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SOME RESULTS ON A NONUNIQUE FIXED POINT

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ABSTRACT. In this paper, we obtain some nonunique fixed point theorems of single valued and multivalued maps in metric and generalized metric spaces, one of which generalized the corresponding results of [5] and [6].

1. Introduction

In [6], Pachpatte obtained some results on a nonunique fixed point complete metric spaces and introduced an inequality as follows;

$$\begin{aligned} &\min\{[d(Tx,Ty)]^2, d(x,y)d(Tx,Ty), [d(y,Ty)]^2\} \\ &\quad - \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \\ &\leqslant r \cdot d(x,Tx)d(y,Ty) \end{aligned} \tag{1.1}$$

for any x, y in X, where r is in $(0,1)$.

In [5], Liu generalized the above result for single valued maps and introduced the following;

$$\begin{aligned} &\min\{[d(Tx,Ty)]^2, d(x,y)d(Tx,Ty), d(x,y)d(y,Ty), \\ &d(x,Tx)d(Tx,Ty), [d(y,Ty)]^2\} \\ &- \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \\ &\leqslant\ r \cdot \max\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \end{aligned}$$

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for any x, y in X, where r is in $(0,1)$.

In the present paper, we obtain some results which generalize Theorem 1 of [6] and Theorem 1 of [5]. Furthermore, we give an example to show that our result indeed generalizes Theorem 1 of [6]. By the way, we show the example in [2] is false.

2. On a nonunique fixed point for single valued maps

Let (X,d) be a metric space and T a self map of X. T is called an orbitally continuous if $\lim_i T^{n_i}x = u$ implies that $\lim_i TT^{n_i}x = Tu$ for each x in X. A metric space X is T-orbitally complete if every Cauchy sequence of the form $\{T^{n_i}x\}_{i\geqslant 1}$ converges in X for x in X. Throughout this paper $\mathbb{R}^+$ denotes the set of nonnegative real numbers.

THEOREM 2.1. *Let (X,d) be a T-orbitally complete metric space and T an orbitally continuous self map of X. If T satisfies the following condition*

$$\begin{aligned}
&\min\{[d(Tx,Ty)]^2, d(x,y)d(Tx,Ty), d(x,y)d(y,Ty),\\
&\quad d(x,Tx)d(Tx,Ty), d(x,Tx)d(y,Ty), d(y,Ty)d(Tx,Ty), [d(y,Ty)]^2\}\\
&\quad - \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\}\\
&\leqslant r\cdot\max\{d(x,y)d(Tx,Ty), d(x,y)d(y,Ty), d(x,Tx)d(Tx,Ty),\\
&\quad d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx), d(y,Tx)d(Tx,Ty), d(y,Tx)d(y,Ty)
\end{aligned} \tag{2.1}$$

for any x, y in X, where r is in $(0,1)$.

Then T has a fixed point and for each x in X the sequence $\{T^nx\}_{n\geqslant 1}$ converges to a fixed point of T.

PROOF. Let x be in X. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for $n\geqslant 0$, where $x_0 = x$. If $x_n = x_{n+1}$ for some $n\geqslant 0$, then the assertion follows immediately. Therefore we assume that $x_n \neq x_{n+1}$ for each $n\geqslant 0$. Put $d_n = d(x_n, x_{n+1})$ for $n\geqslant 0$. By (2.1) we obtain

$$\begin{aligned}
&\min\{(d_{n+1})^2, d_nd_{n+1}, d_nd_{n+1}, d_nd_{n+1}, d_nd_{n+1}, (d_{n+1})^2, (d_{n+1})^2\}\\
&\quad - \min\{d_nd_{n+1}, d(x_n,x_{n+2})d(x_{n+1},x_{n+1})\}\\
&\leqslant r\cdot\max\{d_nd_{n+1}, d_nd_{n+1}, d_nd_{n+1}, d_nd_{n+1}, d(x_n,x_{n+2})d(x_{n+1},x_{n+1}),\\
&\qquad d(x_{n+1},x_{n+1})d_{n+1}, d(x_{n+1},x_{n+1})d_{n+1}\}
\end{aligned}$$

i.e.,

$$(d_{n+1})^2 = \min\{(d_{n+1})^2, d_n d_{n+1}\} \leqslant r \cdot d_n d_{n+1}$$

which implies that $d_{n+1} \leqslant r \cdot d_n$. It is easy to see that $\{x_n\}_{n\geqslant 1}$ is a Cauchy sequence. Since X is orbitally complete, there exists some u in X such that $u = \lim_n T^n x$. By the T-orbitally continuity of T, $Tu = \lim_n TT^n x = u$. This completes the proof.

REMARK 2.1. Theorem 2.1 extend Theorem 1 of [6] and Theorem 1 of [5]. The following example shows that Theorem 2.1 is a proper generalization of Theorem 1 of [6].

EXAMPLE 2.1. Let $X = \{0,1,2,3,4\}, d(x,y) = d(y,x)$ for all x, y in X, $d(x,y) = 0$ if any only if $x = y$ and $d(0,1) = 1$, $d(0,2) = 2.5$, $d(0,3) = 1$, $d(0,4) = 1$, $d(1,2) = 1.5$, $d(1,3) = 2$, $d(1,4) = 1$, $d(2,3) = 2$, $d(2,4) = 1.5$, $d(3,4) = 1$. Obviously, (X, d) is a complete metric space Now let $T : X \to X$, $T0 = 1$, $T1 = 0$, $T2 = 3$, $T3 = 2$, $T4 = 4$. It is easy to verify that the conditions of Theorem 2.1 are satisfied for $r = 0.3$. But Theorem 1 of [6] is not applicable, because T doesn't satisfy (1.1) for $x = 0$, $y = 1$ and all r in $(0,1)$.

REMARK 2.2. In 1990, Ciric [2] gave an example to show that the corresponding results of Dhage [3], Mishra [4] and Pathak [7] are false. Unfortunately the example is false. In fact, through strictly examining the proofs of Dhage, Pathak and Mishra's results we assert that the result of [3], [4] and [7] are true.

Mishra [4], Dhage [3], Pathak [7] assume that T satisfies respectively the following conditions (A), (B) and (C).

$$\begin{aligned} &\min\{d(Tx,Ty), d(x,Tx), d(y,Ty), d(Tx,T^2x), d(y,T^2x)\} \\ &- \min\{d(x,Ty), d(y,Tx), d(x,T^2x), d(Ty,T^2x)\} \leqslant q \cdot d(x,y) \end{aligned} \tag{A}$$

for all x, y in X, where $0 \leqslant q < 1$;

$$\begin{aligned} &\min\{d(Tx,Ty), d(x,Tx), d(y,Ty)\} + a \cdot \min\{d(x,Ty), d(y,Tx)\} \\ &\leqslant q \cdot d(x,y) + p \cdot d(x,Tx) \end{aligned} \tag{B}$$

for all x, y in X, where $0 < p + q < 1$, a is a real number;

$$\begin{aligned} &\min\{d(Tx,Ty), d(y,Ty)\} + a \cdot \min\{d(x,Ty), d(y,Tx)\} \\ &\leqslant q \cdot d(x,y) + p \cdot d(x,Tx) + r \cdot d(x,Ty) \end{aligned} \tag{C}$$

for all x, y in X, where a, p, q and r are numbers such that $0 \leqslant r < 1$, $0 < p + q + 2r < 1$.

The example of Ciric [2] is as follows:

Let $M = \{0,1,3\}$ with the usual metric $d(x,y) = |x - y|$. Define the mapping T by $T0 = 1$, $T1 = 3$, $T3 = 0$. Ciric [2] claimed that T satisfies each of conditions (A), (B) and (C). We find that T doesn't satisfy any one of (A), (B) and (C), because if T satisfies (A), taking $x = 0$, $y = 1$, we have from (A)

$$\begin{aligned} &\min\{d(1,3), d(0,1), d(1,3), d(1,3), d(1,3)\} \\ &- \min\{d(0,3), d(1,1), d(0,3), d(3,3)\} \\ &\leqslant q \cdot d(0,1) \end{aligned}$$

i.e., $1 \leqslant q$. This contradicts the condition $0 \leqslant q < 1$; if T satisfies (B), similarly we have $1 \leqslant q + p < 1$, which is a contradiction, too; if T satisfies (C), we have $2 \leqslant p+q+3r$. Since $0 < p+q+2r < 1$, it follows that $2 \leqslant p + q + 3r < r + 1 < 2$, which is impossible.

THEOREM 2.2. *Let (X,d) be a T-orbitally complete metric space and T an orbitally continuous self map of X. If T satisfies the following condition*

$$\begin{aligned} &a_1[d(Tx,Ty)]^2 + a_2 d(x,y)d(Tx,Ty) + a_3 d(x,y)d(y,Ty) \\ &+ a_4 d(x,Tx)d(Tx,Ty) + a_5 d(x,Tx)d(y,Ty) + a_6 d(y,Ty)d(Tx,Ty) \\ &+ a_7[d(y,Ty)]^2 - \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \\ &\leqslant r \cdot \max\{d(x,y)d(Tx,Ty), d(x,y)d(y,Ty), d(x,Tx)d(Tx,Ty) \\ &, d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx), d(y,Tx)d(Tx,Ty), d(y,Tx)d(y,Ty \end{aligned} \tag{2.2}$$

for all x, y in X, where $\sum_{i=1}^{7} a_i < 1$ and a_i is in R^+ for $i = 1, 2, \cdots, 7$. Then T has a fixed point and the sequence $\{T^n x\}_{n \geqslant 0}$ converges to a fixed point of T for x in X.

PROOF. Note that (2.2) implies (2.1). Theorem 2.2 follows immediately from Theorem 2.1.

3. On a nonunique fixed point for multivalued maps

We recall that (X, d) is a generalized metric space if X is a set and $d : X \times X \to \mathbb{R}^+ \cup \{\infty\}$ satisfies all the properties of being a metric for X besides that d may have "infinite values". An orbit of F at the point x in X is a sequence $\{x_n : x_n \in Fx_{n-1}\}$, where $x_0 = x$. A multivalued map F on X is orbitally upper-semicontinuous if $x_n \to u \in X$ implies $u \in Fu$, whenever $\{x_n\}$ is an orbit of F at each x in X. A space X is F-orbitally complete if every orbit of F at all x in X which is a Cauchy sequence, converges in X. Let A and B be nonempty subsets of X. Denote

$$
\begin{aligned}
&D(A,B) = \inf\{d(a,b) : a \in A, b \in B\} \\
&CL(X) = \{A : A \subset X, A \text{ is closed}\} \\
&N(\varepsilon, A) = \{x \in X : d(x,a) < \varepsilon \text{ for some } a \in A\},\ \varepsilon > 0 \\
&H(A,B) = \begin{cases} \inf\{\varepsilon > 0 : A \subseteq N(\varepsilon, B) \text{ and } B \subseteq N(\varepsilon, A)\}, \\ \qquad\qquad \text{if the infimum exists.} \\ \infty, \text{ otherwise.} \end{cases}
\end{aligned}
$$

Ciric [1] introduced the following inequality:

$$
\begin{aligned}
&\min\{H(Fx,Fy), D(x,Fx), D(y,Fy)\} \\
&\quad - \min\{D(x,Fy), D(y,Fx)\} \leqslant q \cdot d(x,y)
\end{aligned}
$$

for all x, y in M and some $q < 1$. Motivated by it, we obtain the following results.

THEOREM 3.1. *Let (X, d) be a generalized metric space and $F : X \to CL(X)$ an orbitally upper-semicontinuous. If X is F-orbitally* *complete and F satisfies the following condition*

$$\begin{aligned}
&\min\{[H(Fx,Fy)]^2, d(x,y)H(Fx,Fy), d(x,y)D(y,Fy),\\
&\quad D(x,Fx)H(Fx,Fy), D(x,Fx)D(y,Fy), D(y,Fy)H(Fx,Fy),\\
&\quad [D(y,Fy)]^2\}\\
&\quad - \min\{D(x,Fx)D(y,Fy), D(x,Fy)D(y,Fx)\}\\
&\leqslant r\cdot\max\{d(x,y)d(Fx,Fy), d(x,y)D(y,Fy), D(x,Fx)D(Fx,Fy),\\
&\quad D(x,Fx)D(y,Fy), D(x,Fy)D(y,Fx), D(y,Fx)H(Fx,Fy),\\
&\quad D(y,Fx)D(y,Fy)\}
\end{aligned} \tag{3.1}$$

for all x, y in X, where r is in $(0,1)$.

Then F has a fixed point.

PROOF. Let $a > 0$ be a real number less than $1/2$. We define a single valued map $T : X \to X$ by letting $Tx = y \in Fx$ that satisfies

$$d(x,y) \leqslant r^{-a}\cdot D(x,Fx). \tag{3.2}$$

Set $d_n = d(x_{n-1},x_n), D_n = D(x_n,Fx_n)$ and $H_n = H(FX_{n-1},FX_n)$ for $n \geqslant 0$. Now let's consider the following orbit of F at x in $X : x_0 = x$, $x_n = Tx_{n-1}$ for $n \geqslant 0$. We may assume that $x_{n-1} \neq x_n$ for any $n \geqslant 0$, otherwise the result is obtained at once. It follows from $x_n \in Fx_{n-1}$ that $D_n \leqslant H_n$, $D(x_n,Fx_{n-1}) = 0$ and $D_{n-1} \leqslant d_n$. By (3.1) we have

$$\begin{aligned}
&\min\{H_n^2, d_nH_n, d_nD_n, D_{n-1}H_n, D_{n-1}D_n, D_nH_n, D_n^2\}\\
&\quad - \min\{D_{n-1}D_n, D(x_{n-1},Fx_n), D(x_n,Fx_{n-1})\}\\
&\leqslant r\cdot\max\{d_nD(Fx_{n-1}.Fx_n), d_nD_n, D_{n-1}D(Fx_{n-1},Fx_n), D_{n-1}D_n,\\
&\quad D(x_{n-1},Fx_n)D(x_n,Fx_{n-1})H_n, D(x_n,Fx_{n-1})D_n\}
\end{aligned}$$

which implies that

$$\begin{aligned}
\min\{D_n^2, D_{n-1}D_n\} &= \min\{d_n^2, d_nD_n, D_{n-1}D_n\}\\
&\leqslant r\cdot\max\{d_nD_n, D_{n-1}D_n\} = r\cdot d_nD_n
\end{aligned}$$

and by (3.2)

$$\min\{r^{-2a} \cdot D_n^2, r^{-2a} \cdot D_{n-1}D_n\} \leqslant r^{-2a} \cdot d_nD_n \leqslant r^{-2a} \cdot d_nd_{n+1}.$$

On using (3.2)

$$\min\{d_{n+1}^2, d_nd_{n+1}\} \leqslant r^{1-2a} \cdot d_nd_{n+1}.$$

Note that $0 < r^{1-2a} < 1$. If $d_n < d_{n+1}$, then

$$d_nd_{n+1} = \min\{d_{n+1}^2, d_nd_{n+1}\} \leqslant r^{1-2a} \cdot d_nd_{n+1} < d_nd_{n+1}$$

a contradiction. Therefore $d_{n+1} < d_n$ and

$$d_{n+1}^2 = \min\{d_{n+1}^2, d_nd_{n+1}\} \leqslant r^{1-2a} \cdot d_n^2$$

i.e., $d_{n+1} \leqslant b \cdot d_n$, where $b = r^{\frac{1}{2}-a}$. This implies $\{x_n\}_{n\geqslant 1}$ is a Cauchy sequence. Since X is F-orbitally complete, there exists some point u in X such that $\lim_n x_n = u$. Thus the orbitally upper-semicontinuity of F implies $u \in Fu$. This completes the proof.

THEOREM 3.2. *Let (X, d) be a generalized metric space and $F : X \to CL(X)$ an orbitally upper-semicontinuous. If X is F-orbitally complete and F satisfies the following condition*

$$\begin{aligned}
&a_1[H(Fx,Fy)]^2 + a_2d(x,y)H(Fx,Fy) + a_3d(x,y)D(y,Fy)\\
&+ a_4D(x,Fx)H(Fx,Fy) + a_5D(x,Fx)D(y,Fy)\\
&+ a_6D(y,Fy)H(Fx,Fy) + a_7[D(y,Fy)]^2\\
&- \min\{D(x,Fx)D(y,Fy), D(x,Fy)D(y,Fx)\}\\
\leqslant\ & r \cdot \max\{d(x,y)D(Fx,Fy), d(x,y)D(y,Fy),\\
&\quad D(x,Fx)D(y,Fy), D(x,Fy)D(y,Fx), D(y,Fx)H(Fx,Fy),\\
&\quad D(x,Fx)D(Fx,Fy), D(y,Fx)D(y,Fy)\}
\end{aligned} \tag{3.3}$$

for all x, y in X, where $\sum_{i=1}^{7} a_i < 1$ and a_i is in R^+ for $i = 1, 2, \cdots, 7$.

Then F has a fixed point.

PROOF. Since (3.3) implies (3.1), Theorem 3.2 follows immediately from Theorem 3.1.

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