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A REMARK ON SOME INEQUALITIES FOR THE SCHATTEN p-NORM

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Abstract For a closed densely defined linear operator T on a Hilbert space H, let Π denote the function which corresponds to T, the orthogonal projection from $H \oplus H$ onto the graph of T. We extend some ordinary norm ineqralites comparing $||\Pi(A) - \Pi(B)||$ and $||A - B||$ to the Schatten p-norm where A and B are bounded operators on H.

1. Introduction

Let H be a complex Hilbert space, $CD(H)$ denote the faimly of closed densely defined linear operators on H, and $B(H)$ the bounded members of $CD(H)$. Let Π denote the function which assigns to each T in $CD(H)$ the orthogonal projection from $H \oplus H$ onto the graph of T.

For $1 \leq p < \infty$, a compact operator T is said to be in the Schatten p-class C_p if $\sum\limits_{i=1}^{\infty} s_i(T)^p < \infty$, where $s_1(T) \geq s_2(T) \geq \cdots$ are the eigenvalues of the positive operator $(T^*T)^{1/2}$. The Schatten p-norm of T is defined by $||T||_p = (\sum\limits_{i=1}^{\infty} s_i(T)^p)^{1/p}$. It is known that C_p is an ideal in $B(H)$. Moreover, $||.||_p$ defines a norm on C_p which makes it a Banach space see[15] and [14].

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In [8] F. Kittaneh investigated quantitative estimates showing the equivalence between the so-called gap metric and the ordinary operator norm, and in fact obtained various norm inequalities comparing $||\Pi(A) - \Pi(B)||$, $||A - B||$ and some other related quantities. Getting idea from his works, we extend some inequalities to Schatten p-nrom. We begin with some well known lemmas, which is needed to accomplish our goal. Before stating these lemmas, we point out that the notations $\alpha(T) = (1 + T^*T)^{-\frac{1}{2}}$ and $\beta(T) = (1 + T^*T)^{-1/2}$ are used, frequently, in this article. Recall that $0 < \beta(T) \leq 1, \alpha(T) = T\beta(T)$ and $\beta(T) = (1 - \alpha(T)^*\alpha(T)))^{1/2}$ ([5] and [7]).

LEMMA 1. *[13] If $A \in C_p(1 \leq p < \infty)$ and B, C are operators in $B(H)$ then*

$$||CAB||_p \leq ||C||\ ||A||_p||B||.$$

LEMMA 2. *[3] or [9]. If $A, B \in B(H)$ are positive and $A + B \geq C \geq 0$, then*

$$C||A - B||_p \leq ||A^2 - B^2||_p,$$

for $1 \leq p < \infty$.

LEMMA 3. *[11]. If $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an operator in $B(H \oplus H)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

(a) $||T||_p^p \geq ||A||_p^p + ||B||_p^p + ||C||_p^p + ||D||_p^p$ for $2 \leq p < \infty$,

(b) $||T||_p^p \leq ||A||_p^p + ||B||_p^p + ||C||_p^p + ||D||_p^p$ for $1 \leq p \leq 2$,

(c) $||T||_p^q \leq (||A||_p^p + ||D||_p^p)^{q/p} + (||B||_p^p + ||C||_p^p)^{q/p}$ for $2 \leq p < \infty$,

(d) $||T||_p^q \geq (||A||_p^p + ||D||_p^p)^{q/p} + (||B||_p^p + ||C||_p^p)^{q/p}$, for $1 < p \leq 2$,

(e) $2||T||_1 \geq ||A||_1 + ||B||_1 + ||C||_1 + ||D||_1$.

Note that part (e) is not brought in [11] but it can be proved easily, using the same technique as the previous parts. The following lemma is also well known. However, we were unable to find this exact statement any where in the literature.

LEMMA 4. *If $A, B \in B(H)$ then*

$$||A^*A - B^*B||_p \leq ||A - B||_p||A + B||,$$

for $1 \leq p < \infty$.

Proof. If $A - B \notin C_p$ we have nothing to prove Otherwise,

$$\begin{aligned} ||A^*A - B^*B||_p &= ||\frac{1}{2}(A - B)^*(A + B) + \frac{1}{2}(A + B)^*(A - B)||_p \\ &\leq \frac{1}{2}||A - B||_p||A + B|| \\ &\quad + \frac{1}{2}||A + B||\ ||A - B||_p (\text{Lemma } 1) \\ &= ||A - B||_p||A + B||. \end{aligned}$$

2. Main Results

In this section we bring the extensions mentioned above.

THEOREM 1. *If $S, T \in B(H)$, then for $1 \leq p \leq 2$,*

$$\begin{aligned} ||\Pi(S) - \Pi(T)||_p^p &\leq 2||\alpha(S) - \alpha(T)||_p^p[||\alpha(S) + \alpha(T)||^p \\ &\quad + (1 + \frac{1}{4}||\beta(S)^{-2}||||\beta(T)^{-2}||\ ||\alpha(S) + \alpha(T)||^2)^p] \end{aligned}$$

and for $2 \leq p < \infty$,

$$||\Pi(S) - \Pi(T)||_p^q \leq 2^{q/p}||\alpha(S) - \alpha(T)||_p^q[||\alpha(S) + \alpha(T)||^q$$
$$+ (1 + \frac{1}{4}||\beta(S)^{-2}||||\beta(T)^{-2}||||\alpha(S) + \alpha(T)||^2)^q].$$

(where $\frac{1}{p} + \frac{1}{q} = 1$*).*

Proof. By [5, Remark, P.532]

$$\Pi(T) = \begin{pmatrix} \beta(T)^2 & \beta(T)\alpha(T)^* \\ \alpha(T)\beta(T) & \alpha(T)\alpha(T)^* \end{pmatrix},$$

where $T \in B(H)$ for $1 \leq p \leq 2$, by part (b) of Lemma 3,

$$\begin{aligned} ||\Pi(S) - \Pi(T)||_p^p \leq\ & ||\beta(S)^2 - \beta(T)^2||_p^p \\ & + ||\beta(S)\alpha(S)^* - \beta(T)\alpha(T)^*||_p^p \\ & + ||\alpha(S)\beta(S) - \alpha(T)\beta(T)||_p^p \\ & + ||\alpha(S)\alpha(S)^* - \alpha(T)\alpha(T)^*||_p^p \\ =\ & ||\alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T)||_p^p \\ & + 2||\alpha(S)\beta(S) - \alpha(T)\beta(T)||_p^p \\ & + ||\alpha(S)\alpha(S)^* - \alpha(T)\alpha(T)^*||_p^p \end{aligned}$$

Also, Lemma 4, implies that

$$||\alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T)||_p^p \leq ||\alpha(S) - \alpha(T)||_p^p||\alpha(S) + \alpha(T)||^p$$

and

$$||\alpha(S)\alpha(S)^* - \alpha(T)\alpha(T)^*||_p^p \leq ||\alpha(S) - \alpha(T)||_p^p||\alpha(S) + \alpha(T)||^p.$$

Furthermore,

$$
\begin{aligned}
||\alpha(S)\beta(S) - \alpha(T)\beta(T)||_p = {} & ||\frac{1}{2}(\alpha(S) - \alpha(T))(\beta(S) + \beta(T)) \\
& + \frac{1}{2}(\alpha(S) + \alpha(T))(\beta(S) - \beta(T))||_p \\
& \leq \frac{1}{2}(\alpha(S) - \alpha(T)||_p(\beta(S) + \beta(T))|| \\
& + \frac{1}{2}||\alpha(S) + \alpha(T)||\ ||\beta(S) - \beta(T)||_p \text{ (Lemma 1)} \\
& \leq ||\alpha(S) - \alpha(T)||_p + \frac{1}{2}(||\alpha(S) \\
& + \alpha(T)||||\beta(S)\beta(S)^{-1} - \beta(T)^{-1})\beta(T)||_p) \\
& leq||\alpha(S) - \alpha(T)||_p + \frac{1}{2}(||\beta(S)||\ ||\beta(T)||||\alpha(S) \\
& + \alpha(T)||\ ||\beta(S)^{-1} - \beta(T)^{-1}||_p \text{ (Lemma 1)} \\
& \leq ||\alpha(S) - \alpha(T)||_p + \frac{1}{4}(||\alpha(S) \\
& + \alpha(T)||\ ||\beta(S)^{-2} - \beta(T)^{-2}||_p) \text{ (Lemma 2)} \\
& = ||\alpha(S) - \alpha(T)||_p + \frac{1}{4}(||\alpha(S) \\
& + \alpha(T)||\ ||\beta(S)^{-2}(\beta(S)^2 - \beta(T)^2\beta(T)^{-2}||_p) \\
& \leq ||\alpha(S) - \alpha(T)||_p + \frac{1}{4}(||\beta(S)^{-2}||\ ||\beta(T)^{-2}||||\alpha(S) \\
& + \alpha(T)||\ ||\beta(S)^2 - \beta(T)^2||_p) \text{ (Lemma 1)} \\
& = ||\alpha(S) - \alpha(T)||_p + \frac{1}{4}(||\beta(S)^{-2}||\ ||\beta(T)^{-2}||||\alpha(S) \\
& + \alpha(T)||\ ||\alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T)||_p) \\
& \leq ||\alpha(S) - \alpha(T)||_p + \frac{1}{4}(||\beta(S)^{-2}||||\beta(T)^{-2}||\ ||\alpha(S) \\
& + \alpha(T)||^2||\alpha(S) - \alpha(T)||_p) \text{ (Lemma 2).}
\end{aligned}
$$

Thus, we conclude that

$$\begin{aligned}||\Pi(S)-\Pi(T)||_p^p \le 2||\alpha(S)-\alpha(T)||_p^p||\alpha(S)+\alpha(T)||^p \\ + 2(||\alpha(S)-\alpha(T)||_p + \frac{1}{4}||\beta(S)^{-2}||\ ||\beta(T)^{-2}|| \\ ||\alpha(S)+\alpha(T)||^2||\alpha(S)-\alpha(T)||_p)^p.\end{aligned}$$

For $2 \le p < \infty$, by part (c) of Lemma 3, we have

$$\begin{aligned}||\Pi(S)-\Pi(T)||_p^p &\le (||\beta(S)^2-\beta(T)^2||_p^p \\ &+ ||\alpha(S)\alpha(S)^*-\alpha(T)\alpha(T)^*||_p^p)^{q/p} \\ &+ (||\alpha(S)\beta(S)-\alpha(T)\beta(T)||_p^p \\ &+ ||\beta(S)\alpha(S)^*-\beta(T)\alpha(T)^*||_p^p)^{q/p} \\ &= (||\alpha(S)^*\alpha(S)-\alpha(T)^*\alpha(T)||_p^p \\ &+ ||\alpha(S)\alpha(S)^*-\alpha(T)\alpha(T)^*||_p^{q/p} \\ &+ 2^{q/p}||\alpha(S)\beta(S)-\alpha(T)\beta(T)||_p^q.\end{aligned}$$

Similar to the case $1 \le p \le 2$

$$\begin{aligned}||\Pi(S)-\Pi(T)||_p^q \le (2||\alpha(S)-\alpha(T)||_p^p\alpha(S)+\alpha(T)||^p)^{q/p} \\ + 2^{q/p}[||\alpha(S)-\alpha(T)||_p + \frac{1}{4}||\beta(S)^{-2}||\ ||\beta(T)^{-2}|| \\ ||\alpha(S)+\alpha(T)||^2||\alpha(S)-\alpha(T)||_p]^q.\end{aligned}$$

Hence,

$$\begin{aligned}||\Pi(S)-\Pi(T)||_p^q \le 2^{q/p}||\alpha(S)-\alpha(T)||_p^q[||\alpha(S)+\alpha(T)||^q \\ + (1+\frac{1}{4}||\beta(S)^{-2}||||\beta(T)^{-2}||\ ||\alpha(S)+\alpha(T)||^2)^q]\end{aligned}$$

as required.

THEOREM 2. *If $S, T \in B(H)$, then*

$$\begin{aligned}||\alpha(S)-\alpha(T)||_p \le \frac{1}{4}||\Pi(S)-\Pi(T)||_p(2||\beta(S)^{-1}+\beta(T)^{-1}|| \\ + ||\beta(S)^{-2}||\ ||\beta(T)^{-2}||).\end{aligned}$$

for $1 < p < \infty$. *and*

$$||\alpha(S) - \alpha(T)||_1 \leq \frac{1}{2}||\Pi(S) - \Pi(T)||_1(2||\beta(S)^{-1} + \beta(T)^{-1}|| + ||\beta(S)^{-2}||\,||\beta(T)^{-2}||).$$

Proof. First. by **[8, P.793]** we know that $||\alpha(S)\beta(S)|| < \frac{1}{2}$ and using parts (a) and (d) of lemma 3. note that if $1 < p < \infty$. then

$$\begin{aligned}
||\alpha(S) - \alpha(T)||_p &= ||\alpha(S)\beta(S)\beta(S)^{-1} - \alpha(T)\beta(T)\beta(T)^{-1}||_p \\
&\leq \frac{1}{2}||\alpha(S)\beta(S) - \alpha(T)\beta(T)||_p||\beta(S)^{-1} + \beta(T)^{-1}|| \\
&\quad + \frac{1}{2}||\alpha(S)\beta(S) + \alpha(T)\beta(T)||\,||\beta(S)^{-1} \\
&\quad - \beta(T)^{-1}||_p \text{ (Lemma1)} \\
&\leq \frac{1}{2}||\Pi(S) - \Pi(T)||_p||\beta(S)^{-1} + \beta(T)^{-1}|| \\
&\quad + \frac{1}{2}\beta(S)^{-1} - \beta(T)^{-1}||_p^* \\
&\leq \frac{1}{2}||\Pi(S) - \Pi(T)||_p||\beta(S)^{-1} + \beta(T)^{-1}|| \\
&\quad + \frac{1}{4}||\beta(S)^{-2} - \beta(T)^{-1}||_p \text{ (Lemma 2)} \\
&= \frac{1}{2}||\Pi(S) - \Pi(T)||_p||\beta(S)^{-1} + \beta(T)^{-1}|| \\
&\quad + \frac{1}{4}||\beta(S)^{-2}(\beta(S)^2 - \beta(T)^2)\beta(T)^{-1}||_p \\
&\leq \frac{1}{2}||\Pi(S) - \Pi(T)||_p||\beta(S)^{-1} + \beta(T)^{-1}|| \\
&\quad + \frac{1}{4}||\beta(S)^{-2}||\,||\beta(T)^{-2}||\,||\beta(S)^2 - \beta(T)^2||_p \text{ (Lemma 1)} \\
&\leq \frac{1}{2}||\Pi(S) - \Pi(T)||_p||\beta(S)^{-1} + \beta(T)^{-1}|| \\
&\quad + \frac{1}{4}||\beta(S)^{-2}||\,||\beta(T)^{-2}||\,||\Pi(S) - \Pi(T)||_p \\
&\qquad \text{(parts } (a) \text{ and } (d) \text{ of Lemma 3)}
\end{aligned}$$

For the case $p = 1$, the proof is similar by using part (e) of Lemma 3.

THEOREM 3. *If $S, T \in B(H)$ then for $1 \leq p \leq 2$,*

$$||\Pi(S) - \Pi(T)||_p^p \leq 2||S - T||_p^p[||S + T||^p + (1 + \frac{1}{2}||S + T||^2)^p]$$

and for $2 \leq p < \infty$,

$$||\Pi(S) - \Pi(T)||_p^q \leq 2^{q/p}||S - T||_p^q[||S + T||^q + (1 + \frac{1}{2}||S + T||^2)^q].$$

Proof. In can be easily seen that

$$\alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T) = \beta(S)^2(S^*S - T^*T)\beta(T)^2$$

and

$$\alpha(S)\alpha(S)^* - \alpha(T)\alpha(T)^* = \beta(S^*)^2(SS^* - TT^*)\beta(T^*)^2.$$

Then for $1 \leq p < \infty$,

$$\begin{aligned}||\alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T)||_p &\leq ||\beta(S)||^2||S^*S - T^*T||_p||\beta(T)||^2\\ &\leq ||S^*S - T^*T||_p \leq ||S - T||_p||S + T||, \quad \text{(Lemma 4)}\end{aligned}$$

and

$$\begin{aligned}||\alpha(S)\alpha(S)^* - \alpha(T)\alpha(T)^*||_p &\leq ||\beta(S^*)||^2||SS^* - TT^*||_p||\beta(T^*)||^2\\ &\leq ||SS^* - TT^*||_p \leq ||S - T||_p||S + T||, \quad \text{(Lemma 4)}\end{aligned}$$

also,

$$\begin{aligned}||\alpha(S)\beta(S) - \alpha(T)\beta(T)||_p &= ||S\beta(S)^2 - T\beta(T)^2||_p\\ &\leq \frac{1}{2}||(S - T)(\beta(S)^2 + \beta(T)^2)||_p\\ &+ \frac{1}{2}||(S + T)(\beta(S)^2 - \beta(T)^2)||_p\\ &\leq \frac{1}{2}||S - T||_p(||\beta(S)||^2 + ||\beta(T)||^2)\\ &+ \frac{1}{2}||S + T||\ ||\beta(S)^2 - \beta(T)^2||_p\\ &\leq ||S - T||_p + \frac{1}{2}||S + T||\ ||\alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T)||_p\\ &\leq ||S - T||_p + \frac{1}{2}||S + T||^2||S - T||_p.\end{aligned}$$

For $1 \le p \le 2$, by the proof of Theorem 1

$$\begin{aligned}||\Pi(S) - \Pi(T)||_p^p \le\, & ||\alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T)||_p^p \\ & + 2||\alpha(S)\beta(S) - \alpha(T)\beta(T)||_p^p \\ & + ||\alpha(S)\alpha(S)^* - \alpha(T)\alpha(T)^*||_p^p.\end{aligned}$$

So,

$$\begin{aligned}||\Pi(S) - \Pi(T)||_p^p \le\, & ||S - T||_p^p||S + T||^p + 2(||S - T||_p \\ & + \frac{1}{2}||S + T||^2||S - T||_p)^p + ||S - T||_p^p||S + T||^p \\ & = 2||S - T||_p^p[||S + T||^p + (1 + \frac{1}{2}||S + T||^2)^p].\end{aligned}$$

Also for $2 \le p < \infty$, again by the proof of Theorem 1

$$\begin{aligned}||\Pi(S) - \Pi(T)||_p^q \le\, & [||\alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T)||_p^p \\ & + ||\alpha(S)\alpha(S)^* - \alpha(T)\alpha(T)^*||_p^p]^{q/p} \\ & + 2^{q/p}||\alpha(S)\beta(S) - \alpha(T)\beta(T)||_p^q.\end{aligned}$$

Therefore,

$$\begin{aligned}||\Pi(S) - \Pi(T)||_p^q \le\, & (2||S - T||_p^p||S + T||^p)^{q/p} \\ & + 2^{q/p}[||S - T||_p + \frac{1}{2}||S + T||^2||S - T||_p]^q \\ & = 2^{q/p}||S - T||_p^q[||S + T||^q + (1 + \frac{1}{2}||S + T||^2)^q],\end{aligned}$$

as required.

THEOREM 4. *If $S, T \in B(H)$, then we have*

$$\begin{aligned}||S - T||_p \le\, & \frac{1}{2}||\Pi(S) - \Pi(T)||_p(||\beta(S)^{-2} + \beta(T)^{-2}|| \\ & + ||\beta(S)^{-2}||\ ||\beta(T)^{-2}||),\end{aligned}$$

for $1 < p < \infty$, and

$$\begin{aligned}||S - T||_1 \le\, & ||\Pi(S) - \Pi(T)||_1(||\beta(S)^{-2} + \beta(T)^{-2}|| \\ & + ||\beta(S)^{-2}||\ ||\beta(T)^{-2}||).\end{aligned}$$

Proof. Note that $T = \alpha(T)\beta(T)\beta(T)^{-2}$, and so

$$\begin{aligned}
||S-T||_p &= ||\alpha(S)\beta(S)\beta(S)^{-2} - \alpha(T)\beta(T)\beta(T)^{-2}||_p \\
&\leq \frac{1}{2}||(\alpha(S)\beta(S) - \alpha(T)\beta(T))(\beta(S)^{-2} + \beta(T)^{-2})||_p \\
&\quad + \frac{1}{2}||(\alpha(S)\beta(S) + \alpha(T)\beta(T))(\beta(S)^{-2} - \beta(T)^{-2})||_p \\
&\leq \frac{1}{2}||\alpha(S)\beta(S) - \alpha(T)\beta(T)||_p||\beta(S)^{-2} + \beta(T)^{-2}|| \\
&\quad + \frac{1}{2}||\alpha(S)\beta(S) + \alpha(T)\beta(T)||\ ||\beta(S)^{-2} - \beta(T)^{-2}||_p \\
&\leq \frac{1}{2}||\Pi(S) - \Pi(T)||_p||\beta(S)^{-2} + \beta(T)^{-2}|| \\
&\quad + \frac{1}{2}||\beta(S)^{-2} - \beta(T^{-2})||_p \\
&= \frac{1}{2}||\Pi(S) - \Pi(T)||_p||\beta(S)^{-2} + \beta(T)^{-2}|| \\
&\quad + \frac{1}{2}||\beta(S)^{-2}(\beta(S)^2 - \beta(T)^2)\beta(T)^{-2}||_p \\
&\leq \frac{1}{2}||\Pi(S) - \Pi(T)||_p||\beta(S)^{-2} + \beta(T)^{-2}|| \\
&\quad + \frac{1}{2}||\beta(S)^{-2}||\ ||\beta(T)^{-2}||\ ||\beta(S)^2 - \beta(T)^2||_p \\
&\leq \frac{1}{2}||\Pi(S) - \Pi(T)||_p||\beta(S)^{-2} + \beta(T)^{-2}|| \\
&\quad + \frac{1}{2}||\beta(S)^{-2}||\ ||\beta(T)^{-2}||\ ||\Pi(S) - \Pi(T)||_p
\end{aligned}$$

(the parts (a) and (d) of Lemma 3),

which completes the proof of the theorem for $1 < p < \infty$. Also for $p = 1$, use a similar method and part (e) of Lemma 3.

THEOREM 5. *If $S, T \in B(H)$ and $1 \leq p < \infty$, then*

$$||\alpha(|S|) - \alpha(|T|)||_{2p} \leq ||\alpha(S) - \alpha(T)||_{2p}^{1/2}||\alpha(S) + \alpha(T)||_{2p}^{1/2}.$$

Proof. By Lemma 3 of [6] and Theorem 2.1 of [12] we see that

$$\begin{aligned}
||\alpha(|S|) - \alpha(|T|)||_{2p} &= ||\ |\alpha(S)| - |\alpha(T)|\ ||_{2p} \\
&\leq ||\alpha(S) - \alpha(T)||_{2p}^{1/2}||\alpha(S) + \alpha(T)||_{2p}^{1/2}.
\end{aligned}$$

THEOREM 6. *If $S, T \in B(H)$, the*

$$||\alpha(|S|) - \alpha(|T|)||_{2p} \le ||S - T||_p^{1/2} ||S + T||^{1/2},$$

where $1 \le p < \infty$.

Proof. Lemma 3 [6] implies that

$$\begin{aligned} ||\alpha(|S|) - \alpha(|T|)||_{2p} &= ||\ |\alpha(S)| - |\alpha(T)|\ ||_{2p} \\ &= ||(\alpha(S)^*\alpha(S))^{1/2} - (\alpha(T)^*\alpha(T))^{1/2}||_{2p} \\ &\le ||\alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T)||_p^{1/2} \quad \text{[10, Corollary 2]} \\ &= ||\beta(S)^2(S^*S - T^*T)\beta(T)^2||_p^{1/2} \\ &\le ||\beta(S)||\ ||S^*S - T^*T||_p^{1/2} ||\beta(T)|| \\ &\le ||S^*S - T^*T||_p^{1/2} \\ &\le ||S - T^*T||_p^{1/2} ||S + T||^{1/2} \quad \text{(Lemma 4)} \end{aligned}$$

We recall that a real-valued continuous function f on $[0, \infty)$ is said to be operator monotone if for any positive opeartor $A, B \in B(H)$, the relation $A \le B$ implies that $f(A) \le f(B)$. It is known that $f(t) = t^r$ is operator monotone for $0 < r \le 1$, [1].

THEOREM 7. *If $A, B \in B(H)$ are positive, then for any real-valued continuous function f on $[0, \infty)$ with $f(0) = 0$ and f^2 operator monotone, we have*

$$||\alpha(f(A)) - \alpha(f(B))||_{2p} \le ||f(|A - B|)||_{2p},$$

for $1 \le p < \infty$.

Proof. Since

$$f(A)^* = f(A^*) \text{ and}$$
$$\beta(S)^2(S^*S - T^*T)\beta(T)^2 = \alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T)$$

we have

$$\begin{aligned}
||\alpha(f(A)) - \alpha(f(B))||_{2p} &= ||f(A)(1+f(A)^2)^{-1/2} \\
&\quad - f(B)(1+f(B)^2)^{-1/2}||_{2p} \\
&\leq ||(f(A)(1+f(A)^2)^{-1/2})^2 \\
&\quad - (f(B)(1+f(B)^2)^{-1/2})^2||_p^{1/2} \text{ [10, Corollary 2]} \\
&= ||(1+f(A)^2)^{-1}(f(A)^2 - f(B)^2)(1+f(B)^2)^{-1}||_p^{1/2} \\
&\leq ||f(A)^2 - f(B)^2||_p^{1/2} \quad (||(1+f(A)^2)^{-1}|| \\
&= ||\beta(f(A))^2|| \leq 1) \\
&\leq ||f(|A-B|)^2||_p^{1/2} \quad (\text{[2, Theorem 1] applied to} f^2) \\
&\leq ||f(|A-B|)||_{2p}^{1/2} ||f(|A-B|)||_{2p}^{1/2} \ (by [28, P.254] \\
&= ||f(|A-B|)||_{2p}.
\end{aligned}$$

Corollary 1. *If $A, B \in B(H)$ are positive, then for any real number $r, 0 \leq r \leq \frac{1}{2}$, we have*

$$||\alpha(A^r) - \alpha(B^r)||_{2p} \leq || \, |A-B|^r ||_{2p},$$

for $1 \leq p < \infty$.

Proof. Put $f(t) = t^r (0 \leq r \leq \frac{1}{2})$ and apply Theorem 7.

Theorem 8. *If $A, B \in B(H)$ are positive, then for any non-negative operator monotone function f on $[0, \infty)$ with $f(0) = 0$ and $1 \leq p < \infty$, we have*

$$||\alpha(f(A)) - \alpha(f(B))||_p \leq ||f(|A-B|)||_p (1 + (f(\frac{1}{2}||A+B||))^2).$$

Proof. By Theorems 1 and 9 and [2, Theorem1] we have

$$\begin{aligned}
||\alpha(f(A)) - \alpha(f(B))||_p &\leq ||f(A) - f(B)||_p (1 \\
&\quad + \frac{1}{4}||f(A) + f(B)||^2) \\
&\leq ||f(|A-B|)||_p (1 + \frac{1}{4}||f(A) + f(B)||^2).
\end{aligned}$$

Since f is an operator monotone function, then it is concave [1, P.26]. Thus

$$\frac{1}{2}(f(A)+f(B)) \leq f(\frac{1}{2}(A+B)).$$

and by [4, Theorem 2.3]

$$||f(\frac{1}{2}(A+B))|| \leq f(\frac{1}{2}||A+B||).$$

Then

$$||\alpha(f(A)) - \alpha(f(B))||_p \leq ||f(|A-B|)||_p(1 + (f(\frac{1}{2}||A+B||))^2).$$

Corollary 2. *If $A, B \in B(H)$ are positive, then for any real number $r, 0 \leq r \leq 1$, and $1 \leq p < \infty$, we have*

$$||\alpha(A^r) - \alpha(B^r)||_p \leq ||\, |A-B|^r||_p(1 + \frac{||A+B||^{2r}}{2^{2r}}).$$

Proof. Put $f(t) = t^r (0 \leq r \leq 1)$ in the preceding theorem.

Since $||T||_p = ||\, |T|\, ||_p$ for $T \in B(H)$, if $r = 1$ in the preceding corollary, we have

$$||\alpha(A) - \alpha(B)||_p \leq ||A-B||_p(1 + \frac{||A+B||^2}{4}).$$

Corollary 3. *If $A, B \in B(H)$, then*

$$||\alpha(|A|) - \alpha(|B|)||_{2p} \leq ||A-B||_{2p}^{1/2}||A + B||_{2p}^{1/2}(1 + \frac{||\,|A|+|B|\,||^2}{4}),$$

for $1 \leq p < \infty$.

Proof. Applying Corollary 2 to $|A|, |B|$, and $r = 1$ we have

$$\begin{aligned}||\alpha(|A|) - \alpha(|B|)||_{2p} &\leq ||\,|: |A| - |B|\,|\,||_{2p}(1 + \frac{||\,|A|+|B|\,||^2}{4}) \\ &\leq ||A-B||_{2p}^{1/2}||A+B||_{2p}^{1/2}(1 + \frac{||\,|A|+|B|\,||^2}{4}) \text{ [12, Theorem 2.1]}\end{aligned}$$

as required.

COROLLARY 4. *If $A, B \in B(H)$, then*

$$||\alpha(|A|) - \alpha(|B|)||_{2p} \leq ||A - B||_p^{1/2}||A + B||^{1/2}(1 + \frac{||\,|A| + |B|\,||^2}{4}),$$

for $1 \leq p < \infty$.

Proof. In the preceding Corollary we see that

$$||\alpha(|A|) - \alpha(|B|)||_{2p} \leq ||\,|A| - |B|\,||_{2p}(1 + \frac{||\,|A| + |B|\,||^2}{4}).$$

But $||\,|A| - |B|\,||_{2p} \leq ||\,|A|^2 - |B|^2||_p^{1/2}$ (see[10, Corollary 2]), thus by Lemma 4, $||\,|A| - |B|\,||_{2p} \leq ||A - B||_p^{1/2}||A + B||^{1/2}$. Hence

$$||\alpha(|A|) - \alpha(|B|)||_{2p} \leq ||A - B||_p^{1/2}||A + B||^{1/2}(1 + \frac{||\,|A| + |B|\,||^2}{4}).$$

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