $i \to \infty$. That is $b = fp \in fB$. This proves $fB \supseteq B$. Hence $fB = B$. That is $fA = A$

Similarly we obtain that $D = \cap_{n\in N} g^n X$ and $gD = D$ and A, D are nonempty compact sets. By the compactness of $A \times D$, there exist $u \in A$ and $v \in D$ such that $d(u,v) = \delta(A, D)$. From $f^r A = A$ and $g^s D = D$, we infer that there exist $a \in A$ and $b \in D$ such that $u = f^r a$ and $v = g^s b$, respectively. We assert that $u = v$. If not

$$\begin{aligned} d(u,v) &= d(f^r a, g^s b) \\ &< \delta(\cup_{n\in H_f} O(a,b,h),\ \cup_{t\in H_g} O(a,b,t)) \\ &\leq \delta(A, D) = d(u,v), \end{aligned}$$

which is a contradiction. Hence $u = v$. That is $A = B = \{w\}$ for some $w \in X$. Obviously w is a common fixed point of f and g. If z is another common fixed point of f and g, then $z \in \cap_{n\in N} f^n X = \{w\}$. That is $z = w$. Therefore w is the only fixed point of f. Note that $f \in C_{\Im}$ and $g \in C_{\Re}$. Thus we have

$$fpw = pfw = pw, \quad p \in \Im \quad \text{and} \quad gqw = qgw = qw, \quad q \in \Re.$$

Since f and g have a unique fixed point w, we have $pw = w = qw$ for $p \in \Im$ and $q \in \Re$. That is w is a common fixed point of $\Im$ and $\Re$. This completes the proof.

REMARK 2.3. Taking $\Im = \Re = \{f\}$ in Theorem 2.2, we obtain the result which improves Theorem of Jungck [8].

COROLLARY 2.3. *Let $\Im$ and $\Re$ be families of self-mappings of a bounded metric space (X,d). If there exist continuous compact self-mappings $f \in C_{\Im}$, $g \in C_{\Re}$ and $r, s \in N$ such that* (2.6) *holds for every $x, y \in X$ with $f^r x \neq g^s y$, then $\Im$ and $\Re$ have a unique common fixed point.*

REMARK 2.4. Taking $\Im = \{f\}$ and $\Re = \{g\}$ or $\Im = \Re = \{f\}$, Corollary 2.3 reduce the results which extend Theorems of Fisher [1]-[5], Theorems of Kashara and Rhoades [10], Corollary 3 of Leader [12], Theorem 2.5 and Corollary 2.6 of Liu [14].

THEOREM 2.3. *Let $\Im$ and $\Re$ be families of self-mappings of a metric space (X,d). Then the following statements are equivalent.*

(i) *$\Im$ and $\Re$ have a common fixed point;*

(ii) *There exist $x_0 \in X$, $m,n \in N$ and continuous densifying mappings $f \in C_\Im$ and $g \in C_\Re$ such that $O(x_0,f)$ and $O(x_0,g)$ are bounded and*

$$d(f^m x, g^n y) < \delta(O(x,f),O(y,g)) \tag{2.7}$$

for all $x,y \in X$ for which the right-hand side of (2.7) is positive;

(iii) *There exist $x_0, y_0 \in X$, $m,n \in N$ and continuous commuting densifying mappings $f \in C_\Im$ and $g \in C_\Re$ such that $\cup_{i,j\geq 0} O(x_0,y_0,f^i g^j)$ are bounded and*

$$d(f^m x, g^n y) < \delta(\cup_{i,j\geq 0} O(x,y,f^i g^j)) \tag{2.8}$$

for all $x,y \in X$ for which the right-hand side of (2.8) is positive.

PROOF. Let (i) hold and w be a common fixed point of $\Im$ and $\Re$. Define $f,g : X \to X$ by $fx = gx = w$ for all $x \in X$. It is easy to show that $f \in C_\Im$, $g \in C_\Re$, (2.7) and (2 8) hold.

Assume that (ii) holds. Set $B = O(x_0,f)$, $C = O(x_0,g)$ and $A = \cap_{n\in N} f^n \bar{B}$ and $D = \cap_{n\in N} g^n \bar{C}$. Since f is densifying and

$$\alpha(B) = \max\{\alpha\{x\}, \alpha(fB)\} = \alpha(fB),$$

we have $\alpha(B) = 0$, which implies that B is precompact. Since X is complete, $\bar{B}$ is compact. By the continuity of f we get $f\bar{B} \subseteq \overline{fB} \subseteq \bar{B}$. As in the proof of Theorem 2.2, we can conclude that A is a nonempty compact subset of X and $fA = A$. Similarly we have $gD = D$. Now we claim that $\delta(A,D) = 0$. Otherwise $\delta(A,D) > 0$. From the compactness of A and D, there exist $u \in A$, $v \in D$ such that $d(u,v) = \delta(A,D)$. Since $f^m A = A$ and $g^n D = D$, there exist $a \in A$ and $b \in D$ such that $u = f^m a$ and $v = g^n b$, respectively. Using (2.7), we infer that

$$\begin{aligned} 0 < d(u,v) &= d(f^m a, g^n b) \\ &< \delta(O(a,f),O(b,g)) \leq \delta(A,D) = d(u,v), \end{aligned}$$

which is a contradiction and hence $\delta(A, D) = 0$. So $A = D = \{w\}$ for some $w \in X$. Clearly w is a common fixed point of f and g. Suppose that f and g have another common fixed point k different from w.Then by (2.7) we have

$$d(k, w) = d(f^m k, g^n w) < \delta(O(k, f), O(w, g)) = d(k, w),$$

which is impossible. Consequently w is a unique common fixed point of f and g. It is easy to check that w is a common fixed point of $\Im$ and $\Re$. That is, (i) holds.

Assume that (iii) holds. Put $A = \cup_{i,j \geq 0} O(x_0, f^i g^j)$. Then $A = \{x_0\} \cup fA \cup gA$. Since f and g are densifying and

$$\alpha(A) = \max\{\alpha\{x_0\}, \alpha(fA), \alpha(gA)\},$$

we have $\alpha(A) = 0$. Thus $\bar{A}$ is compact, $f\bar{A} \subseteq \bar{A}$ and $g\bar{A} \subseteq \bar{A}$. Put $B = \cap_{n \in N} (fg)^n \bar{A}$. It is easy to verify that $fB = gB = B$ and B is compact. We now claim that B is a singleton. If not, then $\delta(B) > 0$. Since B is compact, there exist $a, b \in B$ such that $d(a, b) = \delta(B)$. In view of $B = f^m B = g^n B$, there exist $x, y \in B$ such that $a = f^m x$ and $b = g^n y$. By (2.8) we have

$$\delta(B) = d(f^m x, g^n y) < \delta(\cup_{i,j \geq 0} O(x, y, f^i g^j)) \leq \delta(B),$$

which is impossible and hence $\delta(B) = 0$. That is, $B = \{w\}$ for some $w \in X$. Obviously w is a common fixed point of f and g. Suppose that f and g have another common fixed point v different from w. Using (2.8) again we get that

$$d(v, w) = d(f^m v, g^n w) < \delta(\cup_{i,j \geq 0} O(v, w, f^i g^j)) = d(v, w),$$

which is a contradiction and therefore f and g have a unique fixed point w. Since $f \in C_{\Im}$ and $g \in C_{\Re}$, it follows that w is a common fixed point of $\Im$ and $\Re$. That is (i) holds. This completes the proof.

COROLLARY 2.4. *Let $\Im$ and $\Re$ be famılies of self-mappings of a bounded metrıc space (X,d). If there exist continuous densıfying self-mappings $f \in C_{\Im}$ and $g \in C_{\Re}$ such that (2.7) or (2.8) holds for every $x,y \in X$ with $f^m x \neq g^n y$, then $\Im$ and $\Re$ have a unique common fixed point.*

REMARK 2.5. By taking $\Im = \{f\}$ and $\Re = \{g\}$ or $\Im = \Re = \{f\}$ in Corollary 2.4, we get the results which improve the theorems of Sharma and Strivastava [16], Furi and Vignoli [6] and Iséki [7].

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Zeqing Liu and Cui Feng
Department of Mathematics
Liaoning Normal University
P. O. Box 200, Dalian, Liaoning 116029
People's Republic of China
E-mail: zeqingliu@sina.com.cn

Shin Min Kang and Yong Soo Kim
Department of Mathematics
Gyeongsang National University
Chinju 660-701, Korea
E-mail: smkang@nongae.gsnu.ac.kr