Honam Mathematical J. **24**(2002), No. 1,pp. 121–130

A FUNCTIONAL CENTRAL LIMIT THEOREM FOR ASSOCIATED RANDOM FIELD

TAE-SUNG KIM, MI-HWA KO

Division of Mathematics & Statistics and Institute of Basic Natural Science, WonKwang University Iksan, Jeonbuk 570-749, Korea.
E-mail: starkim@wonkwang.ac.kr.
E-mail: songhack@wonkwang.ac.kr.

Abstract In this paper we prove a functional central limit theorem for a field $\{X_{\underline{j}} : \underline{j} \in Z_+^d\}$ of nonstationary associated random variables with $EX_{\underline{j}} = 0$, $E|X_{\underline{j}}|^{r+\delta} < \infty$ for some $r > 2, \delta > 0$ and $u(n) = O(n^{-\nu})$ for some $\nu > 0$, where $u(n) := \sup_{\underline{i} \in Z_+^d} \sum_{\underline{j}:|\underline{j}-\underline{i}| \geq n} cov(X_{\underline{i}}, X_{\underline{j}})$, $|\underline{x}| = \max(|x_1|, \cdots, |x_d|)$ for $\underline{x} \in \mathbb{R}^d$. Our investigation implies an analogous result in the case of associated random measure.

1. Introduction

Let $\{X_{\underline{j}} : \underline{j} \in Z_+^d\}$ be a random field on some probability space $(\Omega, \mathcal{F}, P)$ with $EX_{\underline{j}} = 0$, $EX_{\underline{j}}^2 < \infty$. For $n \in Z_+$ put $S_{n\underline{1}} = \sum_{\underline{1} \leq \underline{j} \leq n \cdot \underline{1}} X_{\underline{j}}$, assume $n^{-d} E(S_{n\underline{1}}^2) \to_n \sigma^2 \in (0, \infty)$, and define

$$W_n(\underline{t}) = (\sigma n^{d/2})^{-1} \sum_{j_1=1}^{[nt_1]} \cdots \sum_{j_d=1}^{[nt_d]} X_{\underline{j}}, \tag{1}$$

where $W_n(\underline{t}) = 0$ for some $t_i = 0$. Then W_n is a measurable map from $(\Omega, \mathcal{F})$ into, $(\mathcal{D}_d, \mathcal{B}(\mathcal{D}_d))$, where $\mathcal{D}_d$ is the set of all functions on $[0,1]^d$ which have left limits and are continuous from the right, and $\mathcal{B}(\mathcal{D}_d)$ is the Borel σ-field induced by the Skorohod topology. $\{X_{\underline{j}} : \underline{j} \in Z_+^d\}$ fulfills the functional central limit theorem if W_n converges weakly to the d-parameter Wiener process W on $\mathcal{D}_d$.

Received March 28, 2002.
1991 AMS Subject Classification : 60F17, 60G10.
Key words and phrases : associated, random field, functional central limit theorem, random measure.

Our aim is to investigate the functional central limit theorem for random fields satisfying a condition of strong positive dependence called association.

A finite family $\{X_j : 1 \le j \le n\}$ of random variables is said to be associated if for any two coordinatewise nondecreasing functions f and g on R^n such that f and g have finite variance, there holds $Cov(f(X_1, \cdots, X_n), g(X_1, \cdots, X_n)) \ge 0$. An infinite family is associated if every finite subfamily is associated. This definition was introduced by Esary, Proschan and Walkup(1967).

Under some covariance restrictions a number of limit theorems have been proved for associated sequences. In the stationary case, Newman(1980) proved the central limit theorem and Newman and Wright(1981) extended this to a functional central limit theorem. Burton and Waymire(1985) extended the notion of association to the random measure and proved the central limit theorem. Burton and Kim(1988) obtained the functional central limit theorem for a stationary random field of associated random variables. In the nonstationary case Birkel(1988) derived a functional central limit theorem for one dimensional associated processes.

In this paper we derive a functional central limit theorem for a random field of nonstationary associated random variables by applying Bulinskii(1993) moment inequality and finite δ-susceptibility criterion which is a result of Bickel and Wichura(1971) allowing them to conclude tightness.

2. Preliminaries

A random field is associated if whenever $A \subseteq Z_+^d$ is a finite subset and $f, g : \mathbb{R}^A \to R$ are coordinatewise nondecreasing then $Cov[f(X_{\underline{j}} : \underline{j} \in A),\ g(X_{\underline{j}} : \underline{j} \in A)]$ is nonnegative where the covariance is defined (see [5] and [6]).

Bulinskii(1993) obtained the following lemma using the covariance coefficient (see Cox and Grimmett(1984))

$$u(n) = \sup_{\underline{i} \in Z_+^d} \sum_{\underline{j} : |\underline{j} - \underline{i}| \ge n} cov(X_{\underline{i}}, X_{\underline{j}}), \text{ where } n \in Z_+ \cup \{0\}.$$

LEMMA 2.1. *Let* $\{X_{\underline{j}} : \underline{j} \in Z_+^d\}$ *be a field of associated random variables with zero mean. Assume*

$$\sup_{\underline{j} \in Z_+^d} E|X_{\underline{j}}|^{r+\delta} < \infty \text{ for some } r > 2, \delta > 0, \tag{2}$$

$$u(n) = O(n^{-\nu}) \text{ for some } \nu > 0. \tag{3}$$

Then

$$supE|\sum_{\underline{1} \le \underline{j} \le \underline{n}} X_{\underline{j}}|^r = O(\|\underline{n}\|)^{\frac{r}{2}} \text{ for some } r > 2, \tag{4}$$

where $\|\underline{n}\| = n_1 \times n_2 \times \cdots \times n_d$ *for* $\underline{n} = (n_1, \cdots, n_d)$.

LEMMA 2.2. *Let* $\{X_{\underline{j}} : \underline{j} \in Z_+^d\}$ *be a random field of associated random variables with* $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$ *and define* $W_n(\cdot)$ *as in (1). Assume*

$$E\{W_n^2(\underline{t})\} \to_n \|\underline{t}\| \text{ for } \underline{0} \le \underline{t} \le \underline{1}. \tag{5}$$

Then the following conditions are equivalent:

$$E\{W_n(\underline{s})W_n(\underline{t})\} \to_n \|\underline{s}\| \; for \; \underline{0} \le \underline{s} \le \underline{t} \le \underline{1}, \tag{6}$$

$$E\{(W_n(\underline{t}) - W_n(\underline{s}))(W_n(\underline{v}) - W_n(\underline{u}))\} \to_n 0 \quad for \; \underline{0} \le \underline{s} \le \underline{t} \le \underline{u} \le \underline{v} \le \underline{1}. \tag{7}$$

Proof. It follows from (6) that (7) holds. Next, we have

$$\begin{aligned} &E\{W_n(\underline{s})W_n(\underline{t})\} \\ &= E\{(W_n(s) - W_n(\underline{0}))(W_n(\underline{t}) - W_n(\underline{s}) + W_n(\underline{s}) - W_n(\underline{0}))\} \\ &= E\{(W_n(s) - W_n(\underline{0}))(W_n(\underline{t}) - W_n(\underline{s}))\} \\ &\qquad + E\{(W_n(\underline{s}) - W_n(\underline{0}))^2\} \to_n \|\underline{s}\| \end{aligned}$$

according to (5) and (7). Hence (7) implies (6).

For block $B = (\underline{s}, \underline{t}] = \Pi_{i=0}^d (s_i, t_i], \underline{s} = (s_1, \cdots, s_d), \underline{t} = (t_1, \cdots, t_d)$ let

$$W_n(B) = (\sigma n^{\frac{d}{2}})^{-1} \sum_{\underline{j} \in nB} X_{\underline{j}} \tag{8}$$

where $nB = (n\underline{s}, n\underline{t}] = \Pi_{i=1}^d (ns_i, nt_i]$ for $B = (\underline{s}, \underline{t}]$.

LEMMA 2.3. *Let $\{X_{\underline{j}} : \underline{j} \in Z_+^d\}$ be a field of associated random variables with $EX_{\underline{j}} = 0$ and define $W_n(\cdot)$ as in (1). If (2) and (3) hold, then*

$$supE|W_n(\underline{t})|^{2+\delta} \leq C\|\underline{t}\|^{1+\frac{\delta}{2}} \text{ for } \delta > 0. \tag{9}$$

$$supE|W_n(B)|^{2+\delta} \leq C\|B\|^{1+\frac{\delta}{2}}, \tag{10}$$

where $\|B\|$ denotes the Lebesgue measure of B and $B = (\underline{s}, \underline{t}]$ for $\underline{0} \leq \underline{s} \leq \underline{t} \leq \underline{1}$.

Proof. It follows from (1) and Lemma 2.1 that

$$\begin{aligned} supE|W_n(\underline{t})|^{2+\delta} &= sup(\sigma n^{\frac{d}{2}})^{-(2+\delta)} E|\sum_{\underline{0}<\underline{j}\leq[n\underline{t}]} X_{\underline{j}}|^{2+\delta} \\ &\leq C\|\underline{t}\|^{1+\frac{\delta}{2}}. \end{aligned}$$

Similarly, from (8) and Lemma 2.1 we have

$$\begin{aligned} \sup E|W_n(B)|^{2+\delta} &= \sup(\sigma n^{\frac{d}{2}})^{-(2+\delta)} E|\sum_{[n\underline{s}]<\underline{j}\leq[n\underline{t}]} X_{\underline{j}}|^{2+\delta} \\ &\leq C\|\underline{t} - \underline{s}\|^{1+\frac{\delta}{2}} \\ &= C\|B\|^{1+\frac{\delta}{2}} \end{aligned}$$

3. A functional central limit theorem

THEOREM 3.1. *Let $\{X_{\underline{j}} : \underline{j} \in Z_+^d\}$ be a field of associated random variables with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$ and define $W_n(\cdot)$ as in (1). If (2), (3) and (6) hold then $\{X_{\underline{j}} : \underline{j} \in Z_+^d\}$ fulfills the functional central limit theorem.*

Proof. First note that we obtain (9) and (10) by Lemma 2.3. By Lemma 2 of Deo (1975) it is sufficient to show that $W_n(\cdot)$ converges weakly in the Skorohod topology to a stochastic process W which has the following properties :

(a) $E\{W(\underline{t})\} = 0, E\{W(\underline{t})^2\} = \|\underline{t}\|, \underline{0} \leq \underline{t} \leq \underline{1}$.

(b) W has continuous paths.

(c) Increments of W around any collection of strongly seperated blocks in $[0,1]^d$ are independent random variables.

Note that for a block $B = (\underline{s}, \underline{t}] \subset [0,1]^d$

$$W_n(B) = (\sigma n^{\frac{d}{2}})^{-1} \sum_{\underline{j} \in nB} X_{\underline{j}}$$

where $nB = \Pi_{i=1}^d (ns_i, nt_i]$ for $B = (\underline{s}, \underline{t}]$.

From Chebyshev's inequality, Schwaz inequality and (10) it follows that, for neighboring bocks B and F,

$$\begin{aligned}
&P[min(|W_n(B), |W_n(F)|) \geq \lambda] \\
&\leq \lambda^{-(2+\delta)} E[\{min(|W_n(B)|, |W_n(F)|)\}^{2+\delta}] \\
&\leq \lambda^{-(2+\delta)} [E\{|W_n(B)|^{2+\delta}\} E\{|W_n(F)|^{2+\delta}\}]^{1/2} \\
&\leq \lambda^{-(2+\delta)} [\mathrm{C}\|B\|^{1+\frac{\delta}{2}} C\|F\|^{1+\frac{\delta}{2}}]^{1/2} \\
&\leq \lambda^{-(2+\delta)} \mathrm{C}[(\|B\|\|F\|)^{1/2}]^{1+\frac{\delta}{2}} \\
&\leq \lambda^{-(2+\delta)} \mathrm{C}[(\|B\| + \|F\|)]^{1+\frac{\delta}{2}} \\
&= \lambda^{-(2+\delta)} \mathrm{C}\|B \cup F\|^{1+\frac{\delta}{2}}.
\end{aligned} \tag{11}$$

Thus by Theorem 3 of Bickel and Wichura(1971) the following tightness condition (12) is in force

$$\lim_{n \in Z_+} supP\{w(W_n, \delta) > \epsilon\} \to 0 \ as \ \delta \downarrow 0. \tag{12}$$

where $w(W_n, \delta) = \sup_{|\underline{s}-\underline{t}|<\delta} |W_n(\underline{s}) - W_n(\underline{t})|$ and $|\underline{s}-\underline{t}| = max(|s_1 - t_1|, \cdots, |s_d - t_d|)$ and thus the sequence $\{W_n\}$ is tight.

It should be noted that Bickel and Wichura (1971) assumed that $W_n(\cdot)$ vanished along the lower boundary of $[0,1]^d$:

$$\sum_{1 \leq p \leq d} [0,1] \times \cdots \times [0,1] \times \{0\} \times [0,1] \times \cdots \times [0,1]$$

where $\{0\}$ is in pth position. But by (10) $P(\sum_{\underline{j} \in B} X_{\underline{j}} = 0) = 1$ if $\|B\| = 0$, so a version of W_n exists which is zero along the lower boundary. Let X be a limit in distribution of a subsequence of $\{W_n : n \in Z_+\}$. Then it follows from (12) and Theorem 15.5 of Billingsley(1968) that X is continuous with probability one. It suffices to show that X is distributed like W. From the assumption it is easily seen that

$$EW_n(\underline{t}) \to_n 0, \; EW_n^2(\underline{t}) \to_n \|\underline{t}\|. \tag{13}$$

By (9) for n large enough,

$$E|W_n(\underline{t})|^{2+\delta} <= textC\|\underline{t}\|^{1+\delta/2}. \tag{14}$$

Also $\{W_n^2(\underline{t}), n \in Z_+\}$ and $\{W_n(\underline{t}) : n \in Z_+\}$ are uniformly integrable for every $\underline{0} \leq \underline{t} \leq \underline{1}$. As

$$W_n(\underline{t}) \to_n X(\underline{t}), \qquad W_n^2(\underline{t}) \to_n X^2(\underline{t})$$

in distribution (for a subsequence), Theorem 5.4 of Billingsley (1968) and (13) imply $EX(\underline{t}) = 0, EX^2(\underline{t}) = \|\underline{t}\|$. According to Theorem 19.1 of Billingsley (1968), X is distributed like W if X has independent increments, that is for the strongly separated blocks, $B_1, B_2, \cdots, B_k$,

$$X(B_1), X(B_2), \cdots, X(B_k) \text{ are independent for all } k \in Z_+ \tag{15}$$

where $B_k = (\underline{t}_{k-1}, \underline{t}_k], \underline{0} < \underline{t}_0 \leq \cdots \leq \underline{t}_k \leq \underline{1}$.

It remains to show (15). Since $(W_n(B_1), \cdots, W_n(B_k)) \to_n (X(B_1), \cdots, X(B_k))$ in distribution and since $W_n(B_i)'s$ are associated by (P_4) of Esary, Proschan and Walkup(1967) $X(\underline{t}_1) - X(\underline{t}_0), \cdots, X(\underline{t}_k) - X(\underline{t}_{k-1})$ are associated, according to (P_5) of Esary et al. (1967). A similar argument as above (using Theorem 5.4 of Billingsley (1968) and the fact that associated random variables are nonnegatively correlated) yields, for $i \neq j, B_i = (\underline{s}, \underline{t}]$ and $B_j = (\underline{u}, \underline{v}]$,

$$\begin{aligned} &Cov(X(B_i), X(B_j)) \\ &= \lim_{n\to\infty} Cov(W_n(B_i), W_n(B_j)) \\ &= \lim_{n\to\infty} Cov(W_n((\underline{s}, \underline{t}]), W_n((\underline{u}, \underline{v}])) \\ &\leq \lim_{n\to\infty} Cov(W_n(\underline{t}) - W_n(\underline{s}), W_n(\underline{v}) - W_n(\underline{u})) \\ &= 0 \qquad \underline{0} \leq \underline{s} \leq \underline{t} \leq \underline{u} \leq \underline{v} \leq \underline{1}. \end{aligned}$$

according to (7) of Lemma 2.2. Hence the $X(B_i)'s$ are associated and uncorrelated random variables and thus independent by Corollary 3 of Newman (1984). This proves that $X(B_1), \cdots, X(B_k)$ are independent and therefore the proof of Theorem 3.1 is complete.

4. Random measures

In this section we will apply the notions of associated random fields to the random measures, that is, a simple argument using Chebyshev's inequality allows us to extend the functional central limit theorem for associated random fields to random measure. $\mathcal{B}^d$ denotes the collection of Borel subsets of d-dimensional Euclidean space R^d. The space M of all nonnegative measure μ defined on $(R^d, \mathcal{B}^d)$ and finite on bounded sets will be equipped with the smallest σ-field containing basic sets of the form $\{\mu \in M : \mu(A) \leq r\}$ for $A \in \mathcal{B}^d$, $0 \leq r < \infty$.

A random measure X is a measurable map from a probability space $(\Omega, \mathcal{F}, P)$ into $(M, \mathcal{M})$, the induced measure $P_X = P \circ X^{-1}$ on $(M, \mathcal{M})$ is the distribution of X and if X is a random measure and $\mathcal{B}^d$ is a Borel subset of R^d, then $X(B)$ represent the random mass of the region B (see [6]).

For the random measure X, define the K-renonmalization of X to be signed random mesure X_K, where

$$X_K(B) = \frac{X(KB) - EX(KB)}{\sigma K^{d/2}} \tag{16}$$

and let $X_K(\underline{t}) = X_K(t_1, \cdots, t_d)$ be defined by

$$X_K(\underline{t}) = X_K([0, t_1] \times \cdots \times (0, t_d]) \tag{17}$$

for $\underline{t} \in [0, \infty)^d$. Let $\{X_K\}$ be sequence of random measure on R^d. A set function X_K satisfies the central limit theorem if any bounded $B \in \mathcal{B}^d, X_K(B)$ converges in distribution to $N(0, \|B\|)$ and $K \longrightarrow \infty$ where $X_K(B)$ is defined in (16) and $\|B\|$ denotes the Lebesque measure of B and the random measure X satisfies the functional central limit theorem if X_K converges weakly to the d-dimensional Wiener measure W.

DEFINITION 4.1(BURTON, WAYMINE 1985). A random measure X is associated if and only if the family of random variables $\mathcal{F} = \{X(B) : B \text{ is a Borel set }\}$ is associated.

By applying Lemma 2.1 we obtain the following result :

LEMMA 4.2. *Let X be an associated random measure with $EX(B)$, $EX^2(B) < \infty$ and define $X_K(B)$ and $X_K(\underline{t})$ as in (16) and (17). Assume that*

$$\sup_{\underline{j}\in Z_+^d} E|X(I_{\underline{j}})|^{r+\delta} < \infty \text{ for some } r > 2, \delta > 0, \tag{18}$$

$$v(n) = O(n^{-\nu}) \text{ for some } \nu > 0, \tag{19}$$

where $v(n) = \sup_{\underline{i}\in Z_+^d} \sum_{\underline{j}:|\underline{j}-\underline{i}|\geq n} Cov(X(I_{\underline{i}}), X(I_{\underline{j}}))$, $I_{\underline{j}} = (\underline{j} - \underline{1}, \underline{j}]$ *for* $\underline{1} \leq \underline{j} \in Z_+^d$ *and* $n \in Z_+ \cup \{0\}$. *Then*

$$\sup E|X_K(B)|^r = O(\|B\|)^{\frac{r}{2}} \text{ for some } r > 2 \tag{20}$$

and

$$\sup E|X_K(\underline{t})| = O(\|\underline{t}\|)^{\frac{r}{2}} \text{ for some } r > 2. \tag{21}$$

THEOREM 4.3. *Let X be an associated random measure with $EX(B)$ and $EX^2(B) < \infty$ and define $X_K(\underline{t})$ as in (17). Assume that (18), (19) and*

$$E\{X_K(\underline{s})X_K(\underline{t})\} \longrightarrow_K |\underline{s}| \text{ for } 0 \leq \underline{s} \leq \underline{t} \leq \underline{1} \tag{22}$$

hold. Then X satisfies the functional central limit theorem.

Proof. Note that for a block $B \subset [0,1]^d$,

$$X_K(B) = \frac{X(KB) - EX(KB)}{\sigma K^{d/2}} \tag{23}$$

where if $B = \Pi_{i=1}^d (s_i, t_i]$ then $KB = \Pi_{i=1}^d (Ks_i, Kt_i]$. As in (11) it follows from (20) that for neighboring blocks B and F.

$$P\{min(|X_K(B)|.|X_K(F)|) \geq \lambda] \leq \lambda^{-(2+\delta)} c|B \cup F|^{1+\frac{\delta}{2}}$$

and thus by Theorem 3 of Bickel and Wichura(1971) the sequence $\{X_K\}$ is tight. As in the proof of Theorem 3.1, by (20) $P(X(A) = 0) = 1$ if $\|A\| = 0$, so a version of X_K exists, which is 0 along the lower boundary.

Suppose X is the limit in distribution of a subsequence. Then X is continuous with probability one by the similar arguments in the proof of Theorem 3.1. It suffices to show that X is distributed as W. From (23) and condition (22), it is easily seen that

$$EX_K(\underline{t}) = 0, \qquad EX_K^2(\underline{t}) \longrightarrow_K \|\underline{t}\|. \tag{24}$$

By (20), for K large enough,

$$E(|X_K(\underline{t})|^{2+\delta}) \leq \frac{1}{(\sigma K^{d/2})^{2+\delta}} C'(\sigma^2 K^d \|\underline{t}\|)^{1+\frac{\delta}{2}} \tag{25}$$

and so $\{X_K(\underline{t})\}$ and $\{X_K^2(\underline{t})\}$ are uniformly integrable for every $\underline{t} \in [0,1]^d$. As

$$X_K(\underline{t}) \longrightarrow_K X(\underline{t}), \qquad X_K^2(\underline{t}) \longrightarrow_K X^2(\underline{t})$$

in distribution, Theorem 5.4 of Billingsley (1968) and (24) imply that

$$EX(\underline{t}) = 0, \qquad EX^2(\underline{t}) = \|\underline{t}\|.$$

Finally, let $B_1, \cdots, B_m \subset [0,1]^d$ be strongly separated blocks, and let $B_i = (\underline{s}, \underline{t}]$, $B_j = (\underline{u}, \underline{v}]$, where $\underline{0} \leq \underline{s} \leq \underline{t} \leq \underline{u} \leq \underline{v} \leq \underline{1}$. Since random variables $X(I_{\underline{j}})$ are nonnegative correlated, it follows from (22) that

$$\begin{aligned} Cov(X_K(B_i), X_K(B_j)) &\leq Cov(X_K(\underline{t}) - X_K(\underline{s}), \\ &\quad X_K(\underline{v}) - X_K(\underline{u})) \longrightarrow_K 0 \end{aligned} \tag{26}$$

according Lemma 2.2, where $I_{\underline{j}} = (\underline{j} - 1, \underline{j}]$ for $\underline{1} \leq \underline{j} \in Z_+^d$.

Since $X_K(B_{\underline{j}})$'s are associated by Corollary 3 of Newman(1984) and (26) the $X_K(B_{\underline{j}})$'s are independent as $K \longrightarrow B$. Hence, X must have independent increments. Thus every subsequence $\{X_{K'}\}$ of $\{X_K\}$ has further subsequence of $\{X_{K''}\}$ which converge weakly to the Wiener measure W on $[0,1]^d$.

It follows that X_K converges weakly to the d-dimensional Wiener measure W.

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