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NIELSEN TYPE NUMBERS FOR PERIODIC POINTS ON THE COMPLEMENT

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Abstract A Nielsen number $\overline{N}(f : X - A)$ is a homotopy invariant lower bound for the number of fixed points on $X - A$ where X is a compact connected polyhedron and A is a connected subpolyhedron of X. This number is extended to Nielsen type numbers $\overline{NP_n}(f : X - A)$ of least period n and $\overline{N\phi_n}(f : X - A)$ of the nth iterate on $X - A$ where the subpolyhedron A of a compact connected polyhedron X is no longer path connected.

1. Introduction

The Nielsen number $N(f)$ is a homotopy invariant lower bound for the number of fixed points on a compact connected polyhedron X for a map $f : X \to X$. In [8], Nielsen fixed point theory was extended to a map $f : (X, A) \to (X, A)$ on a pair of polyhedra. A Nielsen number $\overline{N}(f : X - A)$ was introduced in [7], which is a homotopy invariant lower bound for the number of fixed points on $X - A$ for a map $f : (X, A) \to (X, A)$.

A Nielsen type number of least period n, denoted by $NP_n(f)$, is a lower bound for the number of periodic points for a map $f : X \to X$, and a Nielsen type number of the nth iterate, denoted by $N\phi_n(f)$, is lower bound for the number of fixed points of the nth iterate $f^n : X \to X$. These Nielsen type numbers were introduced in [6], and studied for the selfmaps of a path connected compact ANR in [2,5] and for the selfmaps of a non-path connected compact ANR in [3].

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The definition of the Nielsen number $\overline{N}(f : X - A)$ suggests that the definitions of Nielsen type numbers on $X - A$ of period n for a map $f : (X, A) \to (X, A)$ have the structure of the following formulae :

$$\overline{NP_n}(f : X - A) = NP_n(f) - NP_n(f, \overline{f})$$

$$\overline{N\phi_n}(f : X - A) = N\phi_n(f) - N\phi_n(f, \overline{f}).$$

In this paper, X is a compact connected polyhedron and the subpolyhedron A of X is no longer path connected, and $f : (X, A) \to (X, A)$ is a given selfmap. We will define the Nielsen type numbers $\overline{NP_n}(f : X - A)$ and $\overline{N\phi_n}(f : X - A)$ and prove their some properties.

2. Preliminaries

Let $f : X \to X$ be a map of a compact connected polyhedron. A point $x \in X$ is called a periodic point of period n if $f^n(x) = x$. We write $Fix(f^n)$ for the set of periodic points of a map $f : X \to X$. Let $P_n(f) = Fix(f^n) - \bigcup_{m<n} Fix(f^m)$ denote the set of periodic points of least period n. We denote the fundamental group of X based at x_0 by $\pi := \pi_1(X, x_0)$. Let $w : I \to X$ be a path from x_0 to $f(x_0)$ and $nw : I \to X$ be a path from x_0 to $f^n(x_0)$ defined by $nw = w + f(w) + \cdots + f^{n-1}(w)$. We define a homomorphism $f^{nw} : \pi \to \pi$ by $f^{nw}(\alpha) = nw + f^n(\alpha) - nw$ for $\alpha \in \pi$. We define an equivalence relation $\sim$ on π by $\alpha \sim \alpha'$ if there exists a $\beta \in \pi$ such that $\alpha = \beta + \alpha' - f^{nw}(\beta)$. The quotient set of this equvalence relation, denoted by $Coker(1 - f^{nw})$, is the set of periodic point classes of f^n and the element is denoted by $[\alpha]^n$ for $\alpha \in \pi$ ([2],[5],[6]). The homomorphism $f^w : \pi \to \pi$ induces an index preserving function $[f^w] : Coker(1 - f^{nw}) \to Coker(1 - f^{nw})$ defined by $[f^w][\alpha]^n = [f^w\alpha]^n$ (See [2], p.96 Corollary 1). Let l be the smallest integer such that $[f^w]^l[\alpha]^n = [\alpha]^n$ for $[\alpha]^n \in Coker(1 - f^{nw})$. Then $< [\alpha]^n >= \{[\alpha]^n, [f^w][\alpha]^n, \cdots, [f^w]^{l-1}[\alpha]^n\}$ is called the orbit (algebraic orbit) of $[\alpha]^n \in Coker(1 - f^{nw})$ and l is called the length of the orbit $< [\alpha]^n >$. For $m|n$, a function $L_{m,n} = 1 + f^{mw} + \cdots + f^{(n-m)w} : \pi \to \pi$ induces a function $[L_{m,n}] : Coker(1 - f^{mw}) \to Coker(1 - f^{nw})$. We say that

$[\alpha]^n \in Coker(1 - f^{nw})$ is reducible to m if there exists a $[\beta]^m \in Corker(1-f^{mw})$ such that $[L_{m,n}][\beta]^m = [\alpha]^n$. The depth of $[\alpha]^n$, denoted by $d([\alpha]^n)$, is the smallest positive integer m to which $[\alpha]^n$ is reducible. Let d be the depth of $[\alpha]^n$ and l be the length of an orbit $< [\alpha]^n >$.

We notice that $d|n$ and $l|d$, and that the depth of any element of an orbit $< [\alpha]^n >$ is d. Now we can define the depth of the orbit $< [\alpha]^n >$ by $d(< [\alpha]^n >) = d([\alpha]^n)$ ([2]). $[\alpha]^n$ is said to be irreducible if $d([\alpha]^n) = n$.

Let $O_n(f)$ denote the set of all orbits and $IEO_n(f)$ denote the set of all irreducible essential orbits in $Coker(1-f^{nw})$. The height of $IEO_n(f)$, denoted by $h(IEO_n(f))$, is defined to be the sum of the depths of all orbits in $IEO_n(f)$. The Nielsen type number of least period n is defined by $NP_n(f) = h(IEO_n(f))$ ([2],[5],[6]).

Let $X_i (i = 1, 2, \cdots, l)$ be components of a compact polyhedron X and $f : X \to X$ be a map. $\{X_1, X_2, \cdots, X_l\}$ is called a f-cycle in X if $f(X_i) \subseteq X_{i+1}$ for $i = 1, 2, \cdots, l-1$ and $f(X_l) \subseteq X_1$. We call l the length of the cycle, and denote the cycle by $[X_i]$. We define an equivalence relation $\sim$ on $J_r = \{1, 2, \cdots, r\}$ by $i \sim j$ if $i = j$ or $[X_i] = [X_j]$. We denote the equivalence class of $i \in J_r$ by $[i]$ and the set of equivalence classes of J_r by $c(f)$. Let $c(i)$ be the length of $[X_i]$. We shall write the selfmap on X_k by ${f_k}^{c(i)} : X_k \to X_k$ for $[X_i]$ and $k \in [i]$. The Nielsen type number of least period n, $NP_n(f)$ is defined by

$$NP_n(f) = \sum_{[i]\in c(f)} c(i) NP_{n/c(i)}({f_k}^{c(i)}) \qquad ([3]).$$

3. The Nielsen type numbers on $X - A$

Let X be a compact connected polyheron and let A be a nonpath-connected subpolyhedron of X. Let $f : (X, A) \to (X, A)$ be a map $f : X \to X$ such that $f(A) \subset A$ and $\overline{f} = f|A : A \to A$. We denote the path component of A by A_j and write $\overline{f_j} = f|A_j : A_j \to A_k$ for the restriction of f to A_j. Let $\nu : A \to X$ be the inclusion. We take a base point $x_0 \in A_k \subset X$ for a cycle $[A_j]$ and $k \in [i]$. Then ν induces a homomorphism $\nu_\pi : \pi_1(A_k, x_0) \to \pi_1(X, x_0)$. Let $w : I \to X$ be a path from x_0 to $f(x_0)$ and $\overline{w} : I \to A$ be a path from

x_0 to $\overline{f}(x_0)$. Then ν_π induces a function $\nu_* : Coker(1 - \overline{f_k}^{n\overline{w}}) \to Coker(1 - f^{nw})$ where $\overline{f_k}^{n\overline{w}} = (\overline{f_k}^{c(i)})^{n/c(i)\overline{w}}$ for any $[i] \in c(f)$ and $k \in [i]$, and ν_* induces a function $< \nu_* >: O_n(\overline{f}) \to O_n(f)$.

DEFINITION 1. $[\alpha]^n \in Coker(1 - f^{nw})$ is called a common n-periodic point class of f and $\overline{f}$ if there exists an essential n-periodic point class $[\overline{\alpha}]^n$ in $Coker(1 - \overline{f_k}^{n\overline{w}})$ such that $\nu_*([\overline{\alpha}]^n) = [\alpha]^n$. It is an irreducible, essential common n-periodic point class which is itself irreducible and essential ([4]).

LEMMA 2. *Let $l = c(i)$ be the length of a $\overline{f}$-cycle $[A_i]$ and k be an integer in $[i]$. For the functions $[f^{lw}] : Coker(1 - f^{nw}) \to Coker(1-f^{nw})$ and $[\overline{f_k}^{l\overline{w}}] : Coker(1-\overline{f_k}^{n\overline{w}}) \to Coker(1-\overline{f_k}^{n\overline{w}})$, we have $[f^{lw}]\nu_* = \nu_*[\overline{f_k}^{l\overline{w}}]$.*

Proof. Let $lm = n$. For any $[\overline{\alpha}]^n \in Coker(1 - \overline{f_k}^{n\overline{w}})$, we have $[f^{lw}]\nu_*[\overline{\alpha}]^n = [f^{lw}][\alpha]^n = [f^{lw}\alpha]^m$ and $\nu_*[\overline{f_k}^{l\overline{w}}][\overline{\alpha}]^n = \nu_*[\overline{f_k}^{l\overline{w}}\overline{\alpha}]^m = [f^{lw}\alpha]^m$. Hence we have the result.

PROPOSITION 3. *Let $[A_i]$ be a $\overline{f}$-cycle with the length $c(i) = l$ and $k \in [i]$. If $[\alpha]^n \in Coker(1 - f^{nw})$ is an irreducible, essential common n-periodic point class of f and $\overline{f}$, then $[f^{lw}][\alpha]^n$ is an irreducible, essential common n-periodic point class of f and $\overline{f}$.*

Proof. The inclusion $\nu : A_k \to X$ is a morphism from $\overline{f_k}^{l} : A_k \to A_k$ to $f^l : X \to X$ (See [4], p.120). By Lemma 2, we have $\nu_*[\overline{f_k}^{l\overline{w}}][\overline{\alpha}]^n = [f^{lw}]\nu_*[\overline{\alpha}]^n = [f^{lw}][\alpha]^n$. So $[f^{lw}][\alpha]^n$ is common. Since $[f^{lw}]$ is an index preserving bijection (See [3] Proposition 2.1), $[f^{lw}][\alpha]^n$ is essential. Since $[\alpha]^n$ is irreducible, by [2, Lemma 1.12] $[f^{lw}][\alpha]^n$ is irreducible.

Let $IEO_n(f,\overline{f})$ denote the set of all irreducible, essential common n-orbits of f and $\overline{f}$ in $Coker(1-f^{nw})$. By Proposition 3, we define

$$NP_n(f,\overline{f}) = n \times \#(IEO_n(f,\overline{f}))$$

where the symbol # denote the cardinality.
Let $EO_n(\overline{f})$ denote the set of all essential n-orbits in $Coker(1 - \overline{f}^{n\overline{w}})$. We notice that for $k \in [i]$

$$IEO_n(f,\overline{f}) = IEO_n(f) \cap (< \nu_* > (EO_n(\overline{f})) \\ = IEO_n(f) \cap (\bigcup_{[i]\in c(f)} l \cdot EO_m(\overline{f_k}^l))$$

where $n = c(i)m = lm$. Let $\overline{T}_n(f : X - A) = IEO_n(f) - IEO_n(f,\overline{f})$.

DEFINITION 4. $\overline{NP_n}(f : X - A) = h(\overline{T}_n(f : X - A)) = NP_n(f) - NP_n(f,\overline{f})$.

THEOREM 5. *(Homotopy invariance) If the maps $f, g : (X, A) \to (X, A)$ are homotopic, then $\overline{NP_n}(f : X - A) = \overline{NP_n}(g : X - A)$.*

Proof. Let $H : f \simeq g : (X, A) \to (X, A)$ be a homotopy. Then H induces a homotopy $H^n : f^n \simeq g^n$. We write $\overline{H} = H|A \times I$ for the restriction of H to $A \times I$. Let $n\nu = nw + H^n(x_0, -)$ be a path from x_0 to $g^n(x_0)$ and $n\overline{\nu} = n\overline{w} + \overline{H}^n(x_0, -)$ be a path from x_0 to $\overline{g}^n(x_0)$. Then H^n induces an index preserving bijection $H^n_* : Coker(1 - f^{nw}) \to Coker(1 - g^{n\nu})$ such that the following diagram is commutative :

$$(D_1)\quad \begin{array}{ccccc} Coker(1-f^{mw}) & \xrightarrow{[L_{m,n}]} & Coker(1-f^{nw}) & \xrightarrow{[f^w]} & Coker(1-f^{nw}) \\ \downarrow H^m_* & & \downarrow H^n_* & & \downarrow H^n_* \\ Coker(1-g^{m\nu}) & \xrightarrow{[L_{m,n}]} & Coker(1-g^{n\nu}) & \xrightarrow{[g^\nu]} & Coker(1-g^{n\nu}) \end{array}$$

(See [1] Proposition 3.3 and [2] Proposition 2.2).

Let $[A_i]$ be a $\overline{f}$-cycle with the length $c(i) = l$. Let $m = ls$ and $n = lt$ for $m|n$. We put $\overline{f_k}^l = \overline{f_0}$ and $\overline{g_k}^l = \overline{g_0}$ for any $k \in [i]$. Then $\overline{H}$ induces a homotopy $\overline{H}_0 : \overline{f_0} \simeq \overline{g_0} : A_k \to A_k$ such that the following diagrams (D_2) and (D_3) are commutative :

$$(D_2)\quad \begin{array}{ccccc} Coker(1-\overline{f_0}^{s\overline{w}}) & \xrightarrow{[L_{s,t}]} & Coker(1-\overline{f_0}^{t\overline{w}}) & \xrightarrow{[\overline{f_0}^{\overline{w}}]} & Coker(1-\overline{f_0}^{t\overline{w}}) \\ \downarrow \overline{H_0}^s_* & & \downarrow \overline{H_0}^t_* & & \downarrow \overline{H_0}^t_* \\ Coker(1-\overline{g_0}^{s\overline{\nu}}) & \xrightarrow{[L_{s,t}]} & Coker(1-\overline{g_0}^{t\overline{\nu}}) & \xrightarrow{[\overline{g_0}^{\overline{\nu}}]} & Coker(1-\overline{g_0}^{t\overline{\nu}}) \end{array}$$

and

$$(D_3)\qquad \begin{array}{ccc} Coker(1-\overline{f_0}^{t\overline{w}}) & \xrightarrow{\overline{H_0}'_*} & Coker(1-\overline{g_0}^{t\overline{\nu}}) \\ \nu_*\downarrow & & \nu_*\downarrow \\ Coker(1-f^{nw}) & \xrightarrow{H^n_*} & Coker(1-g^{n\nu}) \end{array}$$

(See [3] diagrams (6),(7)).

Since H^n_* is an index preserving bijection and the diagram (D_1) is commutative, H^n_* induces a bijection $< H^n_* >: IEO_n(f) \to IEO_n(g)$. So we have $NP_n(f) = NP_n(g)$.

Since the diagrams (D_2) and (D_3) are commutative, H^n_* induces a bijection $< H^n_* >^{\#}: IEO_n(f,\overline{f}) \to IEO_n(g,\overline{g})$. So we have $NP_n(f,\overline{f}) = NP_n(g,\overline{g})$. Hence we get $\overline{NP_n}(f : X - A) = \overline{NP_n}(g : X - A)$.

THEOREM 6. *(Commutativity) If $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$ are maps, then $\overline{NP_n}(g\circ f : X - A) = \overline{NP_n}(f\circ g : Y - B)$.*

Proof. By ([6] III Theorem 3.4 (ii)), f induces an index preserving bijection $f_* : Coker(1-(g\circ f)^{nw}) \to Coker(1-(f\circ g)^{nf(w)})$ such that the following diagram is commutative :

$$\begin{array}{ccc} Coker(1-(g\circ f)^{mw}) & \xrightarrow{f_*} & Coker(1-(f\circ g)^{mf(w)}) \\ [L_{m,n}]\downarrow & & [L_{m,n}]\downarrow \\ Coker(1-(g\circ f)^{nw}) & \xrightarrow{f_*} & Coker(1-(f\circ g)^{nf(w)}) \\ [(g\circ f)^{w}]\downarrow & & [(f\circ g)^{f(w)}]\downarrow \\ Coker(1-(g\circ f)^{nw}) & \xrightarrow{f_*} & Coker(1-(f\circ g)^{nf(w)}) \end{array}$$

where w is a path from x_0 to $g\circ f(x_0)$.

Then f_* induces a bijection $< f_* >: IEO_n(g\circ f) \to IEO_n(f\circ g)$. So we have $NP_n(g\circ f) = NP_n(f\circ g)$.

By the similar argument in Theorem 5, we have $NP_n(g\circ f,\overline{g}\circ\overline{f}) = NP_n(f\circ g,\overline{f}\circ\overline{g})$. Hence we get $\overline{NP_n}(g\circ f : X - A) = \overline{NP_n}(f\circ g : Y - B)$.

THEOREM 7. *(Homotopy type invariance) Let (X, A) and (Y, B) have the same homotopy type. If $f : (X, A) \to (X, A)$ and $g : (Y, B) \to (Y, B)$ are two self-maps, then $\overline{NP_n}(f : X - A) = \overline{NP_n}(g : Y - B)$.*

Proof. There is a homotopy equivalence $h : (X, A) \to (Y, B)$ such that $h \circ f \simeq g \circ h$. Let $k : (Y, B) \to (X, A)$ be a homotopy inverse of h. Then $k \circ h \simeq 1_X$ and $h \circ k \simeq 1_Y$. So we get $(k \circ h) \circ f \simeq f$ and $g \circ (h \circ k) \simeq g$. By Thoerem 5 and Theorem 6, we have the following :

$$\begin{aligned} NP_n(f) &= NP_n((k \circ h) \circ f) = NP_n((h \circ f) \circ k) \\ &= NP_n((g \circ h) \circ k) = NP_n(g) \end{aligned}$$

and

$$\begin{aligned} NP_n(f, \overline{f}) &= NP_n((k \circ h) \circ f, (\overline{k} \circ \overline{h}) \circ \overline{f}) \\ &= NP_n((h \circ f) \circ k, (\overline{h} \circ \overline{f}) \circ \overline{k}) \\ &= NP_n((g \circ h) \circ k, (\overline{g} \circ \overline{h}) \circ \overline{k}) = NP_n(g, \overline{g}). \end{aligned}$$

Hence we have

$$\overline{NP_n}(f : X - A) = \overline{NP_n}(g : Y - B).$$

THEOREM 8. *(Lower bound property) $f : (X, A) \to (X, A)$ has at least $\overline{NP_n}(f : X - A)$ n-periodic points on $X - A$.*

Proof. Every orbit $< [\alpha]^n >$ in $\overline{T}_n(f : X - A)$ is essential. By ([5] Proposition 2.1), $< [\alpha]^n >$ contains at least $d(< [\alpha]^n >)$ n-periodic points. Hence f has at least $h(\overline{T}_n(f : X - A)) = \overline{NP_n}(f : X - A)$ n -periodic points on $X - A$.

EXAMPLE 9. Let X be a simply connected polyhedron and $f : (X, A) \to (X, A)$ be a map. Then $Coker(1 - f^{nw})$ contains only one n-periodic point class and this class is reducible to 1. So we have

(1)

$$\begin{aligned} \overline{NP_1}(f : X - A) &= \overline{N}(f : X - A) \\ &= \begin{cases} 1, & \text{if } N(\overline{f}) = 0 \text{ and } L(f) \neq 0 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

(2) $\overline{NP_n}(f : X - A) = 0 \quad \text{for} \quad n > 1.$

We denote the set of all m-orbits of $f : X \to X$ by $O_{m|n}(f)$ and $\bigcup_{m|n} \overline{T}_m(f : X - A)$ by $\overline{T}_{m|n}(f : X - A)$ for $m|n$.

DEFINITION 11. *A finite subset S of $O_{m|n}(f)$ - $< \nu_* >$ $(EO_{m|n}(\overline{f}))$ is called a set of n-representatives on $X - A$ for $f : (X - A) \to (X - A)$ if every essential orbit in $O_{m|n}(f) - < \nu_* >$ $(EO_{m|n}(\overline{f}))$ is reducible to at least one element of S for $m|n$.*

LEMMA 12. *Let S be a set n- representatives on $X - A$ for $f : (X, A) \to (X, A)$. Then $\overline{T}_{m|n}(f : X - A) \subset S$.*

Proof. Let $< [\alpha]^m >$ be an element of $\overline{T}_{m|n}(f : X - A)$. Then $< [\alpha]^m >$ is an irreducible, essential orbit. By definition 11, there exists a $< [\beta]^r > \in S$ such that $[L_{r,m}](< [\beta]^r >) = < [\alpha]^m >$. Since $< [\alpha]^m >$ is irreducible, we have $< [\alpha]^m > = < [\beta]^r >$. So we get $< [\alpha]^m > \in S$.

DEFINITION 13. $\overline{N\phi_n}(f : X - A) = min\{h(S) \mid S$ is a set of $n -$ representatives on $X - A$ for $f : (X, A) \to (X, A)\}$ is called the Nielsen type number for the nth iterate of $f : (X, A) \to (X, A)$.

Note that this definition implies that $\overline{N\phi_n}(f : X - A) = 0$ if the empty set is a set of n-representatives for $f : (X, A) \to (X, A)$, and that $N\phi_n(f) = \overline{N\phi_n}(f : X - \phi)$.

THEOREM 14. *(Lower bound property) $f : (X, A) \to (X, A)$ has at least $\overline{N\phi_n}(f : X - A)$ periodic points on $X - A$ of period n.*

Proof. Two elements x and y of $Fix(f^n|X - A)$ belong to the same orbit if and only if there is a nonnegative integer q such that $y = f^q(x)$. Let p be the smallest integer of q such that $x = f^p(x)$ and $R^p = \{x \in X - A \mid x = f^p(x)\}$. Then R^p determines an orbit $< [\alpha]^p >$ for each $p|n$ and $p \geq d(< [\alpha]^p >)$. Let $S \subset O_{p|n}(f) - < \nu_* >$ $(EO_{p|n}(\overline{f}))$ be a set of n-representatives on $X - A$ for $f : (X, A) \to (X, A)$ such that $h(S) = \overline{N\phi_n}(f : X - A)$. Then we have

$$\#(Fix(f^n|X - A)) = \sum_{<[\alpha]^p> \in S} p \geq \sum_{<[\alpha]^p> \in S} d(< [\alpha]^p >)$$

$$= h(S) = \overline{N\phi_n}(f : X - A).$$

THEOREM 15. *For a map $f : (X, A) \to (X, A)$, we have $\overline{N\phi_n}(f : X - A) \geq \sum_{m|n} \overline{NP_m}(f : X - A)$.*

Proof. Let S be a set of n-representatives on $X - A$ for a map $f : (X, A) \to (X, A)$ such that $h(S) = \overline{N\phi_n}(f : X - A)$. By Lemma 12, we have $\overline{T}_{m|n}(f : X - A) \subset S$. So we get

$$\sum_{m|n} \overline{NP_m}(f : X - A) = h(\overline{T}_{m|n}(f : X - A)) \leq h(S)$$
$$= \overline{N\phi_n}(f : X - A).$$

THEOREM 16. *Let $f : (X, A) \to (X, A)$ be a map such that X and each component A_i of A are Jiang spaces. Let $c(i) = l$ be the length of a $\overline{f}$-cycle $[A_i]$ and $m|n$ with $l|m$. If $L(f^m) \neq 0$ and $L(\overline{f_k}^m) \neq 0$ for each $K \in [i]$, then $\overline{T}_{m|n}(f : X - A)$ is a set of n -representatives on $X - A$ for f.*

Proof. Let S be a set of n-representatives on $X - A$ for $f : (X, A) \to (X, A)$ such that $h(S) = \overline{N\phi_n}(f : X - A)$. Then we have $\overline{T}_{m|n}(f : X - A) \subset S$ from Lemma 12. Each orbit $< [\alpha] >$ in S is essential by the hypothesis. So there exists a $< [\beta] > \in S$ such that $< [\alpha] >$ reduces to $< [\beta] >$, and $< [\alpha] >$ is irreducible (See [4], p.127). Hence we get $< [\alpha] > \in \overline{T}_{m|n}(f : X - A)$ and $S = \overline{T}_{m|n}(f : X - A)$.

THEOREM 17. *$\overline{T}_{m|n}(f : X - A)$ is a set of n-representatives on $X - A$ for $f : (X, A) \to (X, A)$ if and only if $\overline{N\phi_n}(f : X - A) = \sum_{m|n} \overline{NP_m}(f : X - A)$.*

Proof. Let $\overline{T}_{m|n}(f : X - A)$ be a set of n-representatives. By Lemma 12 and Theorem 15, we have

$$h(\overline{T}_{m|n}(f : X - A)) \leq \overline{N\phi_n}(f : X - A) \leq h(\overline{T}_{m|n}(f : X - A))$$

and

$$\overline{N\phi_n}(f : X - A) = h(\overline{T}_{m|n}(f : X - A)) = \sum_{m|n} \overline{NP_m}(f : X - A).$$

Conversely, assume that $\overline{T}_{m|n}(f : X - A)$ is not a set of n-representatives. Then we can choose a set S of n-representatives such that $h(S) = \overline{N\phi_n}(f : X - A)$ and $\overline{T}_{m|n}(f : X - A) \subsetneqq S$ by Lemma 12. So we have $\overline{N\phi_n}(f : X - A) \gneqq \sum_{m|n} \overline{NP_m}(f : X - A)$. This contradicts and the result follows.

EXAMPLE 18. Let X be the solid torus $X = S^1 \times D^2$ and let $g : S^1 \times \{0\} \to S^1 \times \{0\}$ be a map with $deg(g) = 2$. Then we have $Fix(g^2) = \{e^{\frac{2k}{3}\pi i} \times \{0\} \mid k = 0, 1, 2\}$ and $Coker(1 - g^{2w}) = \{[k]^2 \mid k = 0, 1, 2\}$. Let S_k^1 be the boundary circle of the disk $\{e^{\frac{2k}{3}\pi i}\} \times D^2$ for $k = 0, 1, 2$ and $A = \bigcup_{k=0}^{2} S_k^1$ be a non-path connected subspace of X. Let $\overline{f} : A \to A$ be a map from each circle to a circle and $\overline{f_k}$ be the restriction of $\overline{f}$ to S_k^1 with $deg(\overline{f_k}) = d_k$. From g and $\overline{f}$, we can extend to a map $f : (X, A) \to (X, A)$. Now we put $[A_0] = \{S_0^1\}$ and $[A_1] = \{S_1^1, S_2^1\}$. Let $\overline{f_0} : S_0^1 \to S_0^1$ be the identity and $\overline{f_1} : S_1^1 \to S_2^1$ be a map with $d_1 = 1$, and $\overline{f_2} : S_2^1 \to S_1^1$ be a map with $d_2 = 3$.

Then we will obtain the following :

i) $Coker(1-f^w) = \{[0]^1\} \cong Z_1$ and $O_1(f) = \{< [0]^1 >\}$, where $< [0]^1 >$ is essential and not common.

ii) $Coker(1 - f^{2w}) = \{[0]^2, [1]^2, [2]^2\} \cong Z_3$ and $O_2(f) = \{< [0]^2 >, < [1]^2 >\}$, where $< [0]^2 >$ is reducible, essential and not common, and $< [1]^2 >$ is irreducible, essential common.

iii) $Coker(1 - f^{3w}) = \{[0]^3, [1]^3, [2]^3, [3]^3, [4]^3, [5]^3, [6]^3\} \cong Z_7$ and $O_3(f) = \{< [0]^3 >, < [1]^3 >, < [3]^3 >\}$, where $< [0]^3 > = \{[0]^3\}$ is reducible, essential and not common, and $< [1]^3 > = \{[1]^3, [2]^3, [4]^3\}$ and $< [3]^3 > = \{[3]^3, [6]^3, [5]^3\}$ are irreducible, essential and not common.

iv) $Coker(1 - f^{6w}) = \{[j]^6 \mid j = 0, 1, 2, \cdots, 62\} \cong Z_{63}$ and $O_6(f) = \{< [j]^6 > \mid j = 0, 1, 3, 5, 7, 9, 11, 13, 15, 21, 23, 27, 31\}$, where $< [0]^6 > = \{[0]^6\}$, $< [21]^6 > = \{[21]^6, [42]^6\}$, $< [9]^6 > = \{[9]^6, [18]^6, [36]^6\}$, $< [27]^6 > = \{[27]^6, [54]^6, [45]^6\}$, $< [1]^6 > = \{[k]^6 \mid k = 1, 2, 4, 8, 16, 32\}$, $< [3]^6 > = \{[k]^6 \mid k = 3, 6, 12, 24, 48, 33\}$, $< [5]^6 > = \{[k]^6 \mid k = 5, 10, 20, 40, 17, 34\}$, $< [7]^6 > = \{[k]^6 \mid k = 7, 14, 28, 56, 49, 35\}$, $< [11]^6 > = \{[k]^6 \mid k = 11, 22, 44, 25, 50, 37\}$, $< [13]^6 > = \{[k]^6 \mid k = 13, 26, 52, 41, 19, 38\}$, $< [15]^6 > = \{[k]^6 \mid k =$

$15, 30, 60, 57, 51, 39\}$, $< [23]^6 >= \{[k]^6 \mid k = 23, 46, 29, 58, 53, 43\}$, $< [31]^6 >= \{[k]^6 \mid k = 31, 62, 61, 59, 55, 47\}$.

we obtain $[L_{m,6}]$ for $m|6$ by

$[L_{1,6}] = 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 = 63$,

$[L_{2,6}] = 1 + 2^2 + 2^4 = 21$,

$[L_{3,6}] = 1 + 2^3 = 9$ and $[L_{6,6}] = 1$,

and the depth of orbits by

$d(< [0]^6 >) = 1$, $d(< [9]^6 >) = d(< [27]^6 >) = 3$, $d(< [21]^6 >) = 2$, $d(< [k]^6 >) = 6$ for $k = 1, 3, 5, 7, 11, 13, 15, 23, 31$.

Also, since $[L_{2,6}](< [1]^2 >) =< [21 \times 1]^6 >=< [21]^6 >$, we can see that $< [21]^6 >$ is common.

So $< [0]^6 >$ is reducible, essential and not common. $< [9]^6 >$ and $< [27]^6 >$ are reducible,essential and not common. $< [21]^6 >$ is reducible, essential and common, $< [k]^6 >$ $(k = 1, 3, 5, 7, 11, 13, 15, 23, 31)$ are irreducible, essential and not common.

From i) $\sim$ iv), we get $\overline{NP_m}(f : X - A)$ and $\overline{N\phi_m}(f : X - A)$ for $m|6$ as the following:

(1) Since $NP_1(f) = |1 - 2| = 1$ and $NP_1(f, \overline{f}) = 0$, we have $\overline{NP_1}(X - A) = 1$.

Since $S = \{< [0]^1 >\}$ is a set of 1-representatives, we have $\overline{N\phi_1}(f : X - A) = h(S) = 1$.

(2) Since $NP_2(f) = |1 - 2^2| - |1 - 2| = 2$ and $NP_2(f, \overline{f}) = 2$, we have $\overline{NP_2}(f : X - A) = NP_2(f) - NP_2(f, \overline{f}) = 0$.

Since $S = \{< [0]^1 >\}$ is a set of 2-representatives, we have $\overline{N\phi_2}(f : X - A) = h(S) = 1$.

(3) Since $NP_3(f) = |1 - 2^3| - |1 - 2| = 6$ and $NP_3(f, \overline{f}) = 0$, we have $\overline{NP_3}(X - A) = NP_3(f) - NP_3(f, \overline{f}) = 6$.

Also since $S = \{< [0]^1 >, < [1]^3 >, < [3]^3 >\}$ is a set of 3-representatives, we have $\overline{N\phi_3}(f : X - A) = h(S) = 1 + 2 \times 3 = 7$.

(4) $NP_6(f) = |1 - 2^6| - |1 - 2^3| - |1 - 2^2| + |1 - 2| = 54$ and $NP_6(f, \overline{f}) = 0$ follow from iv). So we get $NP_6(f : X - A) = 54$.

Also we obtain a set of 6-representatives

$S = \{< [0]^1 >, < [1]^3 >, < [3]^3 >, < [1]^6 >, < [3]^6 >, < [5]^6 >, < [7]^6 >, < [11]^6 >, < [13]^6 >, < [15]^6 >, < [23]^6 >, < [31]^6 >\}$

from iv). So we have $\overline{N\phi_6}(f : X - A) = 1 + 2 \times 3 + 9 \times 6 = 61$ and can see $\overline{N\phi_6}(f : X - A) = \overline{NP_1}(f : X - A) + \overline{NP_2}(f : X - A) + \overline{NP_3}(f : X - A) + \overline{NP_6}(f : X - A) = 1 + 0 + 6 + 54 = 61$.

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