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FUZZY DIRECT PRODUCT IN FUZZY SPACES

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ABSTRACT Using the concept of fuzzy spaces, which was introduced by Dib. The fuzzy external and internal product of fuzzy subgroups are defined Further it is obtained the relation between the introduced concept and the direct product of fuzzy subgroups on fuzzy subsets

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [8] in 1965. The concept of fuzzy subgroups was defined by Rosenfeld [7] Anthony and Sherwood [3] reformulated the definition of fuzzy subgroups with respect to t-norm T. Foster [6] introduced the product fuzzy subgroups with respect to minimum. Abu Osman [1] reformulated the product of fuzzy subgroups with respect to T.

Dib [4] in 1994 introduced the concept of fuzzy spaces which is replacing the concept of universal set in ordinary case. Using fuzzy spaces; Dib generalized the Resonfeld-Anthony- Shorwood definition of fuzzy subgroups. This direction was studied in many works [2,4,5,6].

In this paper, we introduce the fuzzy external and internal direct product of fuzzy subgroups in fuzzy subspaces and their basic properties. Moreover we obtain their relation with the direct product of fuzzy subgroups on fuzzy subgroups, given by Abu Osman.

2. Notations and Preliminaries

In this section, we shall give some notations and preliminaries that will be used throughout this paper from [4,5].

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Zadeh in his classical paper [8] defined the fuzzy set as a function from a set X in to the closed interval $I = [0,1]$. The set I^X represents the family of all fuzzy subsets A of X.

The fuzzy space [4], denoted by (X, L) is the set of all ordered pairs $\{(x, L);\ x \in X\}$, i.e.,

$$(X, L) = \{(x, L);\ x \in X\}$$

where L is the lattice of possible membership values.

The sublattice L' of L is said to be *an M-sublattice* if it satisfies that:

(i) L' contains at least one element other than the zero element.

(ii) L' is closed under an arbitrary times of (supermum $\vee$) and (infimum $\wedge$) operations.

The fuzzy subspace U of the fuzzy space (X, L) is defined as $U = \{(x, u_x); x \in U_0\}$, where U_0 is an ordinary subset X and $u_x(x \in X)$ is in an M-sublattice. The fuzzy subset A of X is contained in the fuzzy subspace U if $A(x) \in u_x$, for $x \in U_0$ and $A(x) = 0$ if $x \notin U_0$ and we write $A \in U$. The fuzzy subspace $V = \{(x, v_x);\ x \in V_0\}$ is called *a subspace of the fuzzy subspace* $U = \{(x, u_x);\ x \in U_0\}$, and we write $V \subset U$, if (i) $V_0 \subset U_0$ and (ii) $v_x \subset u_x;\ x \in V_0$.
Let A be a fuzzy subset of the fuzzy space (X, I). The fuzzy subset A introduces the following fuzzy subspaces:

$$\underline{H}(A) = \{x, [0, A(x)]);\ x \in A_0\}$$

$$\bar{H}(A) = \{(x, \{0\} \cup [A(x), 1]);\ x \in A_0\}$$

$$H_0(A) = \{(x, \{0, A(x)\});\ x \in A_0\}$$

where A_0 is the support of A. *The algebra of fuzzy subspaces* $U = \{(x, u_x);\ x \in U_0\}$ and $V = \{(x, v_x); x \in V\}$ of the fuzzy space (X, I) is defined as follows

$$U \cup V = \{(x, u_x \cup v_x);\ x \in U_0 \cup V_0\}$$

$$U \cap V = \{(x, u_x \cap v_x);\ x \in U_0 \cap V_0\}.$$

We denote $L \diamond K$, the vector lattice $L \times K$ with partial ordered defined by:
(i) $(r_1, s_1) \leq (r_2, s_2)$ if $r_1 \leq r_2, s_1 \leq s_2$, where $r_1, r_2 \in L$, $s_1, s_2 \in K$.
(ii) $(0,0) = (r, s)$ whenever $r = 0$ or $s = 0, r \in L$ and $s \in K$.
The fuzzy Cartesian product of the fuzzy spaces (X, L) and (Y, K) is a fuzzy space, denoted by $(X, L) \diamond (Y, K)$ and is defined by

$$(X, L) \diamond (Y, K) = (X \times Y, L \diamond K).$$

$((x, y), L \diamond K)$ is called *the fuzzy element of this fuzzy space.* For any two fuzzy elements (x, L) and (y, K), we can use the notation $((x, y), L \diamond K) = (x, L) \diamond (y, K)$.

The fuzzy Cartesian product of the two fuzzy subspaces $U = \{(x, u_x);\ x \in U_0\}$ and $V = \{(x, v_x);\ x \in V_0\}$ is defined by $U \diamond V = \{((x, y), (u_x, u_y));\ (x, y) \in U_0 \times V_0\}$.

The fuzzy Cartesian product of the fuzzy subsets A of (X, L) and B of (Y, K) is defined by

$$A \diamond B = \{((x, y), (A(x), B(y))); (x, y) \in A_0 \times B_0\}$$

where A_0 and B_0 are the supports of A and B respectively. *A fuzzy function $\underline{F}$* from the fuzzy space (X, L) to the fuzzy space (Y, K) is characterized by the ordered pair $\underline{F} = (F, \{f_x\})$, where $F : X \to Y$ is a function from X to Y and $\{f_x\}$ is a family of onto comembership functions $f_x : I \to I, x \in X$ satisfying the conditions: (i) f_x is nondecreasing on I, (ii) $f_x(0) = 0$ and $f_x(1) = 1$. The action of the fuzzy function $\underline{F}$ on the fuzzy subset A is defined by

$$\underline{F}(A)y = \begin{cases} V_{x \in F^{-1}(y)} f_x(A(x)), & \text{if } F^{-1}(y) \neq \emptyset \\ 0 & \text{if } F^{-1}(y) = \emptyset. \end{cases}$$

Every fuzzy function $\underline{F} = (F, f_x) : X \to Y$, which has a continuous comembership functions $f_x; x \in X$ translates every fuzzy subspace $U = \{(x, u_x); x \in U_0\}$ to a fuzzy subspace $\underline{F}(U) = \{(y = F(x), f_x(u_x));\ y \in F(U_0)\}$ iff $f_x(u_x) = f_{x'}(u_{x'})$ for all $F(x) = F(x')$. Moreover, the fuzzy function $\underline{F} = (F, f_x)$ acts on the fuzzy element (x, L) as follows

$$\underline{F}(x,L)=(F(x),f_x(L))=(F(x),K).$$

Let U and V be fuzzy subspaces of (X,L) and (Y,K) respectively. The fuzzy function $\underline{F}=(F,f_x):X\to Y$ induces a function from U to V iff (i) $F(U_0)\subset V_0$ and (ii) $f_x(u_x)=v_{F(x)}; x\in U_0$. Let $(X,L),(Y,K)$ and (Z,J) be fuzzy spaces.

Every fuzzy function $\underline{F}=(F,f_{xy}):X\times Y\to Z$, with onto comembership functions $f_{xy}:L\diamond K\to J$, defines a function from the fuzzy spaces $(X\times Y, L\diamond K)$ to the fuzzy space (Z,J) by $\underline{F}((x,y),L\diamond K)=(F(x,y),J)$. If U,V are fuzzy subspaces of $(X,L),(Y,K)$ respectively, then $U\diamond V$ is a fuzzy subspace of $(X\times Y, L\diamond K)$ and $\underline{F}$ acts on $U\diamond V$ as follows

$$\underline{F}((x,u_x)\diamond(y,v_y))=(F(x,y),f_{xy}(u_x\diamond v_y))$$

and

$$\underline{F}(U\diamond V)=\{(F(x,y),f_{xy}(u_x\diamond v_y));(x,y)\in U_0\times V_0\}$$

is a fuzzy subspace of (Z,J) iff

$$f_{xy}(u_x\diamond v_y)=f_{x'y'}(u_{x'}\diamond v_{y'})\ \ for\ \ all\ \ F(x,y)=F(x',y').$$

If $W=\{(z,w_z);z\in W_0\}$ is a fuzzy subspace of (Z,J), then $\underline{F}$ is a fuzzy function from $U\diamond V$ into W if (i) $F(U_0\times V_0)\subset W_0$ (ii) $f_{xy}(u_x\diamond v_y)=w_{xFy}$. If $U=H_0(A), V=H_0(B)$ and $W=H_0(C)$ for some fuzzy subsets A,B and C of X,Y and Z respectively, then $\underline{F}$ translates $H_0(A)\diamond H_0(B)$ into $H_0(C)$ if $F(A_0\times B_0)\subset C_0$ and $f_{xy}(A(x),B(y))=C(F(x,y))$.

A Fuzzy binary operation is a fuzzy function $\underline{F}=(F,f_{x,y}):X\times X\to X$ with onto comembership functions f_{xy} such that $f_{xy}(r,s)\neq 0$ iff $r\neq 0$ and $s\neq 0$. The action of $\underline{F}$ on $(x,I)\diamond(y,I)$ of $(X\times X, I\diamond I)$ is

$$(x,I)\underline{F}(y,I)=(F(x,y),I)=(xFy,I).$$

A fuzzy space (X,I) with a fuzzy binary operation $\underline{F}=(F,f_{xy})$ is said to be a *fuzzy groupoid* and is denoted by $((X,I);\underline{F})$. A *fuzzy* *semi-group* is a fuzzy groupoid, which is associative. A *fuzzy monoid* is a fuzzy semi-group, which admits an identity. A *fuzzy group* is a fuzzy monoid in which each fuzzy element has an inverse.

A very important note is that to each fuzzy group $((X,I),\underline{F})$ is associated an ordinary group (X,F) and they are isomorphic to each other by the correspondence $x \leftrightarrow (x,I)$ [4]

$(U;\underline{F})$ is a fuzzy subgroup of a fuzzy group $((X,I);\underline{F})$ if the fuzzy subspace $U = \{(x,u_x); x \in U_0\}$ is closed under the fuzzy binary operation $\underline{F}$. $(U;\underline{F})$ satisfies the axioms of the ordinary group of if (U_0,F) is an ordinary group and $f_{xy}(u_x,u_y) = u_{xFy}$ for all $x,y \in U_0$.

If $U = H_0(A)$ is a fuzzy subspace induced by A and $(H_0(A);\underline{F})$ is a fuzzy subgroup, then we say "the fuzzy subset A induces fuzzy subgroups". The fuzzy subset A induces fuzzy subgroups iff (i) (A_0,F) is an ordinary subgroup (ii) $f_{xy}(A(x),A(y)) = A(xFy)$, for all $x,y \in A_0$ If A induces fuzzy subgroups, then $(\underline{H}(A);\underline{F})$ and $(\bar{H}(A),\underline{F})$ are fuzzy subgroups.

Let $((X,I),\underline{F})$ be a fuzzy group and $(U;\underline{F}); U = \{(x,u_x), x \in U_0\}$, be a fuzzy subgroup of $((X,I);\underline{F})$. The fuzzy elements (x,u_x); $x \in U_0$ of the fuzzy subgroup $(U;\underline{F})$ are not necessary associative with the fuzzy elements (x,I); $x \in X$; i.e.,

$$(\alpha\underline{F}\beta)\underline{F}\gamma \neq A\underline{F}(B\underline{F}\gamma),$$

where some of A,B and γ are fuzzy elements of U and the other are fuzzy elements of (X,I)

The fuzzy subgroup $(U;\underline{F})$ of the fuzzy group $((X,I);\underline{F})$ is called *associative in* $((X,I);\underline{F})$[5] if the fuzzy elements of U are associative with the fuzzy elements of $((X,I);\underline{F})$.

If $U = \{(x,u_x); x \in U_0\}$ is a fuzzy subgroup of the fuzzy group $((X,I);\underline{F})$, then the left and right cosets of the fuzzy subgroup U are defined by

$$(x,I)U = (x,I)\underline{F}U = \{(xFz; f_{xz}(I,u_x)); z \in U_0\}$$

$$U(x,I) = U\underline{F}(x,I) = \{(zFx; f_{zx}(u_x,I)); z \in U_0\}.$$

The fuzzy subgroup $(U;\underline{F})$ of the fuzzy group $((X,I);\underline{F})$ is called a *fuzzy normal subgroup* if [5]

(i) U is an associative in $((x,I);\underline{F})$.
(ii) $(x,I)U = U(x,I); x \in X$.

The fuzzy subset A is called *an associative in the fuzzy group* $((X,I);$ $\underline{F})$ if the fuzzy subspace $H_0(A) = \{(x,\{0,A(x)\}); A(x) \neq 0\}$ is associative in $((X,I);\underline{F})$.

If the fuzzy subspace $H_0(A)$ is associative in $((X,I);\ \underline{F})$, then $\underline{H}(A)$ and $\bar{H}(A)$ are associative also in $((X,I);\underline{F})$.

3. Fuzzy External Direct Product

Let (X,L) and (Y,K) be fuzzy spaces, with fuzzy binary operations $\underline{F} = (F, f_{xy})$ and $\underline{G} = (G, g_{xy})$ respectively. Consider the fuzzy Cartesian product $(X \times Y, L \diamond K)$. Define the fuzzy operation $\underline{M} = (M, m_{xx'yy'})$ on the fuzzy space $(X \times Y, L \diamond K)$ by using $\underline{F}$ and $\underline{G}$ as follows:

$$M = (F,G) \ \text{and} \ m_{xx'yy'} = (f_{xx'}, g_{yy'}).$$

The fuzzy operation $\underline{M} = ((F,G),(f_{xx'}, g_{yy'}))$ acts as

$$\begin{aligned}(x,y,r,s)\underline{M}(x',y',r',s') &= ((x,y)M(x',y'),(r,s)m_{xx'yy'}(r',s'))\\ &= (xFx', yGy', rf_{xx'}r', sg_{yy'}s').\end{aligned}$$

Using the properties of the comembership functions $f_{xx'}, g_{yy'}$ of the fuzzy binary operations $\underline{F}$ and $\underline{G}$, we notice that the functions $m_{xx'yy'}$ satisfy the following conditions:

1. $m_{xx'yy'}$ is non-decreasing function:

$$\begin{aligned}(r,s)m_{xx'yy'}(r',s') = (rf_{xx'}r', sg_{yy'}s') &\leq (r_1 f_{xx'}r'_1, s_1 g_{yy'}s'_1)\\ &= (r_1,s_1)m_{xx'yy'}(r',s'_1)\end{aligned}$$

$$\text{if } (r,s) \leq (r_1,s_1) \text{ and } (r',s') \leq (r'_1,s'_1).$$

2. $(0,0)m_{xx'yy'}(0,0) = 0$, $(1,1)m_{xx'yy'}(1,1) = (1,1)$
3. $(r,s)m_{xx'yy'}(r',s') = (rf_{xx'}r', sg_{yy'}s') \neq 0$ iff $r \neq 0, s \neq 0, r' \neq 0$ and $s' \neq 0$;
4. $m_{xx'yy'}$ are onto:

$$\begin{aligned}(L \diamond K)m_{xx'yy'}(L \diamond K) &= (Lf_{xx'}L, Kg_{yy'}K) \\ &= (L, K) = L \diamond K.\end{aligned}$$

Therefore $m_{xx'yy'}$ satisfies the conditions of the comembership functions of the fuzzy binary operations. Hence if $\underline{F} = (F, f_{xx'})$ and $\underline{G} = (G, g_{yy'})$ are fuzzy binary operations on (X, L) and (Y, K) respectively, then $\underline{M}$ is a fuzzy binary operation on the fuzzy Cartesian product $(X \times Y, L \diamond K)$ and we write $\underline{M} = \underline{F} \diamond \underline{G} = ((F, G), m_{xx'yy'})$. Therefore we can introduce the following:

DEFINITION 1. For every fuzzy groupoids $((X, L); \underline{F})$ and $((Y, K); \underline{G})$, there is a fuzzy groupoid $((X \times Y, L \diamond K); \underline{M})$, where $\underline{M} = \underline{F} \diamond \underline{G}$, that called *the external fuzzy direct product of the two fuzzy groupoids* $((X, L); \underline{F})$ *and* $((Y, K); \underline{G})$. We shall write

$$((X, L); \underline{F}) \oplus ((Y, K); \underline{G}) = ((X, L) \diamond (Y, K), \underline{F} \diamond \underline{G}).$$

Using the above definition and the properties of the fuzzy binary operation $\underline{F} \diamond \underline{G}$, it is interesting to have the following

THEOREM 1. *If* $((X, L); \underline{F})$ *and* $((Y, K), \underline{G})$ *are fuzzy groups, then the external fuzzy direct product* $((X, L); \underline{F}) \oplus ((Y, K); \underline{G})$ *is a fuzzy group, which is isomorphic to the group of the ordinary external direct product of the two groups* (X, F) *and* (Y, G) *by the correspondance*

$$((x, y), L \diamond K) \leftrightarrow (x, y).$$

The proof of Theorem 1 is directly obtained if we notice that

$$((x, y), L \diamond K)\underline{M}((x', y'), L \diamond K) = ((xFx', yGy'), L \diamond K)$$

Let $U = \{(x, u_x); x \in U_0\}$ *and* $V = \{(x, u_x); x \in V_0\}$ *be fuzzy subspaces of the fuzzy groups* $((X, L); \underline{F})$ *and* $((Y, K), \underline{G})$ *respectively (where* $\underline{F} = (F, F_{xy})$ *and* $\underline{G} = (G, g_{xy})$*).*

THEOREM 2. *If $(U;\underline{F})$ and $(V;\underline{G})$ are fuzzy subgroups of the fuzzy groups $((X,L);\underline{F})$ and $((Y,K);\underline{G})$ respectively, then the external fuzzy direct product of the fuzzy subgroups*

$$(U;\underline{F}) \oplus (V;\underline{G}) = (U \diamond V, \underline{F} \diamond \underline{G})$$

is a fuzzy subgroup of the external fuzzy direct product of the groups

$$((X,L);\underline{F}) \oplus ((Y,K);\underline{G}) = ((X \times Y, L \diamond K); \underline{F} \diamond \underline{G}).$$

PROOF. To prove Theorem 2, using [4] and Theorem 1, it is sufficient to prove that

(i) $U_0 \times V_0$ is an ordinary subgroup of $(X \times Y, F \times G)$

(ii) $(u_x, v_y) m_{xx'yy'} (u'_x, v'_y) = (u_{xFx'}, v_{yGy'})$.

Since $(U;\underline{F})$ and $(V;\underline{G})$ are fuzzy subgroups of the fuzzy groups $((X,L);\underline{F})$ and $((Y,K);\underline{G})$ respectively, then we have

a$_1$) (U_0, F) is an ordinary subgroup of (X, F)

a$_2$) $u_x F u_{x'} = u_{xFx'}$

and

b$_1$) (V_0, G) is an ordinary subgroup of (Y, G)

b$_2$) $v_y \ g_{yy'} \ v_{y'} = v_{yGy'}$.

From a$_1$) and b$_1$), it follows that $U_0 \times V_0$ is an ordinary group of $(X \times Y, F \times G)$ which proves (i). From a$_2$) and b$_2$) it follows that

$$\begin{aligned}(u_x, v_y) m_{xx'yy'} (u_{x'}, v_{y'}) &= (u_x f_{xx'} u_{x'}, v_y g_{yy'} v_{y'}) \\ &= (u_{xFx'}, v_{yGy'})\end{aligned}$$

and (ii) is proved.

If the fuzzy subset A induces fuzzy subgroups in the fuzzy group $((X,L);\underline{F})$, it means that the fuzzy subspaces $H_0(A)$, $\underline{H}(A)$ and $\overline{H}(A)$ are fuzzy subgroups relative to the fuzzy binary operation $\underline{F}$.

THEOREM 3. *If A and B induce fuzzy subgroups in $((X,L),\underline{F})$ and $((Y,K);\underline{G})$ respectively, then $A \diamond B$ induces fuzzy subgroups in the external fuzzy direct product $((X,L);\underline{F}) \oplus ((Y,K);\underline{G})$. Moreover $(H_0(A);\underline{F}) \oplus (H_0(B);\underline{G}) = (H_0(A \diamond B);\underline{H} \diamond \underline{G})$.*

PROOF. Let A, B be fuzzy subsets, inducing fuzzy subgroups in $((X,L);\underline{F})$ and $((Y,K);\underline{G})$ respectively. Then $(H_0(A);\underline{F})$ and $(H_0(B);\underline{G})$ are fuzzy subgroups of $((X,L);\underline{F})$ and $((Y,K);\underline{G})$ respectively (Simillarely $\bar{H}(A), \underline{H}(A), \bar{H}(B)$ and $\underline{H}(B)$ can be considered). From Theorem 2, we have

$$\begin{aligned}(H_0(A);\underline{F}) \oplus (H_0(B);\underline{G}) &= (H_0(A) \diamond H_0(B); \underline{F} \diamond \underline{G}) \\ &= (H_0(A \diamond B); \underline{F} \diamond \underline{G}).\end{aligned}$$

COROLLARY 1. *If A, B are fuzzy subsets, inducing fuzzy subgroups in $((X,L);\underline{F})$ and $((Y,K);\underline{G})$ respectively, then*

$$\underline{H}(A) \oplus \underline{H}(B) = \underline{H}(A \diamond B)$$

$$\bar{H}(A) \oplus \bar{H}(B) = \bar{H}(A \diamond B)$$

EXAMPLE 1. Consider the fuzzy group $((X,I);\underline{F})$, where $X = \{1, -1, i, -i\}$, $i^2 = -1$, $I = [0,1]$ and $\underline{F} = (F, f_{xy})$, where F is the usual multiplication and $f_{xy}(r,s) = \sqrt{f(r)f(s)}$, where

$$f(t) = \begin{cases} t, & 0 \le t \le 0.2 \\ \dfrac{1-2t}{3}, & 0.2 < t \le 0.25 \\ \dfrac{-1}{2} + \dfrac{8}{3}t, & 0.25 < t \le 0.3 \\ t, & 0.3 < t \le 0.85 \\ -5.1 + \dfrac{20}{3}t, & 0.85 < t \le 0.9 \\ t, & 0.9 < t \le 1 \end{cases}$$

It is easy to verify that $((X,I);\ \underline{F})$ is a fuzzy group. If we use the notation $A = < U_0, r >$ to define a fuzzy subset $A(x) = r;\ x \in U_0$ and $A(x) = 0;\ x \notin U_0$. Then it is easy to see that the fuzzy subsets

$$A =< \{1, -1\}, 0.2 >$$

and

$$B =< X, 0.5 >$$

induce fuzzy subgroup in $((X, I); \underline{F})$. According to the above corollary, the fuzzy subset $A \times B =< \{1, -1\} \times X, (0.2, 0.5) >$ induces fuzzy subgroups in the external fuzzy direct product of the fuzzy group $((X, I), \underline{F})$ with itself.

The external product of ordinary groups has many properties, which can be extended to the fuzzy case. Let $((X, L); \underline{F})$ and $((Y, K); \underline{G})$ be two fuzzy groups. Denote by $\tilde{X} = X \times \{f\}$ and $\tilde{Y} = \{e\} \times Y$, where e, f are the identities of the ordinary groups (X, F) and (Y, G) respectively. The proof of the following theorem can be obtained straight-forward:

THEOREM 4. *If $((X, L); \underline{F}) \oplus ((Y, K); \underline{G})$ is the external fuzzy direct product of the fuzzy groups $((X, L); \underline{F})$ and $((Y, K); \underline{G})$, then*

(i) *$((\tilde{X}, J); \underline{M})$ and $((\tilde{Y}, J); \underline{M})$ are fuzzy subgroups of the $((X, L), \underline{F}) \oplus ((Y, K); \underline{G})$.*
(ii) *$((\tilde{X}, J); \underline{M})$ and $((\tilde{Y}, J); \underline{M})$ are associative in $((X, L); \underline{F}) \oplus ((Y, K); \underline{G})$.*
(iii) *Each element of $(\tilde{X}, J)$ commuts with each element of $(\tilde{Y}, J)$.*
(iv) $(\tilde{X}, J) \cap (\tilde{Y}, J) = \{((e, f), J)\}$
(v) *each element of $((X \times Y, J); \underline{M})$ can be written as the summation of an element of $(\tilde{X}, J)$ and element of $(\tilde{Y}, J)$ and this representation is unique.*

Therefore

$$((x, y), J) = ((x, f), J)\underline{M}((e, y), J),$$

where $J = L \diamond K$.

From the above theorem, the fuzzy group $((X, L); \underline{F})$ can be embeded in the external fuzzy direct product $((X, L); \underline{F}) \oplus ((Y, K); \underline{G})$ by the correspondence

$$\underline{i}_X = (i_X, i_x) : (X, L) \to (X \times Y, J),$$

where

$$i_X : X \to X \times Y; \quad i_X(x) = (x, f); \quad x \in X$$

$$\imath_x : I \to J; \quad \imath_x(r) = (r, r), \ r \in I, \ x \in X,$$

and f is the identity element of (Y, G). $\underline{i}_X$ is called the fuzzy natural injection of (X, L) into $(X \times Y, J)$.

Similarly (Y, K) is embeded in $(X \times Y, J)$ by the fuzzy natural injection

$$\underline{i}_Y = (i_Y, i_y) : (Y, K) \to (X \times Y, J);$$

$$\imath_Y : Y \to X \times Y; \quad i_Y(y) = (e, y)$$

$$\imath_y : I \to J; \quad \imath_y(r) = (r, r)$$

where e is the identity element of the group (X, F).
The fuzzy projection of $(X \times Y, J)$ on (X, L) is defined by

$$\underline{\Pi}_X = (\Pi_X, \pi_x) : (X \times Y, J) \to (X, L),$$

where

$$\Pi_X : X \times Y \to X, \ \ \pi_x(x, y) = x$$

$$\pi_x, J \to L; \quad \pi_x(r, s) = \sqrt{rs}$$

Similarly the fuzzy projection of $(X \times Y, J)$ into (Y, K) is defined by

$$\underline{\Pi}_Y : (X \times Y, J) \to (Y, K),$$

$$\Pi_Y : X \times Y \to Y; \ \ \Pi_Y(x, y) = y,$$

$$\pi_y : J \to K; \ \ \pi_y(r, s) = \sqrt{rs}.$$

Foster in [6] introduced the concept of product fuzzy subgroups. Abu Osman in [1] generalized this concept using the t-norm T and he obtained the following result;

THEOREM 5 [1]. *Let G_1, G_2 be groups and $G = G_1 \times G_2$ be the external direct product of G_1 and G_2. Let μ_1 be a (classical) fuzzy subgroup of G_1 and μ_2 be a classical fuzzy subgroup of G_2 with respect to T. Then $\mu = \mu_1 \times \mu_2$ is a classical fuzzy subgroup of G with respect to T, where $\mu(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2))$.*

Now we show that this result can be embeded in our fuzzy external direct product.

LEMMA 1. *1. Any fuzzy subset $C : G_1 \times G_2 \to I$ can be embeded as a fuzzy subset $C^* : G_1 \times G_2 \to J$ as $C^*(x, y) = (C(x, y), C(x, y))$.*

2. If $\mu_1 : G_1 \to I$ and $\mu_2 : G_2 \to I$ are classical fuzzy subsets of G_1 and G_2 respectively. Then, for $C = \mu_1 \times \mu_2, C^*$ is classical J-fuzzy subgroup of the group $G_1 \times G_2$. Since $C^*(x_1, x_2) = (C(x_1, x_2), (C(x_1, x_2)) = (T(\mu_1(x_1), \mu_2(x_2)),\ T(\mu_1(x_1), (\mu_2(x_2)))$.
Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ where $x_1, y_1 \in G_1$ and $x_2, y_2 \in G_2$. Then using Theorem 5

$$\begin{aligned} C^*(xy^{-1}) &= (C(xy^{-1}), C(xy^{-1})) \\ &\geq (T(\mu(x), \mu(y)), T(\mu(x), \mu(y))) = T^*(\mu(x), \mu(y)). \end{aligned}$$

4. Internal Fuzzy Direct Product

Let $((X, I); \underline{F})$ be a fuzzy group and $U = \{(x, u_x); x \in U_0\}$ and $V = \{(x, v_x);\ x \in V_0\}$ be fuzzy subspaces.

DEFINITION 2. Let $(U; \underline{F})$ and $(V; \underline{F})$ be fuzzy subgroups of the fuzzy group $((X, I); \underline{F})$ where $\underline{F} = (F, f_{xy})$. Then U and V are said to *have fuzzy internal product* if (i) $W = U\underline{F}V = UV$ is a fuzzy subgroup (ii) the correspondence $\underline{S} : U \oplus V \to W$; where $(x, u_x)\underline{S}(y, v_y) = (xFy, u_x f_{xy} v_y)$ is a homomorphism. And we write $W = U \odot V$.

Using Theorem 4 and Definition 2, we get the following result.

THEOREM 6. *If the fuzzy groups U and V have the internal fuzzy product $G = U \odot V$ in the fuzzy group $((X, I), \underline{F})$, then*

(i) *U and V have the same identity element*

$$E = (e, \varepsilon_x) \text{ and } U \cap V = \{E\}.$$

(ii) *Each fuzzy element of U commutes with each fuzzy element of V.*
(iii) *The collection of the fuzzy elements of U and V is associative.*
(iv) *Each fuzzy subgroup (U or V) is associative in G.*
(v) *Each element $(z, g_z) \in G$ can be written in a unique way: $(z, g_z) = (x, u_x)\underline{F}(y, v_y) = (xFy, u_x f_{xy} v_y)$.*

COROLLARY 2. *If the fuzzy subgroups U and V have fuzzy internal product $U \odot V$, then*

(i) $u_x \cap v_x = \{0\}$ *for all* $x \in X$ *and* $x \neq e$.
(ii) $U \subset G$ *and* $V \subset G$.

It is interesting to notice that some essential reasults for the internal product in the ordinary case are still true for the fuzzy case.

THEOREM 7. *If* $((X, I); \underline{F}) = U \odot V$, *for some groups U and V, then U and V are fuzzy normal subgroups.*

PROOF. Using Theorem 5 and the known properties of homomorphisms we get:

1. $U($ and $V)$ is associative in $((X, I); \underline{F})$.
2. The collection of fuzzy elements of U and V is associative.
3. For every fuzzy elements of $u \in U$ and $(x, I) \in (X, I)$ we have: (x, I) can be written in a unique way as

$$(x, I) = u_0 \underline{F} v_0$$

for some $u_0 \in U$, $v_0 \in V$, then for any $u \in U$ we get

$$\begin{aligned}(x, I)\underline{F}u &= (u_0 \underline{F} v_0)\underline{F}u = u_0 \underline{F}(v_0 \underline{F} u) = u_0 \underline{F}(u \underline{F} v_0) \\ &= ((u_0 \underline{F} u)\underline{F} u_0^{-1})\underline{F}(u_0 \underline{F} v_0) = u_1 \underline{F}(x, I).\end{aligned}$$

i.e.,

$$(x, I)U = U(x, I).$$

From 1,2 and 3, it follows that U is normal fuzzy group in G.

THEOREM 8. *If U and V are fuzzy subgroups of the fuzzy group* $((X, I); \underline{F})$, *for which*

(i) $U \cap V = \{(e, u_e)\}$, *where (e, u_e) is the a common identity between U and V.*
(ii) $U\underline{F}V = (X, I)$.
(iii) *The collection of fuzzy elements of U and V is associative.*
(iv) *U and V are normal fuzzy subgroups.*

Then U, V have an inner product and $U \odot V = (X, I)$.

PROOF. Any fuzzy element (x, I) can be written in a unique way as $(x, I) = uv;\ \ u \in U, v \in V$. Since if $uv = u_1v_1$, then $v_1v^{-1} = u^{-1}u_1 \doteq (e, u_e)$. It follows $u = u_1$ and $v = v_1$.
Define $H : (X, I) \to U \diamond V$

$$H(x, I) \to (u, v) \quad \text{where} \quad (x, I) = u\underline{F}v.$$

This function is well defined by the previous discussion, which also shows that H is one-to-one correspondance. To prove the theorem it is sufficient to show that H is an isomorphism. H is a homorphism between the two fuzzy groups $((X, I); \underline{F})$ and $U \diamond V$: Let $(x_1, I) = u_1v_1$ and $(x_2, I) = u_2v_2$. Then $(x_1, I)(x_2, I) = u_1v_1u_2v_2$. Now we show thar $v_1u_2 = u_2v_1$ for every $u_2 \in U$ and $v_1 \in V$. Since $(v_1u_2v_1^{-1})u_2^{-1} \in U$, since U is normal and $v_1(u_2v_1^{-1}u_2^{-1}) \in V$ since V is normal. But $U \cap V = (e, u_e)$, then $v_1u_2v_1^{-1}u_2^{-1} = (e, u_e)$, $v_1u_2 = u_2v_1$. Therefore

$$\begin{aligned} H((x_1, I)\underline{F}(x_2, I)) &= \underline{F}(u_1v_1u_2v_2) = \underline{F}(u_1u_2v_1v_2) \\ &= (u_1u_2, v_1v_2) = (u_1, v_1)(u_2, v_2) \\ &= H(x_1, I)\underline{F}H(x_2, I). \end{aligned}$$

Theorem is proved.

Let A and B be fuzzy subsets, inducing fuzzy subgroups. As shown in [5] the fuzzy subset A induces fuzzy normal subgroups if $\underline{H}(A)$ is a fuzzy normal subgroup. If $((X, I); \underline{F})$ is a uniform fuzzy group, where $\underline{F} = (F, f)$ and f is a t-norm, then $\underline{H}(A)$ is a fuzzy normal subgroup if

(i) (A_0, F) is an ordinary normal subgroup in (X, F).
(ii) $A(xFy) = f(A(x), A(y));\ \ x, y \in A_0$.
(iii) $A(z) = A(z')$ for all $F(z) = F(z')$.

DEFINITION 3. If A, B are fuzzy subsets of the uniform fuzzy group $((X, I); \underline{F})$ with t-norm comembership function, then we say that A, B induce inner fuzzy product if $(X, I) = \underline{H}(A) \odot \underline{H}(B)$.

Theorem 6 can be reformulated for the fuzzy subsets, which induce inner fuzzy product as follows

THEOREM 9. *If G is a fuzzy subgroup, with a uniform fuzzy binary operation, having t-norm comembership function; and A, B are fuzzy subsets which induce in G inner fuzzy product, then*

(i) $\underline{H}(A), \underline{H}(B)$ *have the same identity element i.e.,* $A(e) = B(e)$.
(ii) $xFy = yFx$ *and* $A(x)fB(y) = B(y)fA(x)$, *for all* $x \in A_0$, $y \in B_0$.
(iii) $A(x) \cdot B(x) = 0$ *for all* $x \in X$.

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REFERENCES

[1] M T. Abu Osman, *On the direct product of fuzzy subgroups*, Fuzzy Sets and Systems **12** (1984), 87–91.
[2] M. Alghamdi, *Notes On Fuzzy Subgroups*, The Pusan-Kyongnam Math. J **13(2)** (1998)
[3] J M. Anthony and H Sherwood, *Fuzzy groups redefined*, J Math. Anal. Appl **69** (1979), 124–130.
[4] K A. Dib, *On Fuzzy spaces and Fuzzy Group Theory*, Information Sciences **80(3-4)** (1994), 253–282.
[5] K.A Dib, N. Galhum and A.A.M Hassan, *Fuzzy ring and Fuzzy ideals*, Journal of Fuzzy Mathematics **4(2)** (1996), 245–261
[6] K A. Dib and A A Hassan, *The Fuzzy normal subgroups*, Fuzzy Sets and Systems **98(3)** (1998), 393–402.
[7] D H. Foster, *Fuzzy Topological groups*, J Math Anal. Appl **67** (1979), 549–564
[8] A. Rosenfeld, *Fuzzy Groups*, J Math Anal Appl **35** (1971), 512–517.
[9] L A. Zadeh, *Fuzzy sets*, Inform and Control **8** (1965), 338–353.

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