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ON THE S^1- EULER CHARACTERISTIC OF THE SPACE WITH A CIRCLE ACTION *ii*

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Abstract The S^1- Euler characteristic of X is defined by $\tilde{\chi}_{S^1}(X) \in HH_1(ZG)$, where G is the fundamental group of connected finite S^1-compact manifold or connected finite S^1-finite complex X and HH_1 is the first Hochsch ild homology group functor. The purpose of this paper is to find several cases which the S^1- Euler characteristic has a homotopic invariant.

1. Introduction and statement of results

In the recent paper [3], Goehegan and Nicas defined the S^1- Euler characteristic of X. And it is denoted by $\tilde{\chi}_{S^1}(X) \in HH_1(ZG)$. Here G is the fundamental group of X and HH_1 is the first Hochschild homology group functor. And further the S^1- Euler characteristic does not have a homotopic invariant property [3]. Thus we have the following question: Under what conditions does the S^1- Euler characteristic have a homotopic property? This paper is primarily concerned with several cases where the S^1- Euler characteristic has a homotopic invariant. These results were motivated by the observation that the certain fundamental groups play a major role in making the homotopy invariant of the S^1-Euler characteristic under certain conditions [Theorem A, Theorem B].

We work in the category of finite connected S^1-CWcomplexes or finite connected S^1-manifolds.

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We shall give the following theorems:

THEOREM A. *If $f : X \to Y$ is a quasi-nilpotent homology equivalence with $\pi_1(X)(\neq 0)$ residually solvable and the Whitehead torsion of f is trivial, then $\tilde{\chi}_{S^1}(X) = \tilde{\chi}_{S^1}(Y)$.*

THEOREM B. *If $f : X \to Y$ is a quasi-nilpotent homology equivalence and X is one of the following cases with $\pi_1(X) \neq 0$:*

(1) *X is the space satisfying condition (T^{**}) with $\pi_1(X)$ finite,*
(2) *$X(\in T_{LN})$ is the space that $\pi_1(X)$ is torsion-free with all proper subgroups of $\pi_1(X)$ nilpotent,*
(3) *$X(\in T_{LN})$ is the space that $\pi_1(X)$ is infinite with the maximal condition on normal subgroups of $\pi_1(X)$.*
(4) *For $X, Y(\in T_{LN})$ and if $\pi_1(X)$ is a finitely generated group which satisfies $\pi_1(X)$ satisfies either max-∞ for non-nilpotent subgroups.*

Further the Whitehead torsion of f is trivial, then the S^1-Euler characteristic also has a homotopic invariant, i.e., $\tilde{\chi}_{S^1}(X) = \tilde{\chi}_{S^1}(Y)$.

2. Some technical facts.

Let R be a commutative ground ring and S be an associative R-algebra with unit. If M is an $S-S$ bimodule, i.e., a left and right S-module satisfying $(s_1m)s_2 = s_1(ms_2)$ for all $m \in M$, and $s_1, s_2 \in S$, then the Hochschild chain complex $(C_*(S,M), d)$ consists of $C_n(S,M) = S^{\otimes n} \otimes M$ where $S^{\otimes n}$ is the tensor product of n copies of S. Thus we have the differential map [3]:

$d_{n-1}(s_1 \otimes \cdots \otimes s_n \otimes m) = s_2 \otimes \cdots \otimes s_n \otimes ms_1 + \Sigma_{i=1}^{n-1}(-1)^i s_1 \otimes \cdots \otimes s_i s_{i+1} \otimes \cdots \otimes s_n \otimes m + (-1)^n s_1 \otimes \cdots \otimes s_{n-1} \otimes s_n m.$

The tensor products are taken over R. The n-th homology of this complex is the n-th Hochschild homology of S with coefficient bimodule M and is denoted by $HH_n(S,M)$. If $M = S$ with the standard $S-S$ bimodule structure then we usually write $HH_n(S)$ for $HH_n(S,M)$. We will be concerned with HH_1 and HH_0 which are computed from

$$\to \cdots S \otimes S \otimes M \xrightarrow[d_1]{} S \otimes M \xrightarrow[d_0]{} M,$$

where $d_1 : s_1 \otimes s_2 \otimes m \to s_2 \otimes ms_1 - s_1s_2 \otimes m + s_1 \otimes s_2m$
$d_0 : s \otimes m \to ms - sm$.

For the purpose of defining the S^1-Euler characteristic of X we recall the following [3]. Let A be a closed S^1- subset of X and n a non-negative integer. We say X is obtained from A by attaching $S^1 - n$ -cells if there is a collection $\{c_j^n | j \in J\}$ of closed n-dimensional S^1- subsets of X such that:

(1) $X = A \cup \{\bigcup_{j \in J} c_j^n\}$. X has the topology coherent with A and $\{c_j^n | j \in J\}$.
(2) For $i \neq j$, $(c_i^n - A) \cap (c_j^n - A) = \varnothing$.
(3) For each $j \in J$, there is a closed subgroup H_j of S^1 (in fact, H_j is finite or $H_j = S^1$) and an S^1-map $f_j : (S^1/H_j \times D^n, S^1/H_j \times S^{n-1}) \to (c_j^n, c_j^n \cap A)$ such that $f_j : (S^1/H_j \times D^n) = c_j^n$ and f_j maps $S^1/H_j \times D^n - S^1/H_j \times S^{n-1}$ homeomorphically onto $c_j^n - A$.

Each c_j^n is called an $S^1 - n-$ cell and f_j is called the S^1-attaching map.

By use of the Hochschild homology group, the S^1-Euler characteristic of X is defined by the Hochschild 1- cycle [3]:

$\tilde{\chi}_{S^1}(X) = \Sigma_{n \geq 0}(-1)^{n+1}\Sigma_{j \in J_n}\Sigma_{i=1}^{|H_j|} g_{j,n} \otimes g_{j,n}^{-1-i} \in HH_1(ZG)$,
where the element $g_{j,n} \in G = \pi_1(X)$.

We introduce the solvable space and recall residually solvable group for the Theorem A. In fact, the free groups of rank greater than one and fundamental groups of closed 2- manifolds of classical Euler characteristic less than zero are not solvable. The fundamental group of the Klein bottle is solvable. But the Klein bottle is not a nilpotent space because it has a centerless free group $\mathbb{Z}_2 * \mathbb{Z}_2$ as a factor group.

Let us say that a group action G on H is solvable if there exists a finite chain: $H = H_1 \supset H_2 \supset H_3 \supset \cdots \supset H_j \supset \cdots \supset H_n = \{e\}$ such that for each j

(1) H_j is closed under the action of G,
(2) H_{j+1} is normal in H_j and H_j/H_{j+1} is abelian.

Definition 1. A space X is solvable if

(1) $\pi_1(X)$ is solvable. and

(2) there is the solvable action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ for all $n \geq 2$.

We say that a group G has the property χ residually if to every element $g(\neq 1) \in G$, there is a normal subgroup N of G such that $g \notin N$ and G/N has the property χ [9].

Let $F \to E \to B$ be a fibration, the elements of $\pi_1(B)$ operate on $H_q(F)$. Thus we get the following [5].

DEFINITION 2. A fibration $F \to E \to B$ is said to be quasi-nilpotent if the action of $\pi_1(B)$ on $H_n(F)$ is nilpotent, $n \geq 0$ [6].

Let us recall the Whitehead torsion. Let $h : (X, x_0) \to (Y, y_0)$ be a homotopy equivalence of pairs. Put $K = \pi_1(Y, y_0)$ and $G = \pi_1(X, x_0)$ and $y_0 = h(x_0)$ then h induces an isomorphism $h_\sharp : G \to K$. Let us recall the Whitehead group in case $R = ZG$. Let $\pm G \subset GL(1, ZG)$ be the subgroup consisting of 1×1 matrices of the form $[\pm g], g \in G$. The cokernel of the natural homomorphism $\pm G \to K_1(ZG)$ is called the Whitehead group of G and it is denoted by $Wh_1(G)$. Recall the torsion of $h : \gamma(h) \in Wh_1(K)$ [9].

We say that a homotopy equivalence of pairs $h : (X, x_0) \to (Y, y_0)$ is simple if $\gamma(h) = 0$.

3. Proofs of the Theorems.

Proof of Theorem A. By the classical homotopy exact sequence of fibration: $F_f \to X \xrightarrow{f} Y$, $\pi_1(f)$ is an epimorphism. And from the quasi-nilpotent homology equivalence of f we get the reduced homology group $\tilde{H}_1(F_f)$ trivial, i.e., $\pi_1(F_f)$ is perfect. Furthermore the homomorphic image of a perfect group is also perfect. Thus $\pi_1(Y) \cong \frac{\pi_1(X)}{P\pi_1(X)}$ where $P\pi_1(X)$ means a perfect normal subgroup of $\pi_1(X)$.

From the residual solvability of $\pi_1(X)$, for any nontrivial element $g(\in \pi_1(X))$ there is a normal subgroup $(g \notin)N$ such that we have $(\pi_1(X)/N)^{(n)} = [(\pi_1(X)/N)^{(n-1)}, (\pi_1(X)/N)^{(n-1)}]$ be trivial for some n where $[,]$ means the commutator subgroup. Thus we get

$P(\pi_1(X)/N) = (P(\pi_1(X)/N))^{(n)} \leq (\pi_1(X)/N)^{(n)}$

trivial. Finally $P(\pi_1(X)/N)$ is trivial, where $P(G)$ means the maximal perfect normal subgroup of group G.

Now let H be a subgroup of $\pi_1(X)$ and let $H_i = G_i \cap H$ for all i where $G_i = (\pi_1(X)/N)^i$ above. Then H_i/H_{i+1} is also abelian for all i. If H is perfect, then $H = H_0 = H_1 = H_2 = \cdots$. But the intersection of all H_i is 1 and so $H = 1$. Thus we get the maximal perfect normal subgroup $P\pi_1(X)$ trivial. Therefore we get $\pi_1(f)$ as an isomorphism. Since f is a quasi-nilpotent homology equivalence with $\pi_1(f)$ isomorphic, we get f a homotopy equivalence.

In fact, $\tilde{\chi}_{S^1}(X)$ decomposes into two pieces [3], the first part is computed from a homotopic invariant of the underlying space X and the second one is computed from a simple homotopy invariant of the underlying space X. Furthermore f is also simple homotopy equivalent from the Whitehead torsion of f trivial[6]. Thus we get our assertion.

We introduce the locally nilpotent space with relation to Theorem B.

Let us denote T_N the category of nilpotent spaces [1] and continuous maps. We make the locally nilpotent space as an extensive concept of a nilpotent space as follows: a space X is said to be locally nilpotent if $\pi_1(X)$ is a locally nilpotent group [8] and, in addition, there is a nilpotent $\pi_1(X)$ - action on $\pi_n(X)$ for all $n \geq 2$ [1].

We adopt the notation T_{LN} for the category of locally nilpotent spaces and continuous maps. Obviously, the category T_N is a full subcategory of T_{LN}.

Whether the given space X is a nilpotent space or not is checked by the following condition (T^{**}) under the nilpotent $\pi_1(X)$ - action.

DEFINITION 3. We say that X satisfies the condition (T^{**}) if for all $g(\neq 1) \in \pi_1(X)$, then $g \notin [g, \pi_1(X)]$ where $[,]$ is a commutator [4].

Even though BS_3 is a solvable space, it does not satisfy the condition (T^{**}), where B is Milnor's classifying space and S_3 is symmetric group with 3 elements.

We recall that a group G satisfies the maximal condition if it has no infinite strictly increasing chain of subgroups [9]. Given a property $\star$ pertaining to subgroups, a group G is said to satisfy the max-∞ property for $\star$-subgroups if G has no infinite ascending chain $H_1 \subset H_2 \subset \cdots$ of $\star$-subgroups in which all indices $|H_{i+1} : H_i|$ are infinite[6].

Let G be the torsion free locally nilpotent group and H a subgroup of G. For each set π of primes, the π-isolator of H in G, which is the set $\{g \in G : g^n \in H \text{ for some } n \in \pi\}$, is a subgroup of G. In the case where π is the set of all primes we refer simply to the isolator of H in G, denoted $I_G(H)$, and H is said to be isolated in G if $I_G(H) = H$. If H is countable then so is $I_G(H)$; this is an easy consequene of the fact that, for $x, y \in G$ and $n \in \mathbb{N}, x^n = y^n$ implies $x = y$. If H is nilpotent of class c then so is $I_G(H)$. Finally, if G is finitely genterated and $\{N_i : i = 1, 2, \cdots\}$ is the set of all normal subgroups of finite index in G then $H = \cap_{i=1}^{\infty} HN_i$ [6].

Proof of Theorem B. For case (1) : we recall that if $\pi_1(X)$ is a nilpotent group then there exists a finite upper central series of $\pi_1(X)$ by virtue of center of $\pi_1(X)$. So assume that $\pi_1(X)$ is not nilpotent, then we do not have a finite upper central series of $\pi_1(X)$. If $Z_n(\pi_1(X))$ denotes the $n-th$ center of $\pi_1(X)$, we can find an integer n such that $Z_{n+1}(\pi_1(X)) = Z_n(\pi_1(X)) \subsetneq \pi_1(X)$. It follows that if $x \notin Z_n(\pi_1(X))$, then $[x, \pi_1(X)] \nsubseteq Z_n(\pi_1(X))$. Choose any $x_1 \notin Z_n(\pi_1(X))$, and we know that $[x_1, \pi_1(X)] \nsubseteq Z_n(\pi_1(X))$. If $x_1 \in [x_1, \pi_1(X)]$ then we have shown that the condition (T^{**}) does not hold, as required, so assume $x_1 \notin [x_1, \pi_1(X)]$. Then choose $x_2 \in [x_1, \pi_1(X)]$, $x_2 \notin Z_n(\pi_1(X))$. Since $[x_1, \pi_1(X)]$ is a normal subgroup of $\pi_1(X), [x_2, \pi_1(X)] \subseteq [x_1, \pi_1(X)]$. If $x_2 \in [x_2, \pi_1(X)]$, we are done.

Otherwise, we have $[x_2, \pi_1(X)] \subsetneq [x_1, \pi_1(X)]$. We noted $[x_2, \pi_1(X)] \nsubseteq Z_n(\pi_1(X))$. So pick $x_3 \in [x_2, \pi_1(X)]$, $x_3 \notin Z_n(\pi_1(X))$ and continue. Since $\pi_1(X)$ is finite, this process must stop. After all we have α for which $x_\alpha (\neq 1) \in [x_\alpha, \pi_1(X)]$. This is a contradiction to the fact that X satisfies the condition (T^{**}). Thus we get $\pi_1(X)$ as a nilpotent group and maximal perfect normal subgroup $P\pi_1(X)$ is trivial.

For case (2) : for $X(\in T_{LN})$ such that $\pi_1(X)$ is torsion-free with all proper subgroups of $\pi_1(X)$ nilpotent then $X \in T_N$ [10]. Thus $P\pi_1(X)$ is trivial.

For case (3) : when $\pi_1(X)$ is infinite and $\pi_1(X)$ has the maximal condition on normal subgroups then $\pi_1(X)$ is a finitely generated nilpotent group. Thus $\pi_1(X)$ has the center subgroup as the infinite normal abelian subgroup which acts nilpotently on $H_*(\tilde{X})$ for the universal covering space $\tilde{X}$ [2] then X is a nilpotent space. Thus $P\pi_1(X)$ is trivial.

For case (4) : when $\pi_1(X)$ satisfies the max-∞ property for non-nilpotent subgroups, assume that $\pi_1(X)$ is not nilpotent. Then $\pi_1(X)$ has a countable non-nilpotent subgroups and hence such an isolated subgroup, which we denote by K. Write $K = \cup\{K_i\}_{i=1}^{\infty}$, where $1 = K_0 \subset K_1 \subset K_2 \subset \cdots$ is a chain of finitely generated subgroups of increasing nilpotency classes. Let $\{p_1, p_2, p_3, \cdots\}$ be an infinite set of primes satisfying the following: Choose a normal subgroup H_1 of K_1 such that the index $|K_1 : H_1|$ is finite and divisible by p_1. Now let N_2 be a normal subgroup of finite index in K_2 such that $|K_2 : N_2K_1|$ is divisible by p_2 and $N_2H_1 \cap K_1 = H_1$. Write $H_2 = N_2H_1$. Inductively, having defined N_i and H_i for some $i \geq 2$, let N_{i+1} be a normal subgroup of finite index in K_{i+1} such that $|K_{i+1} : N_{i+1}K_i|$ is divisible by p_{i+1} and $N_{i+1}H_i \cap K_i = H_i$ and write $H_{i+1} = N_{i+1}H_i$. We obtain an infinite chain $H_1 \subset H_2 \subset \cdots$ such that, in particular, $|K_i : H_i|$ is finite for each i. Thus, setting $H = \cup_{i=1}^{\infty} H_i$. We have $I_K(H) = K$ and hence H non-nilpotent. Now we define $L_0 = H$ and for each $i \geq 1, L_i =< H, K_i >$. We shall establish the following facts:

(1) For each $i \geq 1, H \cap K_i = H_i$

(2) $|L_1 : L_0| = |K_1 : H_1|$ and, for $i \geq 1$ $|L_{i+1} : L_i| = |K_{i+1} : N_{i+1}K_i|$.

From the choice of the subgroups N_i, we see that $|L_{i+1} : L_i|$ is divisible by p_{i+1} for all $i \geq 1$. In particular, the chain $L_0 \subset L_1 \subset \cdots$ is not finite. We now obtain a similar chain where the indices are all infinite. Let π denote the set of all primes p which divide at least one of the indices $|L_{i+1} : L_i|$. Then π is infinite and we may write it as a disjoint union of infinitely many infinite subsets $\{\pi_i\}_{i=1}^{\infty}$. Let I_1 denote the π_1 -isolator of H in K and, for $n \geq 1$, let I_{n+1} denote the π_{n+1} -isolator of I_n in K. Then each of the indices $|I_{n+1} : I_n|$ is infinite. Thus $\pi_1(X)$ does not satisfy the max-∞ property for non-nilpotent subgroups. We thus have a contradiction.

Anyway, $\pi_1(X)$ acts nilpotently on $H_i(\tilde{X})$. Hence the center $Z(\pi_1(X))$ of $\pi_1(X)$ is infinite and finitely generated. Then we can take an infinite cyclic subgroup $Z(\pi_1(X))$ for a torsion free nontrivial normal abelian subgroup.

From the fact that $f : X \to Y$ is an acyclic map and the classical homotopy exact sequence of fibration: $F_f \to X \to Y$, we know that $\pi_1(f)$ is an epimorphism. Furthermore $H_1(F_f) \cong \frac{\pi_1(F_f)}{[\pi_1(F_f),\pi_1(F_f)]} = 0$ where $[,]$ means the commutator subgroup and F_f is the homotopy fiber of f. Thus $P\pi_1(X)$ is perfect normal subgroup of $\pi_1(X)$. Since X is nilpotent, $P\pi_1(X)$ is trivial. Thus $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$ is an isomorphism. By use of the Hurewicz Theorem [4] inductively, $\pi_i(F_f) = 0$. Thus f is a weak homotopy equivalence. By the Whitehead Theorem [4], f is a homotopy equivalence. Since the Euler characteristic number is invariant under the homotopy equivalence, thus by Theorem 3.1, our proof is completed.

At any cases above we get $\pi_1(f)$ isomorphic. From the quasi-nilpotent homology equivalence of f, f is a homotopy equivalence. By the same method of the last part of Theorem A, our assertion is proved.

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