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PERTURBED PROXIMAL POINT ALGORITHMS FOR GENERALIZED MIXED VARIATIONAL INEQUALITIES

JAE UG JEONG

ABSTRACT. In this paper, we study a class of variational inequalities, which is called the generalized set-valued mixed variational inequality By using the properties of the resolvent operator associated with a maximal monotone mapping in Hilbert spaces, we have established an existence theorem of solutions for generalized set-valued mixed variational inequalities, suggesting a new iterative algorithm and a perturbed proximal point algorithm for finding approximate solutions which strongly converge to the exact solution of the generalized set-valued mixed variational inequalities

1. Introduction

Variational inequality theory introduced by Stampacchia[11] has enjoyed vigorous for the last thirty years. Variational inequality theory described a broad spectrum of interesting and important developments involving a link among various fields of mathematics, physics, economics, and engineering sciences[1,5]. Variational inequalities have been extended and generalized in differential directions using novel and innovative techniques both for their own sake and for applications. A useful and an important generalization of variational inequality is a mixed variational inequality containing the nonlinear term. For the

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applications of the mixed variational inequalities, see [1,2]. Due to the presence of the nonlinear term, the projection method cannot be used to study the existence of a solution of the mixed variational inequalities. It is clear that one cannot develop the projection type algorithms for solving the mixed variational inequalities. These facts motivated us to develop another technique. This technique is related to the resolvent of the maximal monotone operator. Hassouni and Moudafi[4] modified and extended this technique for a class of general mixed variational inequalities.

In this paper, we shall study a class of variational inequalities, which is called the generalized set valued mixed variational inequality. This class is the most general and includes the previously studied classes of variational inequalities as special cases By applying the properties of the resolvent operator associated with a maximal monotone mapping in Hilbert spaces, it is shown that the generalized set valued mixed variational inequalities are equivalent to the fixed point problems. A new iterative algorithm and a perturbed proximal point algorithm for finding approximate solutions which strongly converges to the exact solution of the generalized set valued mixed variational inequalities are proposed and analyzed. The main results proved in this paper represent a refinement and improvement of the previously known results in this field.

2. Preliminaries

Let H be a Hilbert space endowed with a norm $\|\cdot\|$ and a inner product $< \cdot,\cdot >$. Let $N : H \times H \to H$ be a nonlinear operator, $T, A : H \to 2^H$ be set valued mappings, $g : H \to H$ be a single valued mapping and $\phi : H \times H \to R \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot, y) : H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $g(H) \cap \mathrm{dom}\partial\phi(\cdot, y) \neq \phi$. Then the problem of finding $x \in H$, $u \in T(x)$, and $v \in A(x)$ such that

$$(2.1) \qquad < N(u,v), y - g(x) > \geq \phi(g(x), x) - \phi(y, x), \quad ^{\forall}y \in H,$$

is called the generalized set valued mixed variational inequality (GSVMIP (N, T, A, g, ϕ)).

Special cases

(1) If $\phi(x,y) = \phi(x)$ for all $y \in H$, then the problem (2.1) reduces to the generalized multivalued mixed variational inequality, which is mainly due to Noor et al [7].

(2) If K is a given closed convex subset of H and $\phi = I_K$ is the indicator function of K,

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{if otherwise,} \end{cases}$$

then the problem (2.1) is equivalent to finding $x \in H$, $u \in T(x)$, and $v \in A(x)$ such that $g(x) \in H$ and

$$< N(u,v), y - g(x) > \geq 0, \quad ^{\forall} y \in K,$$

a problem considered and studied by Noor[6].

(3) If $K : H \to 2^H$ is a set valued mapping such that each $K(x)$ is a closed convex subset of H (or $K(x) = m(x) + K$, where $m : H \to H$ and K is a closed convex subset of H) and for each fixed $y \in H$, $\phi(\cdot, y) = I_{K(y)}(\cdot)$ is the indicator function of $K(y)$,

$$I_{K(y)}(x) = \begin{cases} 0, & \text{if } x \in K(y), \\ +\infty, & \text{otherwise,} \end{cases}$$

then the problem (2.1) is equivalent to finding $x \in H$, $u \in T(x)$, and $v \in A(x)$ such that $g(x) \in K(x)$ and

$$< N(u,v), y - g(x) > \geq 0, \quad ^{\forall} y \in K(x).$$

(4) For $N(u,v) = u - v$, then problem (2.1) is equivalent to finding $x \in H$, $u \in T(u)$, and $v \in A(x)$ such that

$$< u - v, y - g(x) > \geq \phi(g(x), x) - \phi(y, x), \quad ^{\forall} y \in H,$$

a problem studied by X. P. Ding[3].

In order to prove our main theorem, we need the following concepts and results; see Pascali and Sburlan [9].

DEFINITION 2.1. Let H be a Hilbert space and let $G : H \to 2^H$ be a maximal monotone mapping. For any fixed $\rho > 0$, the mapping $J_\rho^G : H \to H$ defined by

$$J_\rho^G(x) = (I + \rho G)^{-1}(x), \quad \forall x \in H,$$

is said to be *the resolvent operator of G* where I is the identity mapping on H.

LEMMA 2.1. *Let X be a reflexive Banach space endowed with a strictly convex norm and $\phi : X \to R \cup \{+\infty\}$ be a proper convex lower semicontinuous function. Then $\partial\phi : X \to 2^{X^*}$ is a maximal monotone mapping.*

LEMMA 2.2. *Let $G : H \to 2^H$ be a maximal monotone mapping. Then the resolvent operator $J_\rho^G : H \to H$ of G is nonexpansive, i.e., for all $x, y \in H$,*

$$\|J_\rho^G(x) - J_\rho^G(y)\| \le \|x - y\|.$$

DEFINITION 2.2. For all $x_1, x_2 \in H$, the operator $N(\cdot, \cdot)$ is said to be *α-strongly monotone and β-Lipschitz continuous* with respect to the first argument if there exist constants $\alpha > 0$, $\beta > 0$ such that

$$< N(u_1, \cdot) - N(u_2, \cdot), x_1 - x_2 >\ \ge \alpha\|x_1 - x_2\|^2,$$

$$\|N(u_1, \cdot) - N(u_2, \cdot)\| \le \beta\|u_1 - u_2\|$$

for all $u_1 \in T(x_1)$, $u_2 \in T(x_2)$.

In a similar way, we can define the strong monotonicity and Lipschitz continuity of the operator $N(\cdot, \cdot)$ with respect to the second argument.

DEFINITION 2.3. The set valued operator $T : H \to 2^H$ is said to be *δ-Lipschitz continuous* if there exists a constant $\delta > 0$ such that

$$M(T(x), T(y)) \le \delta\|x - y\|, \quad \forall x, y \in H,$$

where $M(A, B) = \sup\{\|a - b\| : a \in A, b \in B\}, \forall A, B \in 2^H$.

DEFINITION 2.4. A mapping $g : H \to H$ is said to be *λ-strongly monotone and σ-Lipschitz continuous* if there exists constants $\lambda > 0$, $\sigma > 0$ such that

$$< g(x) - g(y), x - y > \geq \lambda \|x - y\|^2,$$
$$\|g(x) - g(y)\| \leq \sigma \|x - y\|,$$

for all $x, y \in H$.

3. Main Results

In this section, we shall prove an existence theorem of solutions for GSVMIP$(N, T, A, g, \phi)(2.1)$ and suggest a new iterative algorithm for finding approximate solutions of the problem (2.1). And we show that the sequence of approximate solutions strongly converges to the exact solution of the problem (2.1).

THEOREM 3.1. *(x^*, u^*, v^*) is a solution of the problem (2.1) if and only if (x^*, u^*, v^*) satisfies the relation*

$$g(x) = J_\rho^{\partial\phi(\cdot,x)}(g(x) - \rho N(u, v)), \quad \forall x \in H, \tag{3.1}$$

where $\rho > 0$ is a constant, $J_\rho^{\partial\phi(\cdot,x)} = (I + \rho\partial\phi(\cdot, x))^{-1}$ is the resolvent operator of $\partial\phi(\cdot, x)$, and I is the identity mapping on H.

PROOF. Let (x^*, u^*, v^*) satisfy the relation (3.1), that is,

$$g(x^*) = J_\rho^{\partial\phi(\cdot,x^*)}(g(x^*) - \rho N(u^*, v^*)).$$

The equality holds if and only if

$$-N(u^*, v^*) \in \partial\phi(\cdot, x^*)(g(x^*))$$

by the definition of $J_\rho^{\partial\phi(\cdot,x^*)}$. The relation holds if and only if

$$\phi(y, x^*) - \phi(g(x^*), x^*) \geq \ < -N(u^*, v^*), y - g(x^*) >, \quad \forall y \in H,$$

by the definition of the subdifferential $\partial\phi(\cdot, x^*)$. Hence (x^*, u^*, v^*) is the solution of

$$< N(u^*, v^*), y - g(x^*) > \ \geq \phi(g(x^*), x^*) - \phi(y, x^*), \quad \forall y \in H.$$

REMARK 3.1. From Theorem 3.1, we see that the generalized set valued mixed variational inequality (2.1) is equivalent to the fixed point problem (3.1). Equation (3.1) can be written as

$$x = x - g(x) + J_\rho^{\partial\phi(\ ,x)}[g(x) - \rho N(u,v)]. \tag{3.2}$$

This fixed point formulation enables us to suggest the following algorithms.

Algorithm 3.1

For any given $x_0 \in H$, $\bar{u}_0 \in T(x_0)$, and $\bar{v}_0 \in A(x_0)$, let

$$y_0 = (1-\beta_0)x_0 + \beta_0[x_0 - g(x_0) + J_\rho^{\partial\phi(\ ,x_0)}(g(x_0) - \rho N(\bar{u}_0, \bar{v}_0))].$$

Take any fixed $u_0 \in T(y_0)$ and $v_0 \in A(y_0)$, and let

$$x_1 = (1-\alpha_0) + \alpha_0[y_0 - g(y_0) + J_\rho^{\partial\phi(\ ,y_0)}(g(y_0) - \rho N(u_0, v_0))].$$

Continuing this way, we can define sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$,$\{u_n\}_{n=0}^{\infty}$, and $\{v_n\}_{n=0}^{\infty}$ as

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n[y_n - g(y_n) + J_\rho^{\partial\phi(\ ,y_n)}(g(y_n) - \rho N(u_n, v_n))],$$

$$y_n = (1-\beta_n)x_n + \beta_n[x_n - g(x_n) + J_\rho^{\partial\phi(\ ,x_n)}(g(x_n) - \rho N(\bar{u}_n, \bar{v}_n))], \tag{3.3}$$

for $n = 0, 1, 2, \cdots$, where $u_n \in T(y_n)$, $v_n \in A(y_n)$, $\bar{u}_n \in T(x_n)$, and $\bar{v}_n \in A(x_n)$ can be chosen arbitrarily, $0 \le \alpha_n, \beta_n \le 1$, $\sum_{n=0}^{\infty} \alpha_n$ diverges, and $\rho > 0$ is a constant.

Using fixed point formulation (3.2), we have the following algorithm.

Algorithm 3.2

For any given $x_0 \in H$, compute the sequence $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, and $\{v_n\}_{n=0}^{\infty}$ by the iterative schemes

$$x_{n+1} = x_n - g(x_n) + J_\rho^{\partial\phi(\ ,x_n)}(g(x_n) - \rho N(u_n, v_n)), \tag{3.4}$$

for $n = 0, 1, 2, \cdots$, where $u_n \in T(x_n)$ and $v_n \in A(x_n)$ can be chosen arbitrary and $\rho > 0$ is a constant.

To perturb the Algorithm 3.2, we first add, in the right-hand side of (3.4), an error e_n to take into account a possible inexact computation of the proximal point and we consider another perturbation by replacing ϕ in (3.4) by ϕ_n, where each $\phi_n : H \times H \to R \cup \{+\infty\}$ is such that for each fixed $y \in H$, $\phi_n(\cdot, y)$ is a proper convex lower semicontinuous function on H and the sequence $\{\phi_n\}$ approximates ϕ on $H \times H$. Then we obtain the following perturbed proximal point algorithm.

Algorithm 3.3

For any given $x_0 \in H$, computer the sequence $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, and $\{v_n\}_{n=0}^{\infty}$ by the iterative schemes

$$x_{n+1} = x_n - g(x_n) + J_\rho^{\partial\phi_n(\cdot, x_n)}(g(x_n) - \rho N(u_n, v_n)) + e_n, \tag{3.5}$$

where $\{e_n\}_{n=0}^{\infty}$ is an error sequence in H, $u_n \in T(x_n)$, and $v_n \in A(x_n)$ can be chosen arbitrarily, and $\rho > 0$ is a constant.

Now we show the existence of solutions of the GSVMIP (N, T, A, g, ϕ) (2.1).

THEOREM 3.2. *Let the operator $N(\cdot, \cdot)$ be α-strongly monotone and β-Lipschitz continuous with respect to the first argument. Let the operator $N(\cdot, \cdot)$ be γ-Lipschitz continuous with respect to second argument. Let $T : H \to 2^H$ be δ-Lipschitz continuous, $A : H \to 2^H$ be η-Lipschitz continuous, $g : H \to H$ be λ-strongly monotone and σ-Lipschitz continuous, and $\phi : H \times H \to R \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot, y)$ is a proper convex lower semicontinuous function on H, $g(H) \cap dom\partial\phi(\cdot, y) \neq \phi$, and for each $x, y, z \in H$,*

$$\|J_\rho^{\partial\phi(\cdot, x)}(z) - J_\rho^{\partial\phi(\cdot, y)}(z)\| \leq \mu\|x - y\|.$$

Suppose there exists a constant $\rho > 0$ such that

$$k = \nu + 2\sqrt{1 - 2\lambda + \sigma^2},$$

$$\alpha > (1 - k)\eta\gamma + \sqrt{k(\beta^2\delta^2 - \eta^2\gamma^2)(2 - k)},$$

$$\left|\rho - \frac{\alpha + \eta\gamma(k-1)}{\beta^2\delta^2 - \eta^2\gamma^2}\right| \tag{3.6}$$
$$< \frac{\sqrt{[\alpha + \eta\gamma(k-1)]^2 - k(\beta^2\delta^2 - \eta^2\gamma^2)(2-k)}}{\beta^2\delta^2 - \eta^2\gamma^2},$$

$$\rho\gamma\eta < 1 - k.$$

Then the GSVMIP(N, T, A, g, ϕ)(2.1) has a solution (x^*, u^*, v^*).

PROOF. By Theorem 3.1, it is sufficient to prove that there exist $x^* \in H$, $u^* \in T(x^*)$, and $v^* \in A(x^*)$ such that (3.1) holds. Define a set valued mapping $F : H \to 2^H$ by

$$F(x) = \cup_{u \in T(X)} \cup_{v \in A(x)} [x - g(x) + J_\rho^{\partial\phi(\ ,x)}(g(x) - \rho N(u, v))].$$

For arbitrary $x, y \in H$, $a \in F(x)$, and $b \in F(y)$, there exist $u_1 \in T(x)$, $v_1 \in A(x)$, $u_2 \in T(y)$, and $v_2 \in A(y)$ such that

$$a = x - g(x) + J_\rho^{\partial\phi(\ ,x)}(g(x) - \rho N(u_1, v_1)),$$

$$b = y - g(y) + J_\rho^{\partial\phi(\ ,y)}(g(y) - \rho N(u_2, v_2)).$$

By the assumption of ϕ and Lemma 2.1 and 2.2, we have

$$\begin{aligned}
\|a - b\| &\le \|x - y - (g(x) - g(y))\| \\
&\quad + \|J_\rho^{\partial\phi(\ ,x)}(g(x) - \rho N(u_1, v_1)) \\
&\qquad - J_\rho^{\partial\phi(\ ,x)}(g(y) - \rho N(u_2, v_2))\| \\
&\quad + \|J_\rho^{\partial\phi(\ ,x)}(g(y) - \rho N(u_2, v_2)) \\
&\qquad - J_\rho^{\partial\phi(\ ,y)}(g(y) - \rho N(u_2, v_2))\| \\
&\le \|x - y - (g(x) - g(y))\| \\
&\quad + \|g(x) - g(y) - \rho(N(u_1, v_1) - N(u_2, v_2))\| \\
&\quad + \mu\|x - y\| \\
&\le 2\|x - y - (g(x) - g(y))\| \\
&\quad + \|x - y - \rho(N(u_1, v_1) - N(u_2, v_1))\| \\
&\quad + \rho\|N(u_2, v_2) - N(u_2, v_1)\| + \mu\|x - y\|.
\end{aligned} \tag{3.7}$$

Since $g : H \to H$ is λ-strongly monotone and σ-Lipschitz continuous,

$$\begin{aligned}
&\|x - y - (g(x) - g(y))\|^2 \\
&\qquad = \|x - y\|^2 - 2 < g(x) - g(y), x - y > \\
&\qquad\quad + \|g(x) - g(y)\|^2 \\
&\qquad \leq \|x - y\|^2 - 2\lambda\|x - y\|^2 + \sigma^2\|x - y\|^2 \\
&\qquad = (1 - 2\lambda + \sigma^2)\|x - y\|^2. \qquad (3.8)
\end{aligned}$$

Using the γ-Lipschitz continuity of the operator $N(\cdot, \cdot)$ with respect to second argument and the η-Lipschitz continuity of A, we have

$$\begin{aligned}
\|N(u_2, v_2) - N(u_2, v_1)\| &\leq \gamma\|v_2 - v_1\| \\
&\leq \gamma M(A(y), A(x)) \\
&\leq \gamma\eta\|x - y\|. \qquad (3.9)
\end{aligned}$$

Since $N(\cdot, \cdot)$ is α-strongly monotone and β-Lipschitz continuous with respect to the first argument,

$$\begin{aligned}
&\|x - y - \rho(N(u_1, v_1) - N(u_2, v_1))\|^2 \\
&\quad = \|x - y\|^2 - 2\rho < N(u_1, v_1) - N(u_2, v_1), x - y > \\
&\qquad + \rho^2\|N(u_1, v_1) - N(u_2, v_1)\|^2 \\
&\quad \leq \|x - y\|^2 - 2\rho\alpha\|x - y\|^2 + \rho^2\beta^2\|u_1 - u_2\|^2 \\
&\quad \leq \|x - y\|^2 - 2\rho\alpha\|x - y\|^2 + \rho^2\beta^2 M(T(x), T(y))^2 \\
&\quad \leq (1 - 2\rho\alpha + \rho^2\beta^2\delta^2)\|x - y\|^2. \qquad (3.10)
\end{aligned}$$

Combining (3.7), (3.8), (3.9), and (3.10), we get

$$\begin{aligned}
\|a - b\| \leq \{&2\sqrt{1 - 2\lambda + \sigma^2} + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\delta^2} \\
&+ \rho\gamma\eta + \mu\}\|x - y\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
M(F(x), F(y)) &\leq \{2\sqrt{1 - 2\lambda + \sigma^2} + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\delta^2} \\
&\qquad + \rho\gamma\eta + \mu\}\|x - y\| \\
&= [k + t(\rho) + \rho\gamma\eta]\|x - y\| \\
&= \theta\|x - y\|, \qquad (3.11)
\end{aligned}$$

where $k = 2\sqrt{1-2\lambda+\sigma^2} + \mu$, $t(\rho) = \sqrt{1-2\rho\alpha+\rho^2\beta^2\delta^2}$, and $\theta = k + t(\rho) + \rho\gamma\eta$. By the condition (3.6), we have $\theta < 1$. It follows from the condition (3.11) and Theorem 3.1 of Siddiqi and Ansari[10] that F has a fixed point $x^* \in H$. By the definition of F, there exist $u^* \in T(x^*)$ and $v^* \in A(x^*)$ such that

$$g(x^*) = J_\rho^{\partial\phi(\cdot,x^*)}(g(x^*) - \rho N(u^*, v^*)).$$

Therefore (x^*, u^*, v^*) is a solution of the GSVMIP (N, T, A, g, ϕ)(2.1).

THEOREM 3.3. *Let H, N, T, A, g, and ϕ satisfy all conditions in Theorem 3.2. If the condition (3.6) is also satisfies, then the iterative sequences $\{x_n\}_{n=0}^\infty$, $\{u_n\}_{n=0}^\infty$, and $\{v_n\}_{n=0}^\infty$ defined in the Algorithm 3.1 strongly converge to x^*, u^*, and v^*, respectively, and (x^*, u^*, v^*) is a solution of the GSVMIP(N, T, A, g, ϕ)(2.1).*

PROOF. By the Theorem 3.2, the GSVMIP(N, T, A, g, ϕ)(2.1) has a solution (x^*, u^*, v^*) From Theorem 3.1 we have $x^* \in H$, $u^* \in T(x^*)$, $v^* \in A(x^*)$, and for all $n \geq 0$,

$$\begin{aligned} x^* &= x^* - g(x^*) + J_\rho^{\partial\phi(\cdot,x^*)}(g(x^*) - \rho N(u^*, v^*)) \\ &= (1-\alpha_n)x^* + \alpha_n\{x^* - g(x^*) + J_\rho^{\partial\phi(\cdot,x^*)}(g(x^*) - \rho N(u^*, v^*))\} \\ &= (1-\beta_n)x^* + \beta_n\{x^* - g(x^*) + J_\rho^{\partial\phi(\cdot,x^*)}(g(x^*) - \rho N(u^*, v^*))\}. \end{aligned}$$

By the algorithm 3.1, using a similar argument as in the proof of Theorem 3.2, we obtain

$$\|x_n - x^* - (g(x_n) - g(x^*))\| \leq \sqrt{1-2\lambda+\sigma^2}\|x_n - x^*\|,$$

$$\begin{aligned} &\|x_n - x^* - \rho(N(\bar{u}_n, v^*) - N(u^*, v^*))\| \\ &\qquad \leq \sqrt{1-2\rho\alpha+\rho^2\beta^2\delta^2}\|x_n - x^*\|, \end{aligned}$$

$$\|N(\bar{u}_n, \bar{v}_n) - N(\bar{u}_n, v^*)\| \leq \gamma\eta\|x_n - x^*\|,$$

$$\|y_n - x^* - (g(y_n) - g(x^*))\| \leq \sqrt{1-2\lambda+\sigma^2}\|y_n - x^*\|,$$

$$\begin{aligned}
&\|y_n - x^* - \rho(N(u_n, v^*) - N(u^*, v^*))\| \\
&\qquad \le \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\delta^2}\|y_n - x^*\|, \\
&\|N(u_n, v^*) - N(u_n, v_n)\| \le \gamma\eta\|y_n - x^*\|.
\end{aligned}$$

Thus, by the Algorithm 3.1, the assumption of ϕ, and Lemma 2.1 and 2.2, we have

$$\begin{aligned}
&\|y_n - x^*\| \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \beta_n\|J_\rho^{\partial\phi(\ ,x_n)}(g(x_n) - \rho N(\bar{u}_n, \bar{v}_n)) \\
&\qquad - J_\rho^{\partial\phi(\ ,x^*)}(g(x_n) - \rho N(\bar{u}_n, \bar{v}_n))\| \\
&\quad + \beta_n\|J_\rho^{\partial\phi(\ ,x^*)}(g(x_n) - \rho N(\bar{u}_n, \bar{v}_n)) \\
&\qquad - J_\rho^{\partial\phi(\ ,x^*)}(g(x^*) - \rho N(u^*, v^*))\| \\
&\le (1-\beta_n)\|x_n - x^*\| + 2\beta_n\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \beta_n\mu\|x_n - x^*\| + \beta_n\|x_n - x^* - \rho(N(\bar{u}_n, \bar{v}_n) - N(u^*, v^*))\| \\
&\le (1-\beta_n)\|x_n - x^*\| + 2\beta_n\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \beta_n\mu\|x_n - x^*\| + \beta_n\|x_n - x^* - \rho(N(\bar{u}_n, v^*) - N(u^*, v^*))\| \\
&\quad + \beta_n\rho\|N(\bar{u}_n, \bar{v}_n) - N(\bar{u}_n, v^*)\| \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n[2\sqrt{1 - 2\lambda + \sigma^2} + \mu \\
&\quad + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\delta^2} + \rho\gamma\eta]\|x_n - x^*\| \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n\theta\|x_n - x^*\| \\
&\le \|x_n - x^*\|. \qquad (3\ 12)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\qquad \le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|y_n - x^* - (g(y_n) - g(x^*))\| \\
&\qquad\quad + \alpha_n\|J_\rho^{\partial\phi(\ ,y_n)}(g(y_n) - \rho N(u_n, v_n)) \\
&\qquad\qquad - J_\rho^{\partial\phi(\ ,x^*)}(g(x^*) - \rho N(u^*, v^*))\|
\end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|y_n - x^* - (g(y_n) - g(x^*))\| \\
&\quad + \alpha_n\|J_\rho^{\partial\phi(\ ,y_n)}(g(y_n) - \rho N(u_n, v_n)) \\
&\qquad - J_\rho^{\partial\phi(\ ,x^*)}(g(y_n) - \rho N(u_n, v_n))\| \\
&\quad + \alpha_n\|J_\rho^{\partial\phi(\ ,x^*)}(g(y_n) - \rho N(u_n, v_n)) \\
&\qquad - J_\rho^{\partial\phi(\ ,x^*)}(g(x^*) - \rho N(u^*, v^*))\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + 2\alpha_n\|y_n - x^* - (g(y_n) - g(x^*))\| \\
&\quad + \alpha_n\|y_n - x^* - \rho(N(u_n, v^*) - N(u^*, v^*))\| \\
&\quad + \alpha_n\rho\|N(u_n, v^*) - N(u_n, v_n)\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n[2\sqrt{1-2\lambda+\sigma^2} \\
&\quad + \sqrt{1-2\rho\alpha+\rho^2\beta^2\delta^2} + \rho\gamma\eta]\|y_n - x^*\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta\|y_n - x^*\|. \qquad (3.13)
\end{aligned}
$$

It follows from (3.12) and (3.13) that

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta\|x_n - x^*\| \\
&= [1-(1-\theta)\alpha_n]\|x_n - x^*\| \\
&\le \Pi_{i=0}^n[1-(1-\theta)\alpha_i]\|x_0 - x^*\|.
\end{aligned}
$$

Since $\sum_{n=0}^\infty \alpha_n$ diverges and $1-\theta > 0$, we have $\Pi_{i=0}^\infty[1-(1-\theta)\alpha_i] = 0$. Hence the sequence $\{x_n\}$ strongly converges to x^*. By (3.12), the sequence $\{y_n\}$ also strongly converges to x^*. Since $u_n \in T(y_n)$, $u^* \in T(x^*)$, and T is δ-Lipschitz continuous, we have

$$
\begin{aligned}
\|u_n - u^*\| &\le M(T(y_n), T(x^*)) \\
&\le \delta\|y_n - x^*\| \\
&\to 0,
\end{aligned}
$$

and hence the sequence $\{u_n\}$ strongly converges to u^*. Similarly, we can show that the sequence $\{v_n\}$ strongly converges to v^*. This completes the proof.

THEOREM 3.4. *Let H, N, T, A, and g satisfy all conditions in Theorem 3.2, and ϕ, $\phi_n : H \times H \to R \cup \{+\infty\}$, $n = 0, 1, 2, \cdots$, be such that for each fixed $y \in H$, $\phi(\cdot, y)$ and each $\phi_n(\cdot, y)$ are both proper convex lower semicontinuous functions on H, $g(H) \cap dom\partial\phi(\cdot, y) \neq \phi$, and for each $x, y, z \in H$ and for all $n \geq 0$,*

$$\|J_\rho^{\partial\phi_n(\cdot,x)}(z) - J_\rho^{\partial\phi_n(\cdot,y)}(z)\| \leq \mu\|x - y\|.$$

Assume $\lim_{n\to\infty} \|J_\rho^{\partial\phi_n(\cdot,y)}(z) - J_\rho^{\partial\phi(\cdot,y)}(z)\| = 0$ *for all $y, z \in H$,* $\lim_{n\to\infty} \|e_n\| = 0$, *and there exists a constant $\rho > 0$ such that the condition (3.6) in Theorem 3.2 holds. Then the iterative sequences $\{x_n\}$, $\{u_n\}$, and $\{v_n\}$ defined in the Algorithm 3.3 strongly converge to x^*, u^*, and v^*, respectively, and (x^*, u^*, v^*) is a solution of the GSVMIP (N, T, A, g, ϕ) (2.1).*

PROOF. By Theorem 3.2, the GSVMIP(N, T, A, g, ϕ)(2.1) has a solution (x^*, u^*, v^*) such that $u^* \in T(x^*)$, $v^* \in A(x^*)$, and

$$x^* = x^* - g(x^*) + J_\rho^{\partial\phi(\cdot,x^*)}(g(x^*) - \rho N(u^*, v^*)).$$

By setting $h(x^*) = g(x^*) - \rho N(u^*, v^*)$ and by using the Algorithm 3.3 and the assumption of ϕ and ϕ_n, $n = 0, 1, 2, \cdots$, we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\quad \leq \|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\qquad + \|J_\rho^{\partial\phi_n(\cdot,x_n)}(g(x_n) - \rho N(u_n, v_n)) \\
&\qquad\quad - J_\rho^{\partial\phi_n(\cdot,x_n)}(g(x^*) - \rho N(u^*, v^*))\| \\
&\qquad + \|J_\rho^{\partial\phi_n(\cdot,x_n)}(g(x^*) - \rho N(u^*, v^*)) \\
&\qquad\quad - J_\rho^{\partial\phi_n(\cdot,x^*)}(g(x^*) - \rho N(u^*, v^*))\| \\
&\qquad\quad + \|J_\rho^{\partial\phi_n(\cdot,x^*)}(g(x^*) - \rho N(u^*, v^*)) \\
&\qquad\quad - J_\rho^{\partial\phi(\cdot,x^*)}(g(x^*) - \rho N(u^*, v^*))\| \\
&\qquad + \|e_n\|
\end{aligned}$$

$$\begin{aligned}
&\le 2\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \|x_n - x^* - \rho(N(u_n, v_n) - N(u^*, v^*))\| \\
&\quad + \mu\|x_n - x^*\| + \|J_\rho^{\partial\phi_n(\ ,x^*)}(h(x^*)) - J_\rho^{\partial\phi(\ ,x^*)}(h(x^*))\| \\
&\quad + \|e_n\| \\
&\le 2\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \|x_n - x^* - \rho(N(u_n, v^*) - N(u^*, v^*))\| \\
&\quad + \rho\|N(u_n, v_n) - N(u_n, v^*)\| + \mu\|x_n - x^*\| \\
&\quad + \|J_\rho^{\partial\phi_n(\ ,x^*)}(h(x^*)) - J_\rho^{\partial\phi(\ ,x^*)}(h(x^*))\| + \|e_n\| \\
&\le (k + t(\rho) + \rho\gamma\eta)\|x_n - x^*\| \\
&\quad + \|J_\rho^{\partial\phi_n(\ ,x^*)}(h(x^*) - J_\rho^{\partial\phi(\ ,x^*)}(h(x^*))\| + \|e_n\| \\
&= \theta\|x_n - x^*\| + \varepsilon_n, \qquad (3.14)
\end{aligned}$$

where $k = \mu + 2\sqrt{1 - 2\lambda + \sigma^2}$, $t(\rho) = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\delta^2}$, $\theta = k + t(\rho) + \rho\gamma\eta$, and $\varepsilon_n = \|J_\rho^{\partial\phi_n(\ ,x^*)}(h(x^*)) - J_\rho^{\partial\phi(\ ,x^*)}(h(x^*))\| + \|e_n\|$. By the condition (3.6) in Theorem 3.2, we have $\theta < 1$. It follows from (3.14) that

$$\|x_{n+1} - x^*\| \le \theta^{n+1}\|x_0 - x^*\| + \sum_{i=0}^{n} \theta^i \varepsilon_{n-i}.$$

Since $\varepsilon_n \to 0$ by the assumption, it follows from Orgeta and Rheinboldt [8, p.338] that

$$\lim_{n\to 0} \|x_{n+1} - x^*\| = 0,$$

and hence the sequence $\{x_n\}$ strongly converges to x^*. Since $u_n \in T(x_n)$, $v_n \in A(x_n)$, $u^* \in T(x^*)$, and $v^* \in A(x^*)$, we have

$$\|u_n - u^*\| \le M(T(x_n), T(x^*)) \le \delta\|x_n - x^*\|,$$

$$\|v_n - v^*\| \le M(A(x_n), A(x^*)) \le \eta\|x_n - x^*\|.$$

It follows that the sequences $\{u_n\}$ and $\{v_n\}$ also strongly converge to u^* and v^*, respectively. This completes the proof.

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Department of mathematics
Dong-Eui university
Pusan 614-714, South Korea