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A STARLIKENESS CONDITION ASSOCIATED WITH THE RUSCHEWEYH DERIVATIVE

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ABSTRACT Some Miller-Mocanu type arguments are used here in order to establish a general starlikeness condition involving the familiar Ruscheweyh derivative Relevant connections with the various known starlikeness conditions are also indicated. This paper concludes with several remarks and observations in regard especially to the nonsharpness of the main starlike condition presented here

1. Introduction

Let $\mathcal{A}$ denote the class of functions f *normalized* by

$$f(0) = f'(0) - 1 = 0,$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let $\mathcal{S}^*(\gamma)$ denote the subclass of $\mathcal{A}$ consisting of functions which are *starlike of order* γ in $\mathbb{U}$ $(0 \leqq \gamma < 1)$. As usual, we have

$$\mathcal{S}^* := \mathcal{S}^*(0) \tag{1.1}$$

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for the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$. The general class $\mathcal{S}^*(\gamma)$ can be characterized as follows:

$$f \in \mathcal{S}^*(\gamma) \Longleftrightarrow \Re\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \tag{1.2}$$

$$(z \in \mathbb{U};\ f \in \mathcal{A};\ 0 \leqq \gamma < 1).$$

The familiar Ruscheweyh derivative operator $D^\lambda : \mathcal{A} \to \mathcal{A}$ of order λ is defined, in terms of the Hadamard product (or convolution), in the form (*cf.* [7]):

$$D^\lambda f(z) := \left(\frac{z}{(1-z)^{\lambda+1}}\right) * f(z) \qquad (\lambda \geqq -1;\ z \in \mathbb{U}), \tag{1.3}$$

which readily implies that

$$D^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ z \in \mathbb{U}), \tag{1.4}$$

$\mathbb{N}$ being the set of *positive* integers. In fact, Ruscheweyh [7] made use of the derivative operator D^n in order to derive new criteria for univalence for functions in $\mathcal{A}$. Subsequently, while considering a problem of Ruscheweyh and other related results (*cf.* [7]; see also [8]), Obradović [6] established the following criteria for starlikeness.

THEOREM 1 (Obradović [6, p. 229, Theorem 10]). *Let* $f \in \mathcal{A}$, $\alpha \geqq 0$, $\alpha + \beta \geqq 0$, *and* $n \in \mathbb{N}_0$. *If*

$$\left|\frac{D^{n+2} f(z)}{D^{n+1} f(z)} - 1\right|^\alpha \cdot \left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right|^\beta < \frac{1}{(2n+3)^\beta (2n+4)^\alpha} \quad (z \in \mathbb{U}), \tag{1.5}$$

then $f \in \mathcal{S}^*$.

For special choices of n, α, and β, Theorem 1 yields several criteria for starlikeness. Thus, as already observed by Obradović [6, p. 229, Corollary 6], each of the following three conditions:

$$\left|\frac{zf''(z)}{f'(z)}\right|^\alpha \cdot \left|\frac{zf'(z)}{f(z)} - 1\right|^{1-\alpha} < \frac{1}{3}\left(\frac{3}{2}\right)^\alpha \qquad (z \in \mathbb{U};\ \alpha \geqq 0), \tag{1.6}$$

$$\left|\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)}-1\right)\right|<\frac{1}{6} \qquad (z\in\mathbb{U}), \tag{1.7}$$

and

$$\left|\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)}-1\right)^{-1}\right|<\frac{3}{2} \qquad (z\in\mathbb{U}) \tag{1.8}$$

implies that $f\in\mathcal{S}^*$.

Next, by applying Jack's lemma, Li and Srivastava [2] extended the conditions (1.6), (1.7), and (1.8), and some other related results, to the following form:

THEOREM 2 (Li and Srivastava [2, p. 106, Theorem 3]). *Let* $f\in\mathcal{A}$, $\alpha\geqq 0$, $\alpha+\beta\geqq 0$, *and* $\frac{1}{2}\leqq\gamma<1$. *If*

$$\left|\frac{zf''(z)}{f'(z)}\right|^{\alpha}\cdot\left|\frac{zf'(z)}{f(z)}-1\right|^{\beta}<2^{\alpha}(1-\gamma)^{\alpha+\beta} \qquad (z\in\mathbb{U}), \tag{1.9}$$

then $f\in\mathcal{S}^*(\gamma)$.

The object of the present note is to make use of some Miller-Mocanu type arguments (*cf.* [4]) in order to establish a substantially more general result than Theorem 1. We also indicate the relevant connections of our main result (Theorem 3 below) with the starlikeness conditions asserted by (for example) Theorem 1 and Theorem 2. In the concluding section (Section 4), we present several remarks and observations dealing especially with the fact that our main starlike condition (Theorem 3 below) is not sharp in general.

2. A Set of Lemmas

Let $\mu\in\mathbb{C}$ with $\Re(\mu)>-1$ and define

$$\lambda(\rho,\mu):=\inf_{z\in\mathbb{U}}\left\{\Re\big(H(z)\big)\right\} \tag{2.1}$$

$$(-\Re(\mu)\leqq\rho<1),$$

where

$$H(z)=\frac{(1-z)^{2(\rho-1)}}{\int_0^1 t^{\mu}(1-zt)^{2(\rho-1)}\,dt}-\mu. \tag{2.2}$$

Then it is known that (*cf.* [3, p. 88])

$$\lambda(\rho,\mu) \geqq \rho \qquad (-\Re(\mu) \leqq \rho < 1).$$

Moreover, in the case when μ is real and

$$\rho \geqq \max\left\{-\mu, -\frac{1}{2}\mu\right\} \qquad (\mu \in \mathbb{R}),$$

the value of $\lambda(\rho,\mu)$ is given by (*cf.* [3, p. 88])

$$\lambda(\rho,\mu) = H(-1) = \frac{(\mu+1)\,2^{-2(1-\rho)}}{{}_2F_1\big(2(1-\rho),\mu+1;\mu+2;-1\big)} - \mu, \qquad (2.3)$$

where ${}_2F_1$ denotes the Gauss hypergeometric function.

The following results (given by Lemma 1 and Lemma 2) will be required in our present investigation.

LEMMA 1 (Li *et al.* [3, p. 88, Theorem 1]). *If*

$$\frac{1}{n+1} \leqq \delta < 1 \qquad (n \in \mathbb{N}),$$

then

$$\Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \delta \Longrightarrow \Re\left\{\frac{D^n f(z)}{D^{n-1}f(z)}\right\} > \frac{\lambda(\rho,n-1)+n-1}{n} \qquad (2.4)$$

$$(z \in \mathbb{U};\;\; \rho = (n+1)\delta - n;\;\; n \in \mathbb{N}).$$

This result is sharp.

LEMMA 2. *Let* Ω *be a set in the complex plane* $\mathbb{C}$. *Suppose also that the function*

$$\Phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$$

satisfies the following condition:

$$\Phi(ix,y;z) \notin \Omega \qquad \left(z \in \mathbb{U};\;\; x,y \in \mathbb{R} \text{ and } y \leqq -\frac{1}{2}\left(1+x^2\right)\right). \qquad (2.5)$$

If $p(z)$ *is analytic in* $\mathbb{U}$ *with*

$$p(0) = 1 \text{ and } \Phi\big(p(z), zp'(z); z\big) \in \Omega \qquad (z \in \mathbb{U}), \qquad (2.6)$$

then

$$\Re\{p(z)\} > 0 \qquad (z \in \mathbb{U}).$$

REMARK 1. Lemma 2 is a simple consequence of much more general results considered in the work of Miller and Mocanu (*cf.*, *e.g* , [4]).

3. The Main Result and Its Consequences

In order to state our main result (Theorem 3 below) as simply as possible, we begin by introducing the following definitions and notations.

First of all, in terms of $\lambda(\rho, \mu)$ given by (2.3), we let

$$\delta_0 = \frac{n}{n+1} \text{ and } \delta_{k+1} = \frac{\lambda\big((n-k+1)\,\delta_k - n + k, n-k-1\big) + n - k - 1}{n-k} \tag{3.1}$$

$$(k \in \{0, 1, 2, \ldots , n-1\}\,;\ \ n \in \mathbb{N})\,.$$

Then it is easily observed that

$$\frac{n-k-1}{n-k} \leqq \delta_{k+1} < 1 \qquad (k \in \{0, 1, 2, \ldots , n-1\}\,;\ \ n \in \mathbb{N})\,.$$

Next, for $\alpha \geqq 0$, $\alpha + \beta \geqq 0$, and $n \in \mathbb{N}_0$, we set

$$M(\alpha, \beta, n) := \begin{cases} \left(\frac{3}{4}\right)^{\alpha} & (n = 0, \alpha + \beta = 0) \\ \left(\frac{3}{4}\right)^{\alpha} \left(1 + \frac{2\alpha}{m}\right)^{\frac{1}{2}(\alpha+\beta)} \left(1 + \frac{m}{18\alpha}\right)^{\frac{1}{2}\alpha} & (n = 0; \alpha + \beta \neq 0) \\ \left(\frac{2n+1}{2n}\right)^{\alpha} (n+1)^{-\beta}\,(n+2)^{-\alpha} & (n \neq 0)\,, \end{cases} \tag{3 2}$$

where

$$m := \beta + \sqrt{\beta^2 + 36\alpha\,(\alpha + \beta)},$$

and we note that

$$M(\alpha, \beta, n) \ \geqq \ \frac{1}{(2n+3)^{\beta}\,(2n+4)^{\alpha}} \qquad (\alpha \geqq 0;\ \alpha + \beta \geqq 0;\ n \in \mathbb{N}_0)\,. \tag{3.3}$$

THEOREM 3. *Let* $f \in \mathcal{A}$, $\alpha \geqq 0$, $\alpha + \beta \geqq 0$, *and* $n \in \mathbb{N}_0$. *If*

$$\left|\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1\right|^{\alpha} \cdot \left|\frac{D^{n+1}f(z)}{D^{n}f(z)} - 1\right|^{\beta} < M(\alpha, \beta, n) \qquad (z \in \mathbb{U}), \tag{3.4}$$

then $f \in S^*(\delta_n)$, *where* δ_n *and* $M(\alpha, \beta, n)$ *are defined by* (3.1) *and* (3.2), *respectively.*

PROOF. Suppose that

$$\frac{D^{n+1}f(z)}{D^{n}f(z)} = \frac{p(z) + n}{n+1} \qquad (n \in \mathbb{N}_0). \tag{3.5}$$

Then the function $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Also, from the known identity:

$$z\big(D^n f(z)\big)' = (n+1)\, D^{n+1} f(z) - n\, D^n f(z) \qquad (n \in \mathbb{N}_0), \tag{3.6}$$

we have

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = \frac{1}{n+2}\left(\frac{zp'(z)}{p(z)+n} + p(z) + n + 1\right), \tag{3.7}$$

which yields

$$\begin{aligned}\left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1\right)^{\alpha} & \left(\frac{D^{n+1}f(z)}{D^{n}f(z)} - 1\right)^{\beta} \\ &= (n+1)^{-\beta}(n+2)^{-\alpha}[p(z)-1]^{\beta}\left(\frac{zp'(z)}{p(z)+n} + p(z) - 1\right)^{\alpha} \\ &= (n+1)^{-\beta}(n+2)^{-\alpha}\ \Phi_n\big(p(z), zp'(z); z\big),\end{aligned} \tag{3.8}$$

where, for convenience,

$$\Phi_n(z_1, z_2; z) := (z_1 - 1)^{\beta}\left(\frac{z_2}{z_1 + n} + z_1 - 1\right)^{\alpha}. \tag{3.9}$$

In view of (3.8), the hypothesis (3.4) of Theorem 3 is equivalent to

$$\Phi_n\big(p(z), zp'(z); z\big) \in \Omega_n := \{w : w \in \mathbb{C} \text{ and } |w| < R(\alpha, \beta, n)\} \qquad (n \in \mathbb{N}_0), \tag{3.10}$$

where

$$R(\alpha, \beta, n) := (n+1)^{\beta} (n+2)^{\alpha} M(\alpha, \beta, n) \tag{3.11}$$

with $M(\alpha, \beta, n)$ given by (3.2).

For $z \in \mathbb{U}$, $x, y \in \mathbb{R}$, and $y \leqq -\frac{1}{2}(1+x^2)$, we find from (3.9) that

$$|\Phi_n(ix, y; z)|^2 = \left(1+x^2\right)^{\beta} \left[\left(\frac{ny}{x^2+n^2} - 1\right)^2 + x^2 \left(1 - \frac{y}{x^2+n^2}\right)^2\right]^{\alpha} =: G_n\left(x^2, y\right), \tag{3.12}$$

where

$$G_n(\tau, y) := (1+\tau)^{\beta} \left[\left(\frac{ny}{\tau+n^2} - 1\right)^2 + \tau \left(1 - \frac{y}{\tau+n^2}\right)^2\right]^{\alpha} \tag{3.13}$$

$$\left(\tau := x^2 \geqq 0;\ x, y \in \mathbb{R};\ y \leqq -\frac{1}{2}(1+\tau)\right).$$

Since

$$\frac{\partial}{\partial y} G_n(\tau, y) < 0,$$

it is easily seen from (3.13) that

$$\begin{aligned} G_n(\tau, y) &\geqq G_n\left(\tau, -\frac{1+\tau}{2}\right) \\ &= (1+\tau)^{\alpha+\beta} \left(\frac{\tau+1}{4(\tau+n^2)} + \frac{\tau+n}{\tau+n^2} + 1\right)^{\alpha} \\ &= 4^{-\alpha} (1+\tau)^{\alpha+\beta} \left(\frac{9\tau + (2n+1)^2}{\tau+n^2}\right)^{\alpha} \\ &=: H_n(\tau) \end{aligned} \tag{3.14}$$

and that

$$\frac{d}{d\tau}H_n(\tau) = 4^{-\alpha}(1+\tau)^{\alpha+\beta-1}\left(\frac{9\tau+(2n+1)^2}{\tau+n^2}\right)^{\alpha-1}\frac{F_n(\tau)}{(\tau+n^2)^2}, \tag{3.15}$$

where

$$\begin{aligned} F_n(\tau) := 9(\alpha+\beta)\tau^2 &+ \left[9n^2(2\alpha+\beta)+(2n+1)^2\beta\right]\tau \\ &+ \left[(2n+1)^2 n^2(\alpha+\beta)+(n-1)(5n+1)\alpha\right]. \end{aligned} \tag{3.16}$$

In the case when $n \neq 0$, we find from (3.16) that

$$F_n(\tau) \geqq 0 \qquad (\tau \geqq 0;\ n \in \mathbb{N}) \tag{3.17}$$

which, in conjunction with (3.15), yields

$$\frac{d}{d\tau}H_n(\tau) \geqq 0 \qquad (\tau \geqq 0;\ n \in \mathbb{N}), \tag{3.18}$$

so that

$$H_n(\tau) \geqq H_n(0) = 4^{-\alpha}\left(\frac{(2n+1)^2}{n^2}\right)^{\alpha} = \left(\frac{2n+1}{2n}\right)^{2\alpha} \tag{3.19}$$

$$(\tau \geqq 0;\ n \in \mathbb{N}).$$

Thus, from (3.12) and (3.14), we obtain

$$\Phi_n(ix, y; z) \notin \Omega_n = \left\{w : w \in \mathbb{C} \text{ and } |w| < \left(\frac{2n+1}{2n}\right)^{\alpha} = R(\alpha,\beta,n)\right\}. \tag{3.20}$$

Next, in the case when $n = 0$, we observe from (3.16) *with* $\alpha+\beta \neq 0$ that

$$\begin{aligned} F_0(\tau) &= 9(\alpha+\beta)\tau^2 + \beta\tau - \alpha \\ &= 9(\alpha+\beta)(\tau+\tau_1)(\tau-\tau_2), \end{aligned}$$

where

$$\tau_1 = \frac{\sqrt{\beta^2 + 36\alpha(\alpha+\beta)} + \beta}{18(\alpha+\beta)} \text{ and } \tau_2 = \frac{\sqrt{\beta^2 + 36\alpha(\alpha+\beta)} - \beta}{18(\alpha+\beta)} \geqq 0$$

$$(\alpha \geqq 0;\ \alpha + \beta > 0).$$

Thus, from (3.15) and (3.16) *with* $n = 0$, we have

$$\begin{aligned} H_0(\tau) \geqq H_0(\tau_2) &= \left(\frac{9}{4}\right)^{\alpha} \left(1 + \frac{2\alpha}{m}\right)^{\alpha+\beta} \left(1 + \frac{m}{18\alpha}\right)^{\alpha} \\ &= [R(\alpha, \beta, 0)]^2 \qquad (\tau \geqq 0), \end{aligned} \tag{3.21}$$

which, by virtue of (3.12) and (3.14), shows that

$$\Phi_0(ix, y; z) \notin \Omega_0 = \{w : w \in \mathbb{C} \text{ and } |w| < R(\alpha, \beta, 0)\}, \tag{3.22}$$

m being given, as before, with (3.2).

Finally, in the case when $n = 0$ *and* $\alpha + \beta = 0$, it is readily seen from (3.16) that

$$F_0(\tau) = \beta(\tau + 1) = -\alpha(\tau + 1) \leqq 0 \qquad (\alpha \geqq 0;\ \tau \geqq 0),$$

and (3.15) yields

$$\frac{d}{d\tau} H_0(\tau) \leqq 0 \qquad (\tau \geqq 0),$$

so that

$$H_0(\tau) \geqq \lim_{\tau \to \infty} H_0(\tau) = \left(\frac{9}{4}\right)^{\alpha} = [R(\alpha, \beta, 0)]^2 \qquad (\tau \geqq 0), \tag{3.23}$$

which shows that (3.22) holds true in this case as well.

Thus, for $x, y \in \mathbb{R}$ and $y \leqq -\frac{1}{2}(1 + x^2)$, we have established the needed condition that

$$\Phi_n(ix, y; z) \notin \Omega_n = \{w : w \in \mathbb{C} \text{ and } |w| < R(\alpha, \beta, n)\} \qquad (n \in \mathbb{N}_0) \tag{3.24}$$

in *all* cases listed in (3.2). Therefore, in view of (3.10), we deduce from Lemma 2 that

$$\Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} = \frac{\Re\{p(z)\}+n}{n+1} > \frac{n}{n+1} = \delta_0 \qquad (z\in\mathbb{U};\ n\in\mathbb{N}_0), \tag{3.25}$$

where δ_0 is defined by (3.1).

By applying Lemma 1, we find that, if

$$\Re\left\{\frac{D^{n-k+1}f(z)}{D^{n-k}f(z)}\right\} > \delta_k \tag{3.26}$$

$$(z\in\mathbb{U};\ k\in\{0,1,2,\dots,n-1\};\ n\in\mathbb{N}),$$

then

$$\Re\left\{\frac{D^{n-k}f(z)}{D^{n-k-1}f(z)}\right\} > \delta_{k+1} \tag{3.27}$$

$$(z\in\mathbb{U};\ k\in\{0,1,2,\dots,n-1\};\ n\in\mathbb{N}),$$

where δ_k is defined by (3.1). So, if we start with the hypothesis (3.25), we immediately obtain

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} = \Re\left\{\frac{D^1 f(z)}{D^0 f(z)}\right\} > \delta_n \qquad (z\in\mathbb{U}), \tag{3.28}$$

which implies that $f\in\mathcal{S}^*(\delta_n)$. This evidently completes the proof of Theorem 3

REMARK 2. In view of the inequality (3.3), Theorem 3 provides a significant improvement over Theorem 1. Furthermore, by assigning suitable special values to the various parameters involved, we can deduce several simpler consequences of Theorem 3. Thus, for example, we can show that the condition:

$$\left|\frac{zf''(z)}{f'(z)}\right|^{\alpha}\cdot\left|\frac{zf'(z)}{f(z)}-1\right|^{1-\alpha} < \left(\frac{3}{2}\right)^{\alpha}\left[1+\frac{2\alpha}{1-\alpha+\sqrt{\alpha^2+34\alpha+1}}\right]^{\frac{1}{2}} \cdot\left[1+\frac{1-\alpha+\sqrt{\alpha^2+34\alpha+1}}{18\alpha}\right]^{\frac{1}{2}\alpha} \tag{3.29}$$

$$(z \in \mathbb{U};\ \alpha > 0)$$

or, in particular, the condition:

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| < \left(\frac{35 + \sqrt{73}}{72} \right) \sqrt{\frac{19 + \sqrt{73}}{2}} \cong 2.2443697 \cdots \tag{3.30}$$

$$(z \in \mathbb{U})$$

implies that $f \in \mathcal{S}^*$.

REMARK 3. In the special case when $n = 0$ *and* $\alpha + \beta = 0$, both Theorem 3 and Theorem 1 yield the *same* result which does not seem to improve Theorem 2.

4. Concluding Remarks and Observations

Just as Obradović's starlikeness condition (1.5) given by Theorem 1, our main starlikeness condition (3.4) is not sharp in the general form in which it is stated (see Theorem 3). Thus it would seem to be an interesting open problem to determine the best possible constants involved in Theorem 3. It should also be mentioned in this connection that, by applying a certain result of Ruscheweyh and Singh [9] involving confluent hypergeometric functions, Li and Srivastava [2, p. 108, Theorem 5] obtained a partially sharp result of this type for functions $f(z)$ to be starlike of order γ in $\mathbb{U}$ $(0 \leqq \gamma < 1)$.

In its limit case when $\alpha \to 0+$, the starlikeness condition (3.29) readily yields

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \qquad (z \in \mathbb{U}), \tag{4.1}$$

which indeed is a sharp result. Furthermore, in its special case when $\alpha = 1$, the starlikeness condition (3.29) was obtained by Miller and Mocanu [5], thereby improving several known results on this subject given by (among others) Singh and Singh [10] and Anisiu and Mocanu [1] (see also [2]).

Finally, it should be pointed out that both (3.2) and (3.29) hold true in the limit case when $\alpha \to 0+$. For example, we find from (3.2) that

$$\lim_{\alpha \to 0+} \{M(\alpha, \beta, n)\} = \frac{1}{(n+1)^{\beta}} \qquad (n \in \mathbb{N}_0;\ \beta \neq 0), \tag{4.2}$$

and the limit case of (3.29) when $\alpha \to 0+$ is already given by (4.1) above. The exceptional case when

$$\alpha = \beta = n = 0 \tag{4.3}$$

is clearly excluded in each of the above three theorems.

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