Kangweon-Kyungki Math. Jour. **9** (2001), No. 2, pp. 115–121

CONVERGENCE OF EXPONENTIALLY BOUNDED C–SEMIGROUPS

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Abstract. In this paper, we establish the conditions that a mild C-existence family yields a solution to the abstract Cauchy problem. And we show the relation between mild C-existence family and C-regularized semigroup if the family of linear operators is exponentially bounded and C is a bounded injective linear operator.

1. Introduction

Let X be a Banach space and let A be a linear operator from $D(A) \subset X$ into X. The abstract Cauchy problem for A with initial data $x \in X$ is to find a solution $u(t)$ to the following initial value problem

$$\frac{du}{dt} = Au, \quad t > 0, \quad u(0) = x. \qquad \text{(ACP)}$$

It is well known ([5]) that when A is closed, generating a C_0 semigroup $\{T(t) : t \geq 0\}$ guarantees the abstract Cauchy problem to have a unique mild solution for all initial data $x \in X$, and the solution is given by $u(t) = T(t)x$.

If the abstract Cauchy problem does not have a mild solution for all $x \in X$, we may look for initial data in X that produce mild solutions. In [4], a family of linear operators was introduced that produces a solution of the abstract Cauchy problem for all initial data in the range of a bounded linear operator C. It is known [4] that for a bounded linear operator C mild C-existence family yields a mild solution for all initial data in the range of C. C-existence family is a generalization of the classical C_0 semigroup such as integrated semigroups and regularized semigroups (see [1, 2, 3, 4, 6]).

Received July 12, 2001.
2000 Mathematics Subject Classification: Primary 47D05, Secondary 47D10.
Key words and phrases: Existence Family, C-semigroups, mild solution.

In this paper, we establish the conditions that mild C-existence family for A produces a solution of (ACP). It is known that a mild C-existence family for A is a C-regularized semigroup generated by an extension of A if the family of operators and A commute. We will show the relation between C-regularized semigroup and mild C-existence family if the family of operators is exponentially bounded. If $CA \subset AC$, mild C-existence family is a C-regularized semigroup generated by an extension of A.

Throughout this paper, we denote by $D(A)$ the domain of the operator A on X and $R(A)$ the range of A. An operator T in X is said to commute with A if $AT \subset TA$. It means that if $x \in D(T)$, $Ax \in D(T)$ and $TAx = ATx$. By a solution $u(t)$ of (ACP), we mean a continuously differentiable function $u : [0, \infty) \to X$ such that $u(t) \in D(A)$ for all $t \geq 0$ and satisfies (ACP). A mild solution of (ACP) is a continuous function $u : [0, \infty) \to X$ such that $v(t) = \int_0^t u(s)ds \in D(A)$ and

$$\frac{d}{dt}v(t) = Av(t) + x, \quad t \geq 0.$$

section2. Mild C-existence Families

Let A be as in (ACP).

Definition 2.1. The strongly continuous family $\{S(t) : t \geq 0\}$ of bounded linear operators on X is called a mild C-existence family for A if for all $x \in X$, $t > 0$, $\int_0^t S(s)xds \in D(A)$ and

$$A\left(\int_0^t S(s)xds\right) = S(t)x - Cx.$$

Remark. If $\{S(t) : t \geq 0\}$ is a mild C-existence family for A, then $u(t) = S(t)x$ is a mild solution of (ACP) with $u(0) = Cx$.

Theorem 2.2. *Let $\{S(t) : t \geq 0\}$ be a mild C-existence family for A. Suppose that $CA \subset AC$. Then there exits a solution $u(t)$ of (ACP) with $u(0) = x \in C(D(A))$.*

Proof. Let $x \in C(D(A))$. Then there exists $y \in D(A)$ such that $x = Cy$. So $Ax = ACy = CAy$. Define

$$u(t) = x + \int_0^t S(s)Ayds.$$

Then $du(t)/dt = S(t)Ay$ and $Au(t) = Ax + A(\int_0^t S(s)Ayds) = Ax + S(t)Ay - CAy = S(t)Ay$, since $\{S(t) : t \geq 0\}$ is a mild C-existence family for A.. So $u(t)$ is a solution of (ACP) with $u(0) = x$. □

THEOREM 2.3. *Let $\{S(t) : t \geq 0\}$ be a mild C-existence family for A. If $x \in D(A)$ and $Ax \in R(C)$, then there exists a solution $u(t)$ for (ACP) with initial data x.*

Proof. Let $y \in X$ such that $Ax = Cy$. Define

$$u(t) = x + \int_0^t S(s)yds.$$

Then $du(t)/dt = S(t)y$ and $Au(t) = Ax + A\int_0^t S(s)yds = Ax + S(t)y - Cy = S(t)y$, since $\{S(t) : t \geq 0\}$ is a mild C-existence family for A. Thus $u(t)$ is a solution of (ACP) with $u(0) = x$. □

Next, we characterize an exponentially bounded mild C-existence family in terms of Laplace transforms.

THEOREM 2.4. *Let $\{S(t) : t \geq 0\}$ be a strongly continuous family of bounded linear operators on X such that $||S(t)|| \leq Me^{\omega t}$, $t \geq 0$, for some M, $\omega > 0$. Suppose that A is closed and $\lambda - A$ is injective for $\lambda > \omega$. Then the following are equivalent.*

(1) *$\{S(t) : t \geq 0\}$ is a mild C-existence family for A.*
(2) *$R(C) \subset R(\lambda - A)$ for $\lambda > \omega$ and*

$$(\lambda - A)^{-1}Cx = \int_0^\infty e^{-\lambda t}S(t)xdt, \quad x \in X,\ \lambda > \omega.$$

Proof. Suppose that $\{S(t) : t \geq 0\}$ is a mild C-existence family for A. Let $x \in X$. By integration by parts, we have

$$\int_0^\infty e^{-\lambda t}S(t)xdt = \lambda \int_0^\infty e^{-\lambda t}\left(\int_0^t S(s)xds\right)dt.$$

Since A is closed,

$$\begin{aligned} A\int_0^\infty e^{-\lambda t}S(t)xdt &= \lambda A\int_0^\infty e^{-\lambda t}\left(\int_0^t S(s)xds\right)dt \\ &= \lambda\int_0^\infty e^{-\lambda t}A\left(\int_0^t S(s)xds\right)dt \\ &= \lambda\int_0^\infty e^{-\lambda t}(S(t)x-Cx)dt \\ &= \lambda\int_0^\infty e^{-\lambda t}S(t)xdt - Cx. \end{aligned}$$

So $Cx=(\lambda-A)\int_0^\infty e^{-\lambda t}S(t)xdt$ for all $x\in X$, and the result follows.

Suppose that (2) is satisfied. Let $x\in X$. By Post-Widder inversion theorem of Laplace transform and the closedness of A, we have

$$\begin{aligned} \lambda^{-1}A(\lambda-A)^{-1}Cx &= \lambda^{-1}A\int_0^\infty e^{-\lambda t}S(t)xdt \\ &= \int_0^\infty e^{-\lambda t}A\left(\int_0^t S(s)xds\right)dt. \end{aligned}$$

Since $Cx=(\lambda-A)(\lambda-A)^{-1}Cx$,

$$\begin{aligned} \lambda^{-1}A(\lambda-A)^{-1}Cx &= (\lambda-A)^{-1}Cx-\lambda^{-1}Cx \\ &= \int_0^\infty e^{-\lambda t}(S(t)x-Cx)dt. \end{aligned}$$

By the uniqueness of the Laplace transform, we have

$$A\left(\int_0^t S(s)xds\right)=S(t)x-Cx. \qquad \square$$

The following shows the relation between mild C-existence family and regularized semigroup.

Theorem 2.5. *Let $\{S(t):t\geq 0\}$ be an exponentially bounded mild C-existence family for A such that $||S(t)||\leq Me^{\omega t}$ for $t\geq 0$, some M and $\omega>0$, and let C be a bounded injective linear operator. Suppose that A is closed and has no eigenvalue in $(\omega,\ \infty)$ with $CA\subset AC$. Then $\{S(t):t\geq 0\}$ is a C-regularized semigroup generated by an extension of A.*

Proof. Let $x \in X$. Since $\{S(t) : t \geq 0\}$ is a mild C-existence family and $CA \subset AC$,

$$CS(t)x - C^2x = CA\left(\int_0^t S(s)xds\right) = A\left(\int_0^t CS(s)xds\right).$$

So $CS(t)x$ is a mild solution of (ACP) with $u(0) = C^2x$. Clearly, $S(t)Cx$ is a mild solution of (ACP) with $u(0) = C^2x$. By the uniqueness of mild solution (Proposition in [4]), $CS(t) = S(t)C$.

By Theorem 2.4, for $x \in X$,

$$\begin{aligned} C(\lambda - A)^{-1}Cx &= C\int_0^\infty e^{-\lambda t}S(t)xdt \\ &= \int_0^\infty e^{-\lambda t}S(t)Cxdt \ = (\lambda - A)^{-1}C^2x. \end{aligned}$$

So we have

$$\begin{aligned} &(\lambda - A)^{-1}C(\mu - A)^{-1}Cx \\ &= \frac{1}{\lambda}A(\lambda - A)^{-1}C(\mu - A)^{-1}Cx + \frac{1}{\lambda}C(\mu - A)^{-1}Cx \\ &= \frac{1}{\lambda}(\lambda - A)^{-1}CA(\mu - A)^{-1}Cx + \frac{1}{\lambda}(\mu - A)^{-1}C^2x \\ &= \frac{1}{\lambda}(\lambda - A^{-1})C(\mu(\mu - A)^{-1}Cx - Cx) + \frac{1}{\lambda}(\mu - A)^{-1}C^2x \\ &= \frac{\mu}{\lambda}(\lambda - A)^{-1}C(\mu - A)^{-1}Cx - \frac{1}{\lambda}(\lambda - A)^{-1}C^2x + \frac{1}{\lambda}(\mu - A)^{-1}C^2x \end{aligned}$$

Thus

$$(\lambda - \mu)(\lambda - A)^{-1}C(\mu - A)^{-1}Cx = (\mu - A)^{-1}C^2x - (\lambda - A)^{-1}C^2x.$$

By Theorem 2.4 and integration by parts,

$$\begin{aligned}
&\frac{1}{\lambda-\mu}((\mu-A)^{-1}C^2x-(\lambda-A)^{-1}C^2x)\\
&=\int_0^\infty e^{-(\lambda-\mu)t}(\mu-A)^{-1}C^2xdt-\frac{1}{\lambda-\mu}\int_0^\infty e^{-\lambda t}S(t)Cxdt\\
&=\int_0^\infty e^{-(\lambda-\mu)t}(\mu-A)^{-1}C^2xdt-\int_0^\infty \frac{1}{\lambda-\mu}e^{-(\lambda-\mu)t}e^{-\mu t}S(t)Cxdt\\
&=\int_0^\infty e^{-(\lambda-\mu)t}\left(\int_0^\infty e^{-\mu s}S(s)Cxds\right)dt\\
&\quad-\int_0^\infty e^{-(\lambda-\mu)t}\left(\int_0^t e^{-\mu s}S(s)Cxds\right)dt\\
&=\int_0^\infty e^{-(\lambda-\mu)t}\left(\int_t^\infty e^{-\mu s}S(s)Cxds\right)dt\\
&=\int_0^\infty e^{-\lambda t}\left(\int_t^\infty e^{-\mu(s-t)}S(s)Cxds\right)dt\\
&=\int_0^\infty e^{-\lambda t}\left(\int_0^\infty e^{-\mu w}S(t+w)Cxdw\right)dt
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(\lambda-A)^{-1}C(\mu-A)^{-1}Cx&=\int_0^\infty e^{-\lambda t}S(t)(\mu-A)^{-1}Cxdt\\
&=\int_0^\infty e^{-\lambda t}S(t)\left(\int_0^\infty e^{-\mu s}S(s)xds\right)dt\\
&=\int_0^\infty e^{-\lambda t}\left(\int_0^\infty e^{-\mu s}S(t)S(s)xds\right)dt.
\end{aligned}$$

By the uniqueness of the Laplace transform, we have

$$S(t+s)C=S(t)S(s).$$

Therefore $\{S(t):t\geq 0\}$ is a C-regularized semigroup generated by an extension of A. □

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