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구매량에 종속적인 외상기간을 고려한 퇴화성 제품의 경제적 주문정책

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Analysis of Economic Replenishment Policy for Exponentially Deteriorating Products under Trade Credit Depending on the Amount of Purchase

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This paper analyzes how a buyer can determine the economic replenishment policy when the supplier allows delay in payments for an order of a product. For the analysis, it is assumed that the length of credit period is a function of the buyer's amount of purchase, and inventory is depleted not only by buyer's demand but also by deterioration. Investigation of the properties of the model developed allows us to develop an algorithm whose validity is illustrated using an example problem.

Keywords: inventory, EOQ, deterioration, trade credit, length of credit period

1. Introduction

In today's business transactions, it is more and more common to see that the buyers are allowed some grace period before they settle the account with the supplier. As implicitly stated by Mehta (1968), a major reason for the supplier to offer a credit period to the buyers is to stimulate the demand for the product that he produces, and the supplier usually expects that the increased sales volume can compensate the capital losses incurred during the credit period. Also, the buyers who are allowed a period to pay back for the products bought without paying any interest, can earn interest on the sales of the inventory depending on the length of that payments period. And therefore, the buyer's purchasing and marketing decisions are affected by trade credit. In this regard, a number of research papers appeared which deal with the inventory model under a fixed credit period. Chapman *et al.* (1985), Chung (1998), Goyal (1985) and Kingsman (1983) have studied the effects of trade credit on the buyer's economic ordering policy.

A common assumption of the above researches is the availability of a certain fixed length of credit period that is set by the supplier. However, the length of credit period is considered as supplier's dominant strategy against the competitive suppliers in expectation of increasing the sales volume. Among a number of factors that determine the length of the credit period in a given line of business, credit risk, the size of the account, customer type, and market competition are known to be more important. Also, for the sake of better production and inventory control, some manufacturers prefer less frequent orders with larger

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order sizes to frequent orders with smaller order sizes, if the buyer's annual demand is equal. Thus they offer a longer credit period for larger amount of purchase rather than giving some discount on unit selling price. This kind of commercial practice is based on the principle of economies of scale from the supplier's point of view and their policy tends to make the buyer's order size large enough to qualify for a certain credit period break. Based upon the above observations, Shinn and Hwang (2003) analyzed the joint price and lot-size determination problem under order-size-dependent delay in payments.

All the research works mentioned above implicitly assume that inventory is depleted by buyer's demand alone. This assumption is quite valid for non-deteriorating inventory products whose utility remains constant over time. However, there are numerous types of inventory whose utility does not remain constant over time. Deterioration is defined as decay, damage or spoilage so that the product cannot be used for its original purpose. Therefore, inventory is depleted not only by demand but also by deterioration, and so the loss due to deterioration must be taken into consideration while developing the inventory models. Chu *et al.* (1998) and Jaggi and Aggarwal (1994) evaluated the economic ordering policy for deteriorating products in the presence of trade credit. Also, Hwang and Shinn (1997) examined the joint price and lot-size determination problem for exponentially deteriorating products under the condition of permissible delay in payments. Recently, Sarker *et al.* (2000) analyzed the model to determine an EOQ for deteriorating items under inflation and permissible delay in payments.

This paper deals with the problem of determining the buyer's economic replenishment policy for exponentially deteriorating product under trade credit where the length of delay is a function of the amount purchased by the buyer. In Section 2, we describe the assumptions and notations adopted for this study. A relevant mathematical model is developed and its solution algorithm is presented in Section 3. A numerical example is provided in Section 4, which is followed by concluding remarks in Section 5.

2. Assumptions and Notations

The following assumptions and notations are used to develop the model.

Assumptions:

1) The demand for the product is constant with time.
2) Shortages are not allowed.
3) Time period is infinite.
4) Replenishments are instantaneous with a known and constant lead-time.
5) The inventory system deals with only one type of product.
6) The supplier allows a certain period on settling the account for the product supplied where the length of credit period is a function of the buyer's total amount of purchase.
7) The purchasing cost of the products sold during the credit period is deposited in an interest bearing account with rate I. At the end of this period, the account is settled and the buyer starts paying for the interest charges on the items in inventory with rate $R\,(R \geq I)$.
8) Inventory is depleted not only by demand but also by deterioration. Deterioration follows an exponential distribution with parameter λ.

Notations:

D : annual demand rate
Q : order size
T : replenishment cycle time
λ : a positive number representing the inventory deterioration rate
C : unit purchase cost
S : ordering cost
H : inventory carrying cost, excluding the capital opportunity cost
R : capital opportunity cost (as a percentage)
I : earned interest rate (as a percentage)
tc_j : credit period for the amount purchased CQ, $v_{j-1} \leq CQ < v_j$, where $tc_{j-1} < tc_j$, $j = 1, 2, \cdots, m$ and $v_0 < v_1 < \cdots < v_m$, $v_0 = 0$, $v_m = \infty$

3. Development of the Mathematical Model

Let $q(t)$ be the inventory level at time t. For the case of exponential deterioration, the rate at which inventory deteriorates will be proportional to on hand inventory $q(t)$, and depletion due to deterioration and also due to demand will occur simultaneously as stated by Cohen (1977). Accordingly, the differential equation describing the instantaneous state of $q(t)$ over $(0, T)$ is given by

$$\frac{dq(t)}{dt} + \lambda q(t) = -D, 0 \leq t \leq T \qquad (1)$$

Observing that equation (1) is a first order linear differential equation, its solution is

$$q(t) = q(0)e^{-\lambda t} + \frac{D}{\lambda}(e^{-\lambda t} - 1) \qquad (2)$$

Equation (2) gives the inventory level at time t representing the combined effects of demand usage and exponential deterioration. We also know that due to the inventory carrying costs, it is clearly better off to have the inventory level reach zero just before reordering, i.e., $q(T)=0$. Given that $q(T)=0$, we have

$$q(0)e^{-\lambda T}+\frac{D}{\lambda}(e^{-\lambda T}-1)=0 \quad (3)$$

$$q(0)=\frac{D}{\lambda}(e^{\lambda T}-1) \quad (4)$$

Therefore, from equation (2) and (4), the inventory level at time t is

$$q(t)=\frac{D}{\lambda}[e^{\lambda(T-t)}-1], 0\le t\le T \quad (5)$$

Noting that $q(0)=Q$, the quantity ordered each replenishment cycle is

$$Q=\frac{D}{\lambda}(e^{\lambda T}-1) \quad (6)$$

The purpose of this paper is to find an economic replenishment cycle time which minimizes the buyer's total annual variable cost $TC(T)$. For the formulation of $TC(T)$, we consider the inequalities, $v_{j-1}\le CQ<v_j$ for $j=1,2,\cdots,m$. And using equation (6), the inequalities can be reduced to

$$l_{j-1}\le T<l_j,\ j=1,2,\cdots,m$$

$$\text{where } l_j=\frac{1}{\lambda}\ln\left(\frac{\lambda}{CD}v_j+1\right) \quad (7)$$

Then, the total annual variable cost consists of the following elements.

1) Annual ordering cost $=\frac{S}{T}$

2) Annual purchasing cost $=\frac{CQ}{T}=\frac{CD(e^{\lambda T}-1)}{\lambda T}$

3) Annual inventory carrying cost $=\frac{H}{T}\int_0^T q(t)dt=\frac{HD(e^{\lambda T}-\lambda T-1)}{\lambda^2 T}$

4) Annual capital opportunity cost for $l_{j-1}\le T<l_j$:

(i) **Case 1**($tc_j\le T$): (see <Figure 1>) As products are sold, the purchasing cost of the products is used to earn interest with annual rate I during the credit period tc_j. And the average number of products in inventory earning interest during time $(0, tc_j)$ is $\frac{Dtc_j}{2}$ and the interest earned per order becomes $\left(\frac{Dtc_j}{2}\right)tc_j CI$. When the account is settled, the products still in inventory have to be financed with annual rate R. Since the average number of products during time (tc_j, T) becomes $\frac{1}{(T-tc_j)}\int_{tc_j}^{T}q(t)dt$, the interest payable per order can be expressed as $CR\int_{tc_j}^{T}q(t)dt$. Therefore,

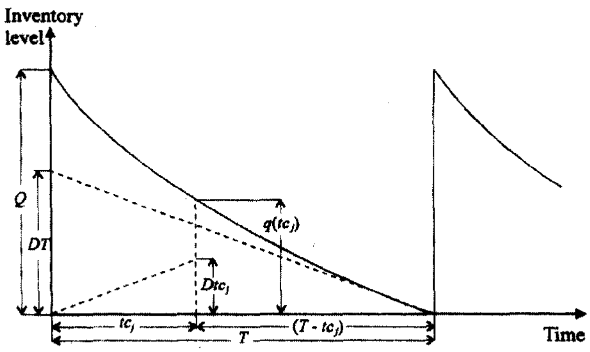

Figure 1. Time-weighted inventory when $T\ge tc_j$.

the annual capital opportunity cost

$$=\frac{CR\int_{tc_j}^{T}q(t)dt-\frac{CIDtc_j^2}{2}}{T}$$

$$=\frac{1}{\lambda^2 T}CRD[e^{\lambda(T-tc_j)}-\lambda(T-tc_j)-1]-\frac{1}{2T}CIDtc_j^2$$

(ii) **Case 2**($tc_j>T$): (see <Figure 2>) For the case of $tc_j>T$, all the purchasing cost of the products is used to earn interest with annual rate I during the credit period tc_j. The average number of products in inventory earning interest during time $(0, T)$ and (T, tc_j) become $\frac{DT}{2}$ and DT, respectively. Therefore,

the annual capital opportunity cost

$$=-\frac{\frac{DT}{2}TCI+DT(tc_j-T)CI}{T}$$

$$=\frac{CIDT}{2}-CIDtc_j$$

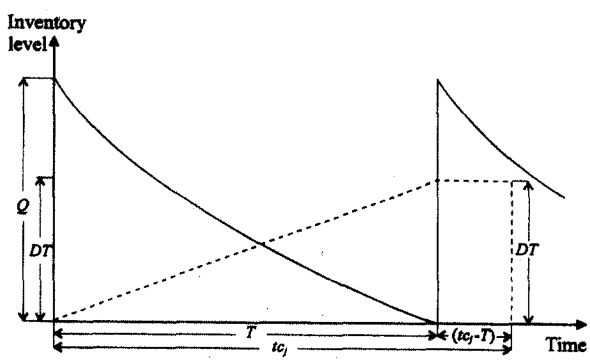

Figure 2. Time-weighted inventory when $T<tc_j$.

Therefore, depending on the relative size of tc_j to T, $TC(T)$ has two different expressions as follows:

1. **Case 1**($tc_j \le T$)

$$TC_{1,j}(T) = \frac{S}{T} + \frac{CD(e^{\lambda T}-1)}{\lambda T} + \frac{HD(e^{\lambda T}-\lambda T-1)}{\lambda^2 T} + \left[\frac{CRD(e^{\lambda(T-tc_j)}-\lambda(T-tc_j)-1)}{\lambda^2 T} - \frac{CIDtc_j^2}{2T}\right], \quad (8)$$
$$l_{j-1} \le T < l_j,\ j = 1, 2, \cdots, m$$

2. **Case 2**($tc_j > T$)

$$TC_{2,j}(T) = \frac{S}{T} + \frac{CD(e^{\lambda T}-1)}{\lambda T} + \frac{HD(e^{\lambda T}-\lambda T-1)}{\lambda^2 T} + \left(\frac{CIDT}{2} - CIDtc_j\right), \quad (9)$$
$$l_{j-1} \le T < l_j,\ j = 1, 2, \cdots, m$$

In order to find the buyer's economic replenishment cycle time T^* which minimizes the total annual variable cost, let us consider the characteristics of $TC(T)$. From equations (8) and (9), it can be shown that $TC(T)$ is continuous at $T = tc_j$. And both $TC_{1,j}(T)$ and $TC_{2,j}(T)$ are strictly decreasing for any fixed value of T as j increases. Therefore, $TC(T)$ has a downward jump discontinuity at l_j, $j = 1, 2, \cdots, m-1$. As a result, l_j and the extreme points of each $TC_{i,j}(T)$ [if it is feasible in its domain corresponding to $TC_{i,j}(T)$]can be candidates for T^*. Theoretically, the extreme point of $TC_{i,j}(T)$ can be obtained from the solution of the equation $dTC_{i,j}(T)/dT = 0$. Although the objective function can be differentiated, the resulting equation is mathematically intractable; that is, it is impossible to find the solution in explicit form. Thus the model will be solved approximately by using a truncated Taylor series expansion for the exponential term, i.e.,

$$e^{\lambda T} \approx 1 + \lambda T + \frac{1}{2}\lambda^2 T^2 \quad (10)$$

which is a valid approximation for smaller values of λT. With the above approximation, the total annual variable cost can be reduced to

$$TC_{1,j}(T) = \frac{S}{T} + CD + \frac{DT(H+C\lambda)}{2} + \left[\frac{C(R-I)Dtc_j^2}{2T} + \frac{CRDT}{2} - CRDtc_j\right] \quad (11)$$

$$TC_{2,j}(T) = \frac{S}{T} + CD + \frac{DT(H+C\lambda)}{2} + \left(\frac{CIDT}{2} - CIDtc_j\right) \quad (12)$$

Note that equation (10) is exact when $\lambda = 0$ so that equation (8) and (9) reduce to the exact formulas equation (11) and (12) for nondeteriorating product. Also, both $TC_{1,j}(T)$ and $TC_{2,j}(T)$ are strictly decreasing for any fixed value of T as j increases.

For the normal condition($R \ge I$) as stated by Goyal (1985), $TC_{i,j}(T)$ is a convex function for every i and j. Thus, there exists a unique value $T_{i,j}$, which minimizes $TC_{i,j}(T)$ and they are:

$$T_{1,j} = \sqrt{\frac{2S + C(R-I)Dtc_j^2}{D(H+C\lambda+CR)}} \quad (13)$$

$$T_{2,j} = \sqrt{\frac{2S}{D(H+C\lambda+CI)}} \quad (14)$$

From equation (13) and (14), it is clear that the value of $T_{1,j}$ is strictly increasing as j increases and the value of $T_{2,j}$ is identical for every j. Also, it can be shown that if $T_{1,j} \ge tc_j$, then $T_{2,j} \ge tc_j$, which implies that $TC_{2,j}(T)$ is decreasing in T for $T < tc_j$. Similarly, if $T_{2,j} < tc_j$, then $T_{1,j} < tc_j$, which implies that $TC_{1,j}(T)$ is increasing in T for $T \ge tc_j$.

Now, from the above results, we can make the following properties about the characteristics of the total annual variable cost function for T, $T \in L_j = \{T | l_{j-1} \le T < l_j\}$, $j = 1, 2, \cdots, m$(The proofs are given in Appendix). These properties simplifies our search process such that only a finite number of candidate values of T needs to be considered to find T^* for the approximate model. Let k be the smallest index such that $T_{2,j} < tc_j$.

Property 1.

For $T \in L_j$, $j \ge k$, we can consider the following three cases for $T_{2,j}$; $T_{2,j} < l_{j-1}$, $l_{j-1} \le T_{2,j} < l_j$, and $l_j \le T_{2,j}$.

(i) If $T_{2,j} < l_{j-1}$, then the total cost becomes minimum at $T = l_{j-1}$.

(ii) If $l_{j-1} \le T_{2,j} < l_j$, then the total cost becomes minimum at $T = T_{2,j}$.

(iii) If $l_j \le T_{2,j}$, then we do not need to consider T to find T^*.

Property 2.

For $T \in L_j$, $j < k$, we can consider the following four cases for $T_{1,j}$; $T_{1,j} < l_{j-1}$, $l_{j-1} \le T_{1,j} < l_j$, $tc_j < l_j \le T_{1,j}$ and $l_j \le tc_j \le T_{1,j}$.

(i) If $T_{1,j} < l_{j-1}$, then the total cost becomes minimum at $T = l_{j-1}$.

(ii) If $l_{j-1} \le T_{1,j} < l_j$, then the total cost becomes minimum at $T = T_{1,j}$.

(iii) If $tc_j < l_j \le T_{1,j}$, then the total cost becomes minimum at $T = l_j^-$, where $l_j^- = l_j - \varepsilon$ and ε is a very small positive number.

(iv) If $l_j \le tc_j \le T_{1,j}$, then we do not need to consider T to find T^*.

Property 3. (Search Stopping Rule)

(i) For $T \in L_j$, if the total cost becomes minimum at $T = T_{1,j}$, then $T^* \ge T_{1,j}$.

(ii) For $T \in L_j$, if the total cost becomes minimum at $T = l_j^-$, then $T^* \ge l_j^-$.

To facilitate the explanation of Properties 1, 2 and 3, we present <Figure 3>, which shows the shape of $TC_{i,j}(T)$ of the example problem introduced in Section 4. Applying Properties 1 and 2 to the problem, the candidates for T^* are l_2, $T_{1,2}$ and l_1^-. Also, Property 3 indicates that T^* is greater than or equal to $T_{1,2}$ and so we do not need to consider l_1^- for T^*. Hence, we only need to consider $T = l_2$ and $T_{1,2}$ in search of T^*.

Based on the above properties, we develop the following solution algorithm to determine T^* for the approximate model.

Solution algorithm

Step 1. Compute $T_{2,j} = \sqrt{\dfrac{2S}{D(H + C\lambda + CI)}}$ and let k be the smallest index such that $T_{2,j} < tc_j$. If $T_{2,j} \ge tc_j$ for all $1 \le j \le m$, then set $k = m+1$ and go to Step 3. Otherwise, find the index n satisfying $l_{n-1} \le T_{2,j} < l_n$ and go to Step 2.

Step 2.

2.1 If $k > n$, then go to Step 2.3. Otherwise, go to Step 2.2.

2.2 Compute the total annual variable cost for $T = T_{2,n}$, l_n, l_{n+1}, $\cdots$, l_{m-2} and l_{m-1} and go to Step 3.

2.3 Compute the total annual variable cost for $T = l_{j-1}$, $j = k, k+1, \cdots, m$ and go to Step 3.

Step 3.

3.1 Set $j = k - 1$.

3.2 If $tc_j \ge l_j$, then go to Step 3.4. Otherwise, go to Step 3.3.

3.3 In case

i) $T_{1,j} < l_{j-1}$, compute the total annual variable cost for $T = l_{j-1}$ and go to Step 3.4.

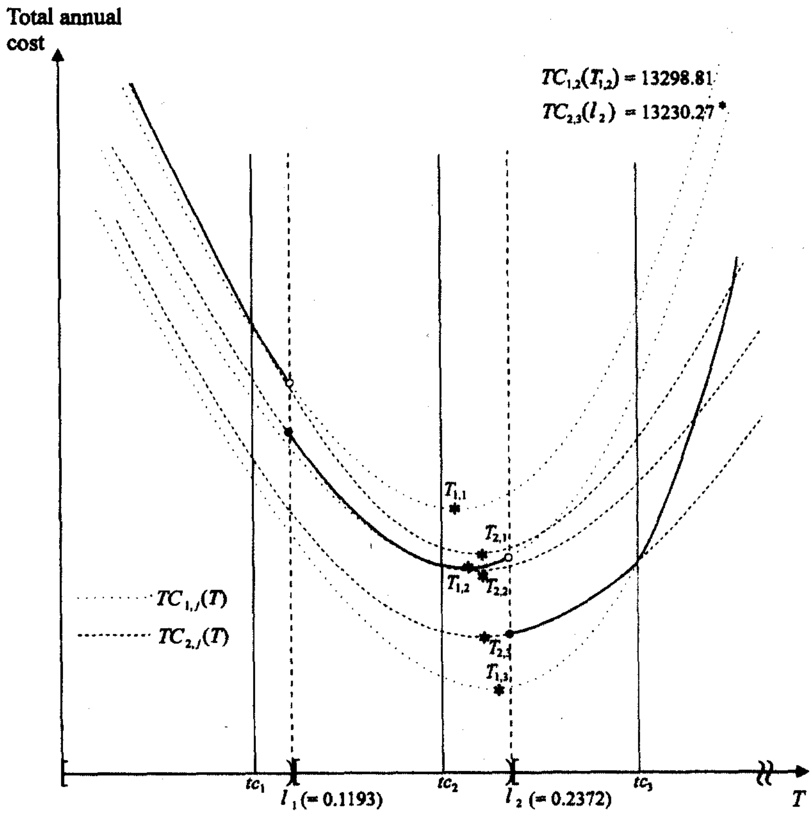

Figure 3. Buyer's total annual cost function of an example problem in Section 4.

ii) $T_{1,j} < l_j$, compute the total annual variable cost for $T = T_{1,j}$ and go to Step 4.

iii) $T_{1,j} \geq l_j$, compute the total annual variable cost for $T = l_j^-$, $l_j^- = l_j - \varepsilon$ where ε is a very small positive number and go to Step 4.

3.4 Reset $j = j-1$ and go to Step 3.2.

Step 4. Select the one that yields the minimum total annual variable cost as T^* for the approximate model and stop.

4. Numerical Example

To illustrate the solution algorithm, the following problem is considered.

(1) $D = 2500$ units, $S = \$100$, $C = \$5$, $H = \$1$, $R = 0.1(=10\%)$, $I = 0.06(=6\%)$ and $\lambda = 0.1$.

(2) Supplier's credit schedule for the buyer's total amount of purchase:

Total amount of purchase	Credit period
$0 \leq CQ < \$1500$	$tc_1 = 0.1$
$\$1500 \leq CQ < \$3000$	$tc_2 = 0.2$
$\$3000 \leq CQ$	$tc_3 = 0.3$

The buyer's replenishment cycle time for the approximate model can be obtained through the following steps.

Step 1. Since $T_{2,j}(=0.2108) < tc_3(=0.3)$, $k = 3$. By definition of l_j, $l_1 = 0.1193$ and $l_2 = 0.2372$. So $T_{2,j} \in (l_1, l_2)$. And therefore, $n = 2$ and go to Step 2.

Step 2.

2.1. Since $k(=3) > n(=2)$, go to Step 2.3.

2.2. Since $l_2(=0.2372) < tc_3(=0.3)$, compute $TC_{2,3}(l_2)$ with equation (12) and go to Step 3.

Step 3.

3.1. Set $j = 3-1 = 2$.

3.2. Since $tc_2(=0.2) < l_2(=0.2372)$, go to Step 3.3.

3.3 Since $T_{1,2}(=0.2098) \in (l_1, l_2)$, compute $TC_{1,2}(T_{1,2})$ with equation (11) and go to Step 4.

Step 4. Since $TC_{2,3}(0.2372) = 13230.27 = \min.$ $[TC_{1,2}(0.2098), TC_{2,3}(0.2372)]$, the buyer's economic replenishment cycle time for the approximate model becomes 0.2372 with its minimum total annual variable cost \$13230.27.

Table 1. Results for the approximate model with various values of λ

λ	Buyer's Economic Replenishment Policy		
	Replenishment Cycle Time	Order Size	Annual Total Cost
0.00	0.2481	620.17	\$13081.23
0.10	0.2372	599.95	\$13230.27
0.15	0.2358	599.88	\$13303.31
0.20	0.2344	599.78	\$13375.54
0.25	0.2331	599.67	\$13446.98
0.30	0.2318	599.53	\$13517.63
0.50	0.1451	375.90	\$13728.41

To study the effect of the inventory deteriorating rate on the replenishment cycle time and total annual variable cost, we solve the same example problem with various values of λ. The results are shown in <Table 1> and we find that as λ increases, T^* decreases while $TC(T^*)$ increases, which are consistent with our expectations.

5. Conclusions

The length of credit period is considered as supplier's dominant strategy against the competitive suppliers. Also, one of the important determinants of the length of the credit period is the size of the account. Some manufacturers offer a longer credit period for larger amount of purchase rather than giving some price discount. This kind of commercial practice can be justified by the principle of economies of scale. And their policy tends to make the buyer's order size larger by inducing him to qualify for a longer credit period in his payments.

In this paper, we analyzed the buyer's economic replenishment policy for an exponentially deteriorating product when the supplier allows the credit period for settling the account for an order of a product and the length of credit period is a function of the buyer's total amount of purchase. For the system presented, a mathematical model was developed. Recognizing that the model has a very complicated structure, a truncated Taylor series expansion is utilized to find a solution algorithm. To illustrate the validity of the solution algorithm, an example problem was solved and the results are consistent with our expectation. The model developed in this paper may help the buyers find an economic replenishment policy when the order-size-dependent delay in payments is available.

Appendix

Proof of Property 1.

(i) In this case, two possible cases occur; $T_{2,j} < tc_j < l_{j-1}$ and $T_{2,j} < l_{j-1} \le tc_j$. For the first case, $T_{2,j} < tc_j$ implies that $T_{1,j} < tc_j$. So, $TC_{1,j}(T)$ is increasing in T for $T \in L_j$. Thus,

$$TC_{1,j}(l_{j-1}) \le TC_{1,j}(T) \text{ for } T \in L_j \quad \text{(A1)}$$

Hence, if $T_{2,j} < tc_j < l_{j-1}$, then the total cost becomes minimum at $T = l_{j-1}$ for $T \in L_j$. For the second case, since $T_{2,j} < l_{j-1}$, $TC_{2,j}(T)$ is increasing in T for $T \ge l_{j-1}$ and $TC_{1,j}(T)$ is also increasing in T for $T \ge tc_j$. Since the total annual variable cost function is continuous at $T = tc_j$, we have

$$\begin{aligned} TC_{2,j}(l_{j-1}) &\le TC_{2,j}(T) \text{ for } l_{j-1} \le T < tc_j && \text{(A2)} \\ &< TC_{2,j}(tc_j) && \text{(A3)} \\ &\le TC_{1,j}(T) \text{ for } tc_j \le T && \text{(A4)} \end{aligned}$$

Hence, if $T_{2,j} < l_{j-1} \le tc_j$, then the total cost becomes minimum at $T = l_{j-1}$ for $T \in L_j$. Therefore, if $T_{2,j} < l_{j-1}$, then the total annual variable cost becomes minimum at $T = l_{j-1}$ for $T \in L_j$.

(ii) By definition of $T_{2,j}$

$$TC_{2,j}(T_{2,j}) \le TC_{2,j}(T) \text{ for any } T \quad \text{(A5)}$$

Since $T_{2,j} < tc_j$ and the total annual variable cost function is continuous at $T = tc_j$,

$$\begin{aligned} TC_{2,j}(T_{2,j}) &\le TC_{2,j}(T) \text{ for } l_{j-1} \le T < tc_j && \text{(A6)} \\ &< TC_{2,j}(tc_j) && \text{(A7)} \\ &\le TC_{1,j}(T) \text{ for } tc_j \le T && \text{(A8)} \end{aligned}$$

Hence, the total annual variable cost becomes minimum at $T = T_{2,j}$ for $T \in L_j$.

(iii) Since $T_{2,j} \ge l_j$, $TC_{2,j}(T)$ is decreasing in T for $T \in L_j$ and thus,

$$TC_{2,j}(l_j^-) \le TC_{2,j}(T) \text{ for } T \in L_j \quad \text{(A9)}$$

where $l_j^- = l_j - \varepsilon$ and ε is a very small positive number.

Also, since $TC_{2,j}(T)$ is strictly decreasing for a fixed T as j increases,

$$\begin{aligned} TC_{2,j}(T) &> TC_{2,j}(l_j) \text{ for } T \in L_j && \text{(A10)} \\ &> TC_{2,j+1}(l_j) && \text{(A11)} \end{aligned}$$

Hence, we do not need to consider T to find T^* for $T \in L_j$. *Q.E.D.*

Proof of Property 2.

(i) Since $T_{1,j} < l_{j-1}$,

$$TC_{1,j}(l_{j-1}) \le TC_{1,j}(T) \text{ for } T \in L_j \quad \text{(A12)}$$

So, the total annual variable cost becomes minimum at $T = l_{j-1}$ for $T \in L_j$.

(ii) In this case, two possible cases occur; $tc_j < l_{j-1} \le T_{1,j}$ and $l_{j-1} \le tc_j \le T_{1,j}$. For the first case, by definition of $T_{1,j}$,

$$TC_{1,j}(T_{1,j}) \le TC_{1,j}(T) \text{ for } T \in L_j \quad \text{(A13)}$$

So, if $tc_j < l_{j-1} \le T_{1,j}$, then the total cost becomes minimum at $T = T_{1,j}$ for $T \in L_j$. For the second case, by definition of $T_{1,j}$,

$$TC_{1,j}(T_{1,j}) \le TC_{1,j}(T) \text{ for any } T \quad \text{(A14)}$$

Because $T_{1,j} \ge tc_j$, $T_{2,j} \ge tc_j$ and $TC_{2,j}(T)$ is decreasing in T for $T < tc_j$. Since the total annual variable cost function is continuous at $T = tc_j$, we have

$$\begin{aligned} TC_{1,j}(T_{1,j}) &\le TC_{1,j}(T) \text{ for } tc_j \le T < l_j && \text{(A15)} \\ &< TC_{2,j}(T) \text{ for } l_{j-1} \le T < tc_j && \text{(A16)} \end{aligned}$$

So, if $l_{j-1} \le tc_j \le T_{1,j}$, the total cost becomes minimum at $T = T_{1,j}$ for $T \in L_j$. Therefore, if $l_{j-1} \le T_{1,j} < l_j$, then the total annual variable cost becomes minimum at $T = T_{1,j}$ for $T \in L_j$.

(iii) Since $T_{1,j} \ge l_j$,

$$TC_{1,j}(l_j^-) \le TC_{1,j}(T) \text{ for } tc_j \le T < l_j \quad \text{(A17)}$$

Because $T_{1,j} > tc_j$, $T_{2,j} > tc_j$ and $TC_{2,j}(T)$ is decreasing in T for $T \le tc_j$. Since the total annual variable cost function is continuous at $T = tc_j$, we have

$$\begin{aligned} TC_{1,j}(l_j^-) &\le TC_{1,j}(T) \text{ for } tc_j \le T < l_j && \text{(A18)} \\ &< TC_{2,j}(T) \text{ for } T < tc_j && \text{(A19)} \end{aligned}$$

So, the total annual variable cost becomes minimum at $T = l_j^-$ for $T \in L_j$.

(iv) Since $l_j \le tc_j \le T_{1,j}$, $tc_j \le T_{2,j}$ and $l_j \le tc_j$. So, by the result of Property 1 (iii), we do not need to consider T to find T^* for $T \in L_j$. *Q.E.D.*

Proof of Property 3.

(i) From Property 2 (ii), if the total cost becomes minimum at $T = T_{1,j}$ for $T \in L_j$, then

$$l_{j-1} \leq T_{1,j} < l_j \text{ and } tc_j \leq T_{1,j} \quad \text{(A20)}$$

By definition of $T_{1,j}$,

$$TC_{1,j}(T_{1,j}) \leq TC_{1,j}(T) < TC_{1,j-1}(T) < \cdots < TC_{1,1}(T) \text{ for any } T \quad \text{(A21)}$$

Also, by definition of the index k, $TC_{2,n}(T)$ is decreasing in T for $T < tc_n$, $n \leq j$. Hence, if the total cost becomes minimum at $T = T_{1,j}$ for $T \in L_j$, then $T^* \geq T_{1,j}$.

(ii) From Property 2 (iii), if the total cost becomes minimum at $T = l_j^-$ for $T \in L_j$, then

$$tc_j < l_j \leq T_{1,j} \quad \text{(A22)}$$

By definition of $T_{1,j}$,

$$TC_{1,j}(l_j^-) \leq TC_{1,j}(T) \text{ for } T < l_j \quad \text{(A23)}$$

$$TC_{1,j}(T) < TC_{1,j-1}(T) < \cdots < TC_{1,1}(T) \text{ for any } T \quad \text{(A24)}$$

Also, by definition of the index k, $TC_{2,n}(T)$ is decreasing in T for $T < tc_n$, $n \leq j$. Hence, if the total cost becomes minimum at $T = l_j^-$ for $T \in L_j$, then $T^* \geq l_j^-$. $Q.E.D.$

References

Chapman, C. B., Ward, S. C., Cooper, D. F. and Page, M. J. (1985), Credit policy and inventory control, *Journal of Operational Research Society*, **35**(12), 1055-1065.

Chu, P., Chung, K. J. and Lan, S. P. (1998), Economic order quantity of deteriorating items under permissible delay in payments, *Computers & Operations Research*, **25**(10), 817-824.

Chung, K. J. (1998), A theorem on the determination of economic order quantity under conditions of permissible delay in payments, *Computers & Operations Research*, **25**(1), 49-52.

Cohen, M. A. (1977), Joint pricing and ordering policy for exponentially decaying inventory with known demand, *Naval Research Logistics Quarterly*, **24**(2), 257-268.

Goyal, S. K. (1985), Economic order quantity under conditions of permissible delay in payments, *Journal of Operational Research Society*, **36**(4), 335-338.

Hwang, H. and Shinn, S. W. (1997), Retailer's pricing and lot sizing policy for exponentially deteriorating products under the condition of permissible delay in payments, *Computers & Operations Research*, **24**(6), 539-547.

Jaggi, C. K. and Aggarwal, S. P. (1994), Credit financing in economic ordering policies of deteriorating items, *International Journal of Production Economics*, **34**, 151-155.

Kingsman, B. G. (1983), The effect of payment rule on ordering and stock holding in purchasing, *Journal of Operational Research Society*, **34**(11), 1085-1098.

Mehta, D. (1968), The formulation of credit policy models, *Management Science*, **15**, B30-B50.

Sarker, B. R., Jamal, A. M. M. and Wang, S. (2000), Supply chain models for perishable products under inflation and permissible delay in payment, *Computers & Operations Research*, **27**(1), 59-75.

Shinn, S. W. and Hwang, H. (2003), Optimal pricing and ordering policies for retailers under order-size-dependent delay in payments, *Computers & Operations Research* to appear.