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# SERIES REPRESENTATIONS FOR THE EULER-MASCHERONI CONSTANT $\gamma$

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ABSTRACT The third important Euler-Mascheroni constant $\gamma$, like $\pi$ and $e$, is involved in representations, evaluations, and purely relationships among other mathematical constants and functions, in various ways. The main object of this note is to summarize some known series representaions for $\gamma$, with comments for their proofs, and to point out that one of those series representaions for $\gamma$ seems to be *incorrectly* recorded. A brief historical comment for $\gamma$ is also provided

## 1. Introduction

The Euler's constant $\gamma$ is defined by

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right), \tag{1.1}$$

where the involved sequence is easily seen to be decreasing and bounded, and so the $\gamma$ is well defined.

The discovery that the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\cdots$ is divergent, is attributed by James Bernoulli to his brother (see Glaisher [10]). Yet

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the connection between $1+\frac{1}{2}+\cdots+\frac{1}{x}$ and $\log x$ was first established by Euler [8] who (see Walfisz [19]) gave the formula

$$1+\frac{1}{2}+\cdots+\frac{1}{x}=\gamma+\log x+\frac{1}{2x}-\frac{B_1}{2x^2}+\frac{B_2}{4x^4}-\frac{B_3}{6x^6}+\cdots,$$

$B_1, B_2, \ldots$ being Bernoulli numbers, in which, by putting $x=10$, he calculated

$$\gamma = 0.57721\,56649\,015325\cdots.$$

The value of Euler's constant was given by Mascheroni in 1790 with 32 figures as follows:

$$\gamma = 0.57721\,56649\,01532\,8606\underline{1}\,811209008239\cdots.$$

In 1809, Soldner computed the value of $\gamma$ as

$$\gamma = 0.57721\,56649\,01532\,8606\underline{0}\,6065\cdots$$

which differs from Mascheroni's value in the twentieth place. In fact, Masheroni's value turned out to be not correct. However, maybe since Mascheroni's error has led to eight additional calculations of this constant, so $\gamma$ is often called the *Euler-Mascheroni* constant. Gauss in 1813 computed the 23 first decimals; in 1860 Adams published the 260 first decimals. For rather recent computations for $\gamma$, we may see Knuth [14], Brent and McMillian [5], Bailey [4], and so on.

The true nature of Euler's constant (whether an algebraic or a transcendental number) has not been known, which is contained in a part of the Hilbert's seventh one (that is, Irrationality and Transcendence of Certain Numbers) among his famous 23 problems announced for the next 20th century in the 2nd International Congress of Mathematicians at Paris in 1990 (see [13]). Appell [2] in 1926 gave a proof that $\gamma$ is irrational. Appell himself's finding an error quickly, he published a retraction within a couple of weeks of his original announcement (see Ayoub [3]).

In addition, it is worthy of noting the following facts: The $\gamma$ is the third important mathematical constant next to $\pi$ and $e$ whose transcendence were shown by Ferdinand Lindemann in 1882 and Charles

Hermite in 1873, respectively. The mathematical constants $\pi$, $e$, and $\gamma$ are often referred to as *holy trinity.*

We can show that the infinite product

$$\prod_{n=1}^{\infty}(1+\frac{z}{n})e^{-\frac{z}{n}} \tag{1.2}$$

converges in the finite complex plane $\mathbb{C}$ to an entire function which has simple zeros at $z=-1,-2,-3,\dots$, this argument yields that the infinite product in (1.3) converges on every compact subset in $\mathbb{C}\setminus\{-1,-2,-3,\dots\}$ to a function with simple poles at $z=-1,-2,\dots$. Using this fact, Weierstrass defines the Gamma function, $\Gamma(z)$, is a meromorphic function on $\mathbb{C}$ with simple poles at $z=0,-1,-2,\dots$, given by

$$\Gamma(z)=\frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{\frac{z}{n}}, \tag{1.3}$$

where $\gamma$ *is a constant chosen so that* $\Gamma(1)=1$ and which is just (1.1). The first thing that must be done is to show that the constant $\gamma$ exists. Substituting $z=1$ in (1.3) yields a finite number

$$c=\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-1}e^{\frac{1}{n}}$$

which is clearly positive. Let $\gamma=\log c$; it follows that with this choice of $\gamma$, the equation (1.3) for $z=1$ gives $\Gamma(1)=1$. This constant $\gamma$ is just the Euler's constant and it satisfies

$$e^{\gamma}=\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-1}e^{\frac{1}{n}}. \tag{1.4}$$

Since both sides of (1.4) involve only real positive numbers and the real logarithm is continuous, we may apply the logarithm function to both sides of (1.4) and obtain

$$\begin{aligned}\gamma &= \sum_{k=1}^{\infty} \log\left[\left(1+\frac{1}{k}\right)^{-1} e^{\frac{1}{k}}\right] \\ &= \sum_{k=1}^{\infty}\left[\frac{1}{k} - \log(k+1) + \log k\right] \\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{k} - \log(k+1) + \log k\right] \\ &= \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \log(n+1)\right].\end{aligned}$$

Subtracting and adding $\log n$ to each term of this sequence and using the fact that

$$\lim_{n\to\infty}\log\left(\frac{n}{n+1}\right) = 0 \tag{1.5}$$

yields the very (1.1).

Like $\pi$ and $e$, the constant $\gamma$ is also involved in representations, evaluations, and purely relationships among other mathematical constants and functions, in various ways. For example, many integral representations for $\gamma$ have been developed (see Choi and Seo [7]; also Srivastava and Choi [17, pp. 4-6]). Here, among other things, we aim at summarizing some known series representaions for $\gamma$, with comments for their proofs. It is also pointed out that some of series representaions for $\gamma$ seem to be *incorrectly* recorded.

## 2. Series Representations

We start by recalling a well-known series representation for $\gamma$:

$$\gamma = \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k}, \tag{2.1}$$

where the Riemann Zeta function $\zeta(s)$ defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\Re(s) > 1) \tag{2.2}$$

is a special case when $a = 1$ of the *Hurwitz* (or *generalized*) *Zeta* function $\zeta(s, a)$ defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty} (k+a)^{-s} \quad (\Re(s) > 1;\ a \neq 0,\ -1,\ -2, \ldots), \tag{2.3}$$

which can be continued meromorphically to the whole complex $s$-plane (except for a simple pole at $s = 1$ with its residue 1) by means of the contour integral representation (see Srivastava and Choi [17, p. 89, Eq.(3)]) or various other integral representations (see Srivastava and Choi [17, pp. 90-93]).

Indeed, using the Maclaurin series expansion of $\log(1 + x)$ and (1.5) for the second and last equalities, respectively, we have

$$\begin{aligned}
\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} &= \sum_{\ell=1}^{\infty} \sum_{k=2}^{\infty} \frac{(-1)^k}{k\ell^k} \\
&= \lim_{n\to\infty} \sum_{\ell=1}^{n} \left( \frac{1}{\ell} - \log\left(1 + \frac{1}{\ell}\right)\right) \\
&= \lim_{n\to\infty} \left( \sum_{\ell=1}^{n} \frac{1}{\ell} - \log(n+1)\right) \\
&= \gamma.
\end{aligned}$$

The following representations may be proved in a similar way:

$$\gamma = 1 - \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k}; \tag{2.4}$$

$$\gamma = 1 - \log 2 + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k) - 1}{k}; \tag{2.5}$$

$$\gamma = 1 - \frac{1}{2}\log 2 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{2k+1}; \tag{2.6}$$

$$\gamma = \log 2 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{k+1}; \tag{2.7}$$

$$\gamma = \log 2 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(2k+1)\, 2^{2k}}; \tag{2.8}$$

$$\gamma = 1 - \log\frac{3}{2} - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{(2k+1)\, 2^{2k}}; \tag{2.9}$$

$$\gamma = 2 - 2\log 2 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{(k+1)(2k+1)}; \tag{2.10}$$

$$\gamma = 1 + \frac{1}{3}\log\frac{8}{15} - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{2k+1}\left(\frac{3}{2}\right)^{2k}; \tag{2.11}$$

$$\gamma = 1 - \frac{1}{2}\log 6 + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)-1}{k}\, 2^{k-1}. \tag{2.12}$$

It is noted that the above-listed representations can be proved by using the familiar result (*cf.* Whittaker and Watson [20, p. 276], Hansen [12, p. 358, Entry (54.11.1)], and Srivastava and Choi [17, p. 18]):

$$\sum_{k=2}^{\infty} \zeta(k,a)\frac{t^k}{k} = \log\Gamma(a-t) - \log\Gamma(a) + t\psi(a) \quad (|t| < |a|), \tag{2.13}$$

where the *Psi* (or *Digamma*) function $\psi(z)$ is defined by

$$\psi(z) := \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{2.14}$$

By noting that

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)(2k+1)} = 2 \log 2 - 1, \tag{2.15}$$

which can be proved by making use the following identities (see Srivastava and Choi [17, Section 1.2]):

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \frac{z}{k(z+k)} \tag{2.16}$$

and

$$\psi\left(-\frac{1}{2}\right) = 2 - \gamma - 2 \log 2, \tag{2.17}$$

the identity (2.10) is equivalent to Galisher's

$$\gamma = 1 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(k+1)(2k+1)}, \tag{2.18}$$

which is commented in the work of Ramanujan [16] who gave a general formula containing (2.18) as a very special case.

Ramanujan [15] also showed that

$$\gamma = \log 2 - 2 \sum_{k=1}^{\infty} k \sum_{j=A_{k-1}+1}^{A_k} \frac{1}{(3j)^3 - 3j}, \tag{2.19}$$

where

$$A_k := \frac{1}{2}(3^k - 1) \quad (k = 0, 1, 2, \ldots).$$

By using a simple representation of $\gamma$ as the sum of the areas of a series of trilaterals, Vacca [18] proved

$$\gamma = \frac{1}{2} - \frac{1}{3} + \sum_{k=2}^{\infty} k \sum_{i=0}^{2^k-1} \frac{(-1)^i}{2^k+i}. \tag{2.20}$$

Addison [1] gave a series representation for $\gamma$ like (2.19) and (2.20):

$$\gamma = \frac{1}{2} + \sum_{k=1}^{\infty} k \sum_{i=2^{k-1}}^{2^k-1} \frac{1}{2i(2i+1)(2i+2)}. \tag{2.21}$$

Gerst [9] derived a simpler series representation of the same type as (2.21), including several alternate forms of (2.21) and (2.22), directly from the standard definition (1.1) of $\gamma$:

$$\gamma = 1 - \sum_{k=1}^{\infty} k \sum_{i=2^{k-1}+1}^{2^k} \frac{1}{(2i-1)(2i)}. \tag{2.22}$$

Campbell [6, p. 200] recorded an interesting series representation for $\gamma$ which can easily be written in the form:

$$\gamma = 1 - \log\frac{3}{2} - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{(2k+1)^{2k}}. \tag{2.23}$$

Note that the expression of $\gamma$ in (2.23) seems to be the most rapidly convergent series among the ever-known series representations of $\gamma$ by observing $1 < \zeta(2k+1) < 2$ for each positive integer $k$ and the following rough estimations:
(2.24)

$$0 < \sum_{k=21}^{\infty} \frac{\zeta(2k+1)-1}{(2k+1)^{2k}} < 10^{-65} \quad \text{and} \quad 0 < \sum_{k=41}^{\infty} \frac{\zeta(2k+1)-1}{(2k+1)^{2k}} < 10^{-153}$$

However, when (2.23) is compared with (2.9), it is carefully concluded that the expression (2.23) is *incorrectly* recorded.

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