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SOME FAMILIES OF INFINITE SERIES SUMMABLE VIA FRACTIONAL CALCULUS OPERATORS

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ABSTRACT. Many different families of infinite series were recently observed to be summable in closed forms by means of certain operators of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order). In this sequel to some of these recent investigations, the authors present yet another instance of applications of certain fractional calculus operators Alternative derivations without using these fractional calculus operators are shown to lead naturally a family of analogous infinite sums involving hypergeometric functions

1. Introduction and Definitions

The subject of *fractional calculus* (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable importance and popularity during the past three decades or so, due mainly to its demonstrated applications in many seemingly diverse fields of science and engineering (see, for details, [5], [7], and [16]). Various operators of fractional calculus are indeed found to be useful in

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such areas of mathematical analysis as (for example) ordinary and partial differential equations, integral equations, summation of series, *et cetera.*

One of the most frequently encountered tools in the theory and applications of fractional calculus is furnished by the Riemann-Liouville *fractional differintegral* (that is, *fractional derivative* and *fractional integral*) operator D_z^μ of order μ, defined by (*cf.*, *e.g.*, [15], [16], and [19])

$$D_z^\mu\{f(z)\} := \begin{cases} \frac{1}{\Gamma(-\mu)}\int_0^z (z-\zeta)^{-\mu-1} f(\zeta)d\zeta & (\mathfrak{R}(\mu) < 0) \\ \frac{d^m}{dz^m} D_z^{\mu-m}\{f(z)\} & (m-1 \leqq \mathfrak{R}(\mu) < m; m \in \mathbb{N}), \end{cases} \tag{1.1}$$

provided that the integral in (1.1) exists, $\mathbb{N}$ being (as usual) the set of *positive* integers. In many recent works (see, for example, [10], [11], [12], [22], and [23]), dealing with the summation of series by means of fractional calculus, an essentially equivalent differintegral operator $\mathcal{N}_z^\nu$ ($\nu \in \mathbb{R}$) was employed fairly successfully. We choose first to recall here the definition of this fractional differintegral operator $\mathcal{N}_z^\nu$ as follows:

DEFINITION (*cf.* [8], [9], and [21]). If the function $f(z)$ is analytic (regular) inside and on $\mathcal{C}$, where

$$\mathcal{C} := \{\mathcal{C}^-, \mathcal{C}^+\},$$

$\mathcal{C}^-$ is a contour along the cut joining the points z and $-\infty + i\mathfrak{I}(z)$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point at $-\infty$, $\mathcal{C}^+$ is a contour along the cut joining the points z and $\infty + i\mathfrak{I}(z)$, which starts from the point at ∞, encircles the point z once counter-clockwise, and returns to the point at ∞,

$$\mathcal{N}_z^\nu\{f(z)\} := \frac{\Gamma(\nu+1)}{2\pi i}\int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta-z)^{\nu+1}}\, d\zeta$$

$$(\nu \in \mathbb{R}\backslash\mathbb{Z}^-;\ \mathbb{Z}^- := \{-1,-2,-3,\dots\}) \tag{1.2}$$

and

$$\mathcal{N}_z^{-n}\{f(z)\} := \lim_{\nu\to -n}(\mathcal{N}_z^\nu\{f(z)\}) \quad (n \in \mathbb{N}), \tag{1.3}$$

where $\zeta \neq z$,

$$-\pi \leqq \arg(\zeta - z) \leqq \pi \quad \text{for} \quad \mathcal{C}^-, \tag{1.4}$$

and

$$0 \leqq \arg(\zeta - z) \leqq 2\pi \quad \text{for} \quad \mathcal{C}^+, \tag{1.5}$$

then $\mathcal{N}_z^\nu\{f(z)\}$ $(\nu > 0)$ is said to be the *fractional derivative of $f(z)$ of order ν* and $\mathcal{N}_z^\nu\{f(z)\}$ $(\nu < 0)$ is said to be the *fractional integral of $f(z)$ of order $-\nu$*, provided that

$$|\mathcal{N}_z^\nu\{f(z)\}| < \infty \quad (\nu \in \mathbb{R}). \tag{1.6}$$

For the sake of completeness and ready reference, we find it to be worthwhile also to recall here each of the following potentially useful lemmas and properties associated with the fractional differintegral operator $\mathcal{N}_z^\nu$ which is defined above (see, for details, [8] and [9]).

LEMMA 1 (Linearity Property). *If the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$\mathcal{N}_z^\nu\{k_1 f(z) + k_2 g(z)\} = k_1 \mathcal{N}_z^\nu\{f(z)\} + k_2 \mathcal{N}_z^\nu\{g(z)\} \quad (\nu \in \mathbb{R}; z \in \Omega) \tag{1.7}$$

for any constants k_1 and k_2.

LEMMA 2 (Index Law). *If the function $f(z)$ is single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$\mathcal{N}_z^\nu(\mathcal{N}_z^\mu\{f(z)\}) = \mathcal{N}_z^{\mu+\nu}\{f(z)\} = \mathcal{N}_z^\mu(\mathcal{N}_z^\nu\{f(z)\}) \tag{1.8}$$

$$(\mathcal{N}_z^\mu\{f(z)\} \neq 0; \quad \mathcal{N}_z^\nu\{f(z)\} \neq 0; \quad \mu, \nu \in \mathbb{R}; \quad z \in \Omega).$$

LEMMA 3 (Generalized Leibniz Rule). *If the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$\mathcal{N}_z^\nu\{f(z) \cdot g(z)\} = \sum_{n=0}^{\infty} \binom{\nu}{n} \mathcal{N}_z^{\nu-n}\{f(z)\} \cdot g^{(n)}(z) \quad (\nu \in \mathbb{R}; \ z \in \Omega), \tag{1.9}$$

where $g^{(n)}(z)$ *is the ordinary derivative of* $g(z)$ *of order* n $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$, *it being tacitly assumed (for simplicity) that* $g(z)$ *is the polynomial part (if any) of the product* $f(z) \cdot g(z)$.

PROPERTY 1. *For constants* c, λ, *and* ν,

$$\mathcal{N}_z^{\nu}\left\{(z-c)^{\lambda}\right\} = e^{-i\pi\nu}\,\frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)}\,(z-c)^{\lambda-\nu} \tag{1.10}$$

$$\left(\nu \in \mathbb{R};\;\; c, z \in \mathbb{C};\;\; c \neq z;\;\; |\Gamma(\nu-\lambda)/\Gamma(-\lambda)| < \infty\right).$$

PROPERTY 2. *For constants* c *and* ν,

$$\mathcal{N}_z^{-\nu}\left\{(z-c)^{-\nu}\right\} = -\frac{e^{i\pi\nu}}{\Gamma(\nu)}\log(z-c) \tag{1.11}$$

$$\left(\nu \in \mathbb{R};\;\; c, z \in \mathbb{C};\;\; c \neq z;\;\; |\Gamma(\nu)| < \infty\right).$$

PROPERTY 3. *For constants* c *and* ν,

$$\mathcal{N}_z^{\nu}\left\{\log(z-c)\right\} = -e^{-i\pi\nu}\,\Gamma(\nu)\,(z-c)^{-\nu} \tag{1.12}$$

$$\left(\nu \in \mathbb{R};\;\; c, z \in \mathbb{C};\;\; c \neq z;\;\; |\Gamma(\nu)| < \infty\right).$$

The main object of this sequel to the aforementioned recent works is to present yet another instance of applications of the fractional differintegral operator $\mathcal{N}_z^{\nu}$. We also show how such infinite sums can be extended naturally to a family of analogous infinite sums involving the generalized hypergeometric ${}_pF_q$ function with p numerator and q denominator parameters, defined by (*cf.* [3, Chapter 4])

$$\begin{aligned} {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) &= {}_pF_q\left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix}\; z\right] \\ &:= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k}\,\frac{z^k}{k!} \end{aligned} \tag{1.13}$$

$$(p, q \in \mathbb{N}_0;\ p \leqq q+1;\ p \leqq q \text{ and } |z| < \infty;$$
$$p = q+1 \text{ and } |z| < 1;\ p = q+1,\ |z| = 1, \text{ and } \Re(\omega) > 0),$$

where (*and in what follows*) $(\lambda)_k$ denotes the Pochhammer symbol (or the *shifted* factorial, since $(1)_k = k!$ $(k \in \mathbb{N}_0)$) given by

$$(\lambda)_k := \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k=0; \lambda \neq 0) \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k \in \mathbb{N}) \end{cases} \tag{1.14}$$

and

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \quad (\beta_j \notin \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}; j = 1, \ldots, q). \tag{1.15}$$

Just as in some other earlier works (*cf.*, *e.g.*, [1], [2], [6], [13], [14], and [18]), we also indicate alternative derivations of these families of infinite sums *without* using fractional calculus operators.

2. Applications of the Fractional Differintegral Operator

First of all, in view of the familiar expansion formula:

$$\log(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k \quad (|z| < 1), \tag{2.1}$$

we readily obtain

$$\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{a-c}{z-c}\right)^k = \log\left(\frac{z-c}{z-a}\right) \tag{2.2}$$

$$(z, c, a \in \mathbb{C};\ z \neq c;\ z \neq a;\ |(a-c)/(z-c)| < 1).$$

Now we multiply both sides of (2.2) by $(z-b)^m$ $(m \in \mathbb{N}_0)$ and operate upon each member of the resulting equation by the fractional differintegral operator $\mathcal{N}_z^\nu$. Making use of the generalized Leibniz rule (1.9), we thus find that

$$\sum_{k=1}^{\infty} \frac{(a-c)^k}{k} \sum_{l=0}^{m} \binom{\nu}{l} \mathcal{N}_z^{\nu-l} \left\{(z-c)^{-k}\right\} \mathcal{N}_z^l \left\{(z-b)^m\right\}$$

$$= \sum_{l=0}^{m} \binom{\nu}{l} \mathcal{N}_z^{\nu-l} \left\{ \log \left(\frac{z-c}{z-a} \right) \right\} \mathcal{N}_z^{l} \left\{ (z-b)^m \right\}, \tag{2.3}$$

which, upon applying the fractional differintegral formulas (1.10) and (1.12), yields our first result in the form:

$$\sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{a-c}{z-c} \right)^k \sum_{l=0}^{m} \binom{m}{l} \frac{(\nu-l)_k}{\nu-l} \left(-\frac{z-c}{z-b} \right)^l$$

$$= \sum_{l=0}^{m} \binom{m}{l} \frac{1}{\nu-l} \left(-\frac{z-c}{z-b} \right)^l \left[\left(\frac{z-c}{z-a} \right)^{\nu-l} - 1 \right] \tag{2.4}$$

$$\left(m \in \mathbb{N}_0;\ z \neq a;\ z \neq b;\ z \neq c;\ \left| (a-c)/(z-c) \right| < 1;\ \nu \notin \mathbb{N}_0 \right).$$

Next, since [3, p. 102, Equation 2.8 (15)]

$$\log(1+z) = z\ {}_2F_1(1,1;2;-z) \tag{2.5}$$

in terms of the Gauss hypergeometric function defined by (1.13) *with*, of course,

$$p - 1 = q = 1,$$

we consider the following immediate consequence of the definition (1.13):

$$\sum_{k=1}^{\infty} \frac{(\alpha_1)_{k-1} \cdots (\alpha_p)_{k-1}}{(k-1)!\,(\beta_1)_{k-1} \cdots (\beta_q)_{k-1}} \left(\frac{a-c}{z-c} \right)^{\lambda+k}$$

$$= \left(\frac{a-c}{z-c} \right)^{\lambda+1} {}_pF_q \left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} \frac{a-c}{z-c} \right] \quad (\lambda \in \mathbb{C}), \tag{2.6}$$

which, in the special case when

$$p - 1 = q = 1 \quad (\alpha_1 = \alpha_2 = 1;\ \beta_1 = 2) \quad \text{and} \quad \lambda = 0, \tag{2.7}$$

reduces at once to the expansion formula (2.2).

By appealing to the fractional differintegral formula (1.10) once again, it is fairly easy to show, for constants c, λ, and μ, that

$$\mathcal{N}_z^\nu \left\{ (z-c)^\lambda \, {}_pF_q \left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} \mu (z-c) \right] \right\}$$

$$= e^{-i\pi\nu} \frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)} (z-c)^{\lambda-\nu} \, {}_{p+1}F_{q+1} \left[\begin{matrix} \lambda+1, \alpha_1, \ldots, \alpha_p; \\ \lambda-\nu+1, \beta_1, \ldots, \beta_q; \end{matrix} \mu(z-c) \right] \tag{2.8}$$

$$\left(\nu \in \mathbb{R},\ \mu, c, z \in \mathbb{C};\ c \neq z;\ |\Gamma(\nu-\lambda)/\Gamma(-\lambda)| < \infty;\ \lambda - \nu \notin \mathbb{Z}^-\right)$$

and

$$\mathcal{N}_z^\nu \left\{ (z-c)^\lambda \, {}_pF_q \left[\begin{matrix} \alpha_1, \ldots, \alpha_p, \\ \beta_1, \ldots, \beta_q; \end{matrix} \frac{\mu}{z-c} \right] \right\}$$

$$= e^{-i\pi\nu} \frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)} (z-c)^{\lambda-\nu} \, {}_{p+1}F_{q+1} \left[\begin{matrix} \nu-\lambda, \alpha_1, \ldots, \alpha_p; \\ -\lambda, \beta_1, \ldots, \beta_q; \end{matrix} \frac{\mu}{z-c} \right] \tag{2.9}$$

$$(\nu \in \mathbb{R};\ \mu, c, z \in \mathbb{C};\ c \neq z;\ |\Gamma(\nu-\lambda)/\Gamma(-\lambda)| < \infty;\ \lambda \notin \mathbb{N}_0).$$

The above-detailed method of derivation of the summation formula (2.4) *via* the fractional differintegral operator $\mathcal{N}_z^\nu$ can be applied *mutatis mutandis*, using (2.6) and (2.9) instead of (2.2) and (1.12), respectively, in order to obtain the following generalization of the summation formula (2.4):

$$\sum_{k=1}^{\infty} \frac{(\alpha_1)_{k-1} \cdots (\alpha_p)_{k-1}}{(k-1)!\,(\lambda+1)_{k-1} (\beta_1)_{k-1} \cdots (\beta_q)_{k-1}} \left(\frac{a-c}{z-c}\right)^k$$

$$\cdot \sum_{l=0}^{m} \binom{m}{l} \frac{(\nu-l)_{\lambda+k}}{\nu-l} \left(-\frac{z-c}{z-b}\right)^l$$

$$= \frac{a-c}{z-c} \sum_{l=0}^{m} \binom{m}{l} (\nu-l+1)_\lambda \left(-\frac{z-c}{z-b}\right)^l$$

$$
{}_{p+1}F_{q+1}\left[\begin{array}{r} \nu+\lambda-l+1,\alpha_1,\ldots,\alpha_p;\, \dfrac{a-c}{z-c} \\ \lambda+1,\beta_1,\ldots,\beta_q; \end{array}\right] \tag{2.10}
$$

$\left(m \in \mathbb{N}_0;\ z \neq a;\ z \neq b;\ z \neq c;\ |(a-c)/(z-c)| < 1;\ \nu \notin \mathbb{N}_0;\ \lambda \notin \mathbb{Z}^-\right).$

In its special case when the constraints in (2.7) are satisfied, the summation formula (2.10) would reduce to (2.4), since [17, p. 462, Entry 7.3.1.125]

$$
{}_2F_1(1,b;2;z) = \frac{1}{(b-1)\,z}\left[(1-z)^{1-b}-1\right] \tag{2.11}
$$

$$
(b \neq 1;\ 0 < |z| < 1).
$$

On the other hand, by applying (2.11) as well as the Chu-Vandermonde theorem [17, p. 489, Entry 7.3.5.4]:

$$
{}_2F_1(-n,b;c;1) = \frac{(c-b)_n}{(c)_n} \quad \left(n \in \mathbb{N}_0;\ c \notin \mathbb{Z}_0^-\right), \tag{2.12}
$$

the *simpler* summation formula (2.4) *with*

$$
b = c \quad \text{and} \quad \nu = m+1 \quad (\text{and} \quad m \longmapsto n) \tag{2.13}
$$

yields the sum:

$$
\sum_{k=n+1}^{\infty} (k-1)\cdots(k-n)\left(\frac{a-c}{z-c}\right)^k = n!\left(\frac{a-c}{z-a}\right)^{n+1} \tag{2.14}
$$

$$
(n \in \mathbb{N}_0;\ z \neq c;\ z \neq a),
$$

which can indeed be proven *directly* by first letting $k \longmapsto k+n+1$ $(n, k \in \mathbb{N}_0)$ and then using the binomial expansion:

$$
{}_1F_0(\lambda;\text{—};z) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!}\, z^k = (1-z)^{-\lambda} \quad (\lambda \in \mathbb{C};\ |z| < 1). \tag{2.15}
$$

We choose to leave, as an exercise for the interested reader, each of the aforementioned derivations of (2 14) directly *and* as a special case of the summation formula (2.4) under the constraints given by (2.13).

3. Remarks and Observations

A closer examination of each of the summation formulas (2.4) and (2.10) would reveal the fact that these results can be derived alternatively (and more simply) *without* using the fractional differintegral operator $\mathcal{N}_z^\nu$. While the summation formula (2.4) is a rather straightforward consequence of the binomial expansion (2.15) in its *equivalent* form:

$$\sum_{k=1}^{\infty} \frac{(\lambda)_k}{k!}\, z^k = (1-z)^{-\lambda} - 1 \quad (\lambda \in \mathbb{C};\ |z| < 1), \tag{3.1}$$

the general result (2.10) is derivable *directly* from the definition (1.13). As a matter of fact, for suitably bounded single and double sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_{m,n}\}_{m,n=0}^{\infty}$ of essentially arbitrary real or complex parameters, we readily obtain

$$\sum_{k=1}^{\infty} A_{k-1} \left(\frac{a-c}{z-c}\right)^k \sum_{l=0}^{m} \binom{m}{l} \frac{(\nu-1)_{\lambda+k}}{\nu-1}\, B_{k,l} \left(-\frac{z-c}{z-b}\right)^l$$

$$= \frac{a-c}{z-c} \sum_{l=0}^{m} \binom{m}{l} (\nu-l+1)_\lambda \left(-\frac{z-c}{z-b}\right)^l$$

$$\cdot \sum_{k=0}^{\infty} (\nu+\lambda-l+1)_k\ A_k\ B_{k+1,l} \left(\frac{a-c}{z-c}\right)^k \tag{3.2}$$

$$(m \in \mathbb{N}_0;\ z \neq a;\ z \neq b;\ z \neq c;\ |(a-c)/(z-c)| < 1;\ \nu \notin \mathbb{N}_0),$$

provided that each member of (3.2) exists.

The summation formula (2.4) follows immediately from (3.2) when we set

$$A_k = \frac{1}{k+1} \quad (k \in \mathbb{N}_0), \quad B_{m,n} = 1 \quad (m, n \in \mathbb{N}_0), \quad \text{and} \quad \lambda = 0, \tag{3.3}$$

and apply the reduction formula (2.11) on the right-hand side of (3.2). In order to deduce the summation formula (2.10) as a special case of (3.2), we simply set

$$A_k = \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{k!\,(\lambda+1)_k\,(\beta_1)_k \cdots (\beta_q)_k} \quad (k \in \mathbb{N}_0) \text{ and } B_{m,n} = 1 \quad (m,n \in \mathbb{N}_0), \tag{3.4}$$

and appeal to the definition (1.13) on the right-hand side of (3.2).

Next we recall the following known reduction formula for a generalized hypergeometric function defined by (1.13) [17, p. 572, Entry 7.10.1.1]:

$$ {}_{r+1}F_r \left[\begin{matrix} \lambda, \mu_1, \ldots, \mu_r; \\ \mu_1+1, \ldots, \mu_r+1; \end{matrix} z \right] = \sum_{k=1}^{r} {}_2F_1(\lambda, \mu_k; \mu_k+1; z) \cdot \prod_{j=1 (j\neq k)}^{r} \left\{ \frac{\mu_j}{\mu_j - \mu_k} \right\} \tag{3.5}$$

$$\left(\mu_j \notin \mathbb{Z}_0^-;\ \mu_j \neq \mu_k;\ j \neq k;\ j,k = 1,\ldots,r;\ |z|<1\right).$$

For the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by (*cf.* [3, p. 27, Equation 1.11 (1)]; see also [20, p. 121, Equation 2.5 (1)])

$$\Phi(z,s,a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s} \tag{3.6}$$

$$\left(a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z|<1;\ \Re(s)>1 \quad \text{when} \quad |z|=1\right),$$

it is easily observed that [20, p. 123, Equation 2.5 (16)]

$$\Phi(z,1,a) = a^{-1}\ {}_2F_1(1,a;a+1;z) \quad (|z|<1). \tag{3.7}$$

Thus, in its special case when $\lambda = 1$, the reduction formula (3.5) assumes the form:

$$ {}_{r+1}F_r \left[\begin{matrix} 1, \mu_1, \ldots, \mu_r; \\ \mu_1+1, \ldots, \mu_r+1; \end{matrix} z \right] = (\mu_1 \cdots \mu_r) \sum_{k=1}^{r} \Phi(z,1,\mu_k)$$

$$\prod_{j=1(j\neq k)}^{r} \left\{(\mu_j - \mu_k)^{-1}\right\} \tag{3.7}$$

$$\left(\mu_j \notin \mathbb{Z}_0^-;\ \mu_j \neq \mu_k;\ j \neq k;\ j,k = 1,\ldots,r;\ |z| < 1\right),$$

which, for $\mu_j \longmapsto \mu_j + 1$ $(j = 1, \ldots, r)$, immediately yields

$$\sum_{k=1}^{\infty} \frac{z^{k-1}}{(k+\mu_1)\cdots(k+\mu_r)} = \sum_{k=1}^{r} \Phi(z,1,\mu_k+1) \prod_{j=1(j\neq k)}^{r} \left\{(\mu_j - \mu_k)^{-1}\right\} \tag{3.8}$$

$$\left(\mu_j \notin \mathbb{Z}^-;\ \mu_j \neq \mu_k;\ j \neq k;\ j,k = 1,\ldots,r,\ |z| < 1\right).$$

In particular, since [17, p. 463, Entry 7.3.1.135]

$${}_2F_1(1,n;n+1;z) = -\frac{n}{z^n}\left(\log(1-z) + \sum_{k=1}^{n-1} \frac{z^k}{k}\right) \quad (n \in \mathbb{N};\ |z| < 1) \tag{3.9}$$

or, equivalently,

$$\Phi(z,1,n) = -\frac{1}{z^n}\left(\log(1-z) + \sum_{k=1}^{n-1} \frac{z^k}{k}\right) \quad (n \in \mathbb{N};\ |z| < 1), \tag{3.10}$$

each of which corresponds (for $n = 1$) to the familiar relationship (2.5), upon setting

$$\mu_j = m_j \quad (m_j \in \mathbb{N}_0;\ j = 1,\ldots,r)$$

in the summation formula (3.8), we get

$$\sum_{k=1}^{\infty} \frac{z^k}{(k+m_1)\cdots(k+m_r)} = -\sum_{k=1}^{r} z^{-m_k}\left(\log(1-z) + \sum_{l=1}^{m_k} \frac{z^l}{l}\right) \cdot \prod_{j=1(j\neq k)}^{r} \left\{(m_j - m_k)^{-1}\right\} \tag{3.11}$$

$$\left(m_j \in \mathbb{N}_0;\ m_j \neq m_k;\ j \neq k;\ j,k = 1,\ldots,r;\ |z| < 1\right).$$

By appealing to the easily verifiable identities (3.9) and (3.10), the summation formula (3.11) can be proven *directly* in a rather elementary way. In much more general settings, the case $z = 1$ of the infinite series occurring on the left-hand side of (3.11) was considered earlier by Al-Saqabi *et al.* [1] and Wu *et al.* [24] (see also Aular de Durán *et al.* [2]). Furthermore, if in (3.11) we set

$$r = n + 1 \quad \text{and} \quad m_j = j - 1 \quad (j = 1, \ldots, n + 1),$$

and multiply each side of the resulting equation by z^n, we obtain

$$\sum_{k=1}^{\infty} \frac{z^{k+n}}{k(k+1)\cdots(k+n)} = -\sum_{k=1}^{n+1} z^{n-k+1} \left(\log(1-z) + \sum_{l=1}^{k-1} \frac{z^l}{l} \right) \cdot \prod_{j=1 (j \neq k)}^{n+1} \left\{ (j-k)^{-1} \right\} \quad (n \in \mathbb{N}_0;\ |z| < 1), \tag{3.12}$$

which provides a virtually simpler version of the *main* result in a recent paper by Tu *et al.* [22, p. 6, Theorem 2]. Closed-form expressions for infinite series of the type occurring in (3.12) can also be found to be listed by Hansen [4, p. 174].

Finally, we remark that some interesting extensions of the straightforward consequence (2.14) of the binomial expansion (2.15), especially when $n = 2$ *and* $n = 3$, were derived recently by Wang *et al.* [23] *via* the fractional differintegral operator $\mathcal{N}_z^{\nu}$ in a manner which we have already illustrated fairly adequately in the preceding section.

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